Classes of Idempotents in Hilbert space

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Abstract

An idempotent operator E in a Hilbert space $\mathcal{H}$ ($E^2 = 1$) is written as a 2×2 matrix in terms of the orthogonal decomposition

$$\mathcal{H} = R(E) \oplus R(E)^{\perp}$$

($R(E)$ is the range of E) as

$$E = \begin{pmatrix} 1_{R(E)} & E_{1,2} \\ 0 & 0 \end{pmatrix}.$$

We study the sets of idempotents that one obtains when $E_{1,2} : R(E)^{\perp} \to R(E)$ is a special type of operator: compact, Fredholm and injective with dense range, among others.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators in $\mathcal{H}$, $\mathcal{Q}$ the set of idempotent operators, i.e. operators E such that $E^2 = E$, and $\mathcal{P}$ the set of orthogonal projections in $\mathcal{H}$ (selfadjoint elements in $\mathcal{Q}$). Given an operator A with closed range, $P_{R(A)}$ and $P_{N(A)}$ will denote the orthogonal projections onto the range $R(A)$ and the nullspace $N(A)$ of A, respectively. Given an orthogonal projection P, operators can be written as 2×2 in terms of the decomposition $\mathcal{H} = R(P) \oplus N(P)$. In particular if $E \in \mathcal{Q}$, in terms of $P_{R(E)}$,

$$E = \begin{pmatrix} 1 & E_{1,2} \\ 0 & 0 \end{pmatrix}.$$

An idempotent E determines, and is determined by, the (non orthogonal) decomposition $\mathcal{H} = R(E) \dot{+} N(E)$ (we shall reserve the symbol $\oplus$ for orthogonal sums, and the symbol $\dot{+}$ for direct sums). There are well known formulas highlighting this correspondence, for instance [2]

$$P_{R(E)} = E(E + E^* - 1)^{-1}, \quad P_{N(E)} = (1 - E)(1 - E - E^*)^{-1} \tag{1}$$

and [7]

$$E = P_{R(E)}(P_{R(E)} - P_{N(E)})^{-1}. \tag{2}$$

Implicit in these formulas are the facts that $E + E^* - 1$ and $P_{R(E)} - P_{N(E)}$ are invertible operators for any given $E \in \mathcal{Q}$.

In this paper we study the following subsets of $\mathcal{Q}$:

1. The set $\mathcal{Q}_d$ of idempotents E such that E^*E is diagonalizable (we say the A is diagonalizable if there exists an orthonormal system $\{f_n\}_{n\geq 1}$ and complex numbers α_n such that $A\xi = \sum_{n\geq 1} \alpha_n \langle \xi, f_n\rangle f_n$, for any $\xi \in \mathcal{H}$).

2. The set $\mathcal{Q}_k$ of idempotents E such that in the matrix form above, $E_{1,2}$ is compact.

3. The set $\mathcal{Q}_g$ of idempotents E such that $R(E)$ and $N(E)$ are in generic position. Two subspaces $\mathcal{S}, \mathcal{T} \subset \mathcal{H}$ are in generic position [13] if

$$\mathcal{S} \cap \mathcal{T} = \mathcal{S}^\perp \cap \mathcal{T} = \mathcal{S} \cap \mathcal{T}^\perp = \mathcal{S}^\perp \cap \mathcal{T}^\perp = \{0\}.$$

4. The set $\mathcal{Q}_f$ of idempotents E such that the pair $(P_{R(E)}, P_{N(E)})$ is a Fredholm pair of projections [5], [1]. A pair of projections (P, Q) is a Fredholm pair if

$$PQ|_{R(Q)} : R(Q) \to R(P)$$

is a Fredholm operator in $\mathcal{B}(R(Q), R(P))$. The index of this operator is the index of the pair, and is the integer

$$ind(P, Q) = \dim(R(P) \cap N(Q)) - \dim(N(P) \cap R(Q)).$$

5. The set $\mathcal{Q}_c$ of idempotents E such that the selfadjoint contraction $A = P_{R(E)} - P_{N(E)}$ has a cyclic vector in $\mathcal{H}$.

The contents of the paper are the following. In Section 2 we recall some preliminary facts, concerning the Halmos' decomposition of $\mathcal{H}$ induced by a pair of projections. In Section 3 we study the set $\mathcal{Q}_d$, we give characterizarions and compute its connected components. $\mathcal{Q}_d$ is shown to be dense in $\mathcal{Q}$. In Section 4 we study the set $\mathcal{Q}_k$, also here we compute the connected components. These are closed submanifolds of $\mathcal{B}(\mathcal{H})$, not necesarilly complemented. Moreover, it is shown that $\mathcal{Q}_k$ admits the action of the linear Fredholm group

$$Gl_\infty(\mathcal{H}) = \{G \in \mathcal{B}(\mathcal{H}) : G \text{ is invertible and } G - 1 \text{ is compact}\}.$$

The connected components of $\mathcal{Q}_k$ are the orbits of this action. In Section 5 we study the set $\mathcal{Q}_g$. Elements $E \in \mathcal{Q}_g$ are characterized by the property that there exists a unique minimal geodesic of $\mathcal{P}$ joining $P_{R(E)}$ and $P_{N(E)}$. $\mathcal{Q}_g$ is connected. In Section 6 we study $\mathcal{Q}_f$. Elements in $\mathcal{Q}_f$ have naturally an index. It is shown that the connected components of $\mathcal{Q}_f$ are open in $\mathcal{Q}$, and are parametrized by the index. In Section 7 we introduce three symmetries (=selfadjoint unitaries in $\mathcal{H}$) with remarkable properties with respect to the classes considered. In Section 8 we study $\mathcal{Q}_c$.

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2 preliminary facts

Let us recall the following facts concerning the theory of two projections (see for instance [13] or [1] or [6]). Let $P_1, P_0 \in \mathcal{P}$. We shall consider the special case $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$, for some $E \in \mathcal{Q}$, which corresponds with the property $P_1 - P_0$ invertible, due to the formulas above. For arbitrary P_1, P_0 denote

$$\mathcal{H}_{11} = R(P_1) \cap R(P_0)\ ,\ \ \mathcal{H}_{00} = N(P_1) \cap N(P_0)\ ,\ \mathcal{H}_{10} = R(P_1) \cap N(P_0)\ ,\ \ \mathcal{H}_{01} = N(P_1) \cap R(P_0)$$

and $\mathcal{H}_0$ the orthogonal complement of the sum of the above. This last subspace is usually called the generic part of the pair P_1, P_0. Note also that

$$N(P_1 - P_0) = \mathcal{H}_{11} \oplus \mathcal{H}_{00} \ , \quad N(P_1 - P_0 - 1) = \mathcal{H}_{10} \ \text{ and } N(P_1 - P_0 + 1) = \mathcal{H}_{01},$$

so that the generic part depends in fact of the difference $P_1 - P_0$. In the case $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$, $\mathcal{H}_{11} = \mathcal{H}_{00} = \{0\}$, therefore Halmos' decomposition consists of three subspaces. We shall refer it as the *three space decomposition* induced by E

Halmos proved that there is an isometric isomorphism between $\mathcal{H}_0$ and a product Hilbert space $\mathcal{L} \times \mathcal{L}$ such that in the above decomposition (putting $\mathcal{L} \times \mathcal{L}$ in place of $\mathcal{H}_0$), the *generic parts* of the projections P_1 and P_0 are, respectively

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = cos(X)$ and $S = sin(X)$ for some operator $0 < X \leq \pi/2$ in $\mathcal{L}$ with trivial nullspace. Therefore, in our case $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$, one has (in the three space decomposition $\mathcal{H} = \mathcal{H}_{10} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_0$)

$$P_1 = 1 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \ \ P_0 = 0 \oplus 1 \oplus \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}.$$

In particular,

$$(P_1 - P_0)^2 = 1 \oplus 1 \oplus \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix},$$

so that in this case ($P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$) S and X are invertible in $\mathcal{L}$. In the three space decomposition of $\mathcal{H}$, E is of the form

$$E = 1 \oplus 0 \oplus \begin{pmatrix} 1 & -S^{-1}C \\ 0 & 0 \end{pmatrix}.$$

This follows after straightforward matrix computations, using formula (2).

The following lemma applies in any of the subsets of $\mathcal{Q}$ studied here, and will be useful in the study of their connected componentes.

Lemma 2.1. *Suppose that E and F are in the same connected component of $\mathcal{Q}$, and in the same class $\mathcal{Q}_x$ ($x = d, k, g, f$ or c). Then there exists a unitary operator U in $\mathcal{H}$ such that E and UFU lie again in the same component of $\mathcal{Q}$, the same class $\mathcal{Q}_x$, and have the same range.*

Proof. The first two assertions are true for any unitary operator: F and UFU^* are in the same componwent of $\mathcal{Q}$ (the unitary group of $\mathcal{H}$ is connected), and in the same class $\mathcal{Q}_x$ (unitary conjugation trivially preserves these classes). Then it only remains to find a unitary operator U such that $R(E) = R(UFU^*)$. Since E and F are in the same component of $\mathcal{Q}$, and the map $E \mapsto P_{R(E)}$ is continuous in $\mathcal{Q}$ (using the first of the formulas in (1)). Then $P_{R(E)}$ and $P_{R(F)}$ lie in the same connected component of $\mathcal{P}$. It is known that the connected components of $\mathcal{P}$ coincide with the orbits of the unitary conjugation. Then there exists a unitary operator U such that

$$UP_{R(E)}U^* = P_{R(F)}.$$

The proof follows noting that $UP_{R(E)}U^* = P_{R(UEU^*)}$. □

3 Diagonalizable idempotents

In this section we study the set

$$\mathcal{Q}_d = \{E \in \mathcal{Q} : E^*E \text{ is diagonalizable } \}.$$

Remark 3.1. If $E \in \mathcal{Q}_d$, then there exist orthonormal systems $\{v_n\}_{n\geq 1}$ and $\{w_n\}_{n\geq 1}$ and real numbers $s_n \geq 1$ such that

$$E\xi = \sum_{n\geq 1} s_n \langle \xi, v_n\rangle w_n,$$

where $\langle w_i, v_j\rangle = \frac{1}{s_i}\delta_{ij}$. Moreover, $s_i = 1$ if and only if $v_i = w_i$.

Indeed, this follows from the polar decomposition of E, $E = V(E^*E)^{1/2}$. Since E^*E is diagonalizable, there exists an orthonormal system $\{v_n\}$, and $s_n \geq 0$ (the singular values of E) such that

$$(E^*E)^{1/2}\xi = \sum_{n\geq 1} s_n \langle \xi, v_n\rangle v_n.$$

Then $E\xi = \sum_{n\geq 1} s_n\langle \xi, v_n\rangle V v_n$. Clearly $w_n = Vv_n$ form an orthonormal system. Also, since $w_j \in R(E)$,

$$w_j = E(w_j) = \sum_{n\geq 1} s_n \langle w_j, v_n\rangle w_n,$$

and thus $s_n\langle w_j, v_n\rangle = \delta_{jn}$. Note that

$$1 = \|w_j\| = s_j\langle w_j, v_j\rangle,$$

and $0 \leq \langle w_j, v_j\rangle \leq 1$. Equality occurs in and only if v_j is a multiple of w_j, and thus they are equal. Apparently, any operator E of this form is an idempotent in $\mathcal{Q}_d$.

Remark 3.2. The expression obtained above implies that $E \in \mathcal{Q}_d$ if and only if $E^* \in \mathcal{Q}_d$. Indeed, if $E \in \mathcal{Q}_d$, using the usual notation $w \otimes v$ for the rank one operator $w \otimes v(\xi) = \langle \xi, v\rangle w$, one has

$$E = \sum_{n\geq 1} s_n w_n \otimes v_n,$$

(the series considereed in the strong operator topology) with $\{v_n\}, \{w_n\}$ orthonormal system satisfying $\langle w_i, v_j\rangle = \frac{1}{s_i}\delta_{ij}$. Then

$$E^* = \sum_{n\geq 1} s_n v_n \otimes w_n$$

is an idempotent operator of the same type.

Note the following elementary fact:

Lemma 3.3. *Let $A \in \mathcal{B}(\mathcal{H})$ be selfadjoint. Then A is diagonalizable if and only if A^2 is diagonalizable.*

Proof. A diagonalizable implies A^2 diagonalizable (with the same basis). Suppose A^2 diagonalizable. Then

$$A^2 = \sum_{n\geq 1} \lambda_n P_n,$$

with $\lambda_n > 0$ ($\lambda_n \neq \lambda_m$ if $n \neq m$) and $\{P_n\}_{n\geq 1}$ pairwise orthogonal. Since A commutes with A^2, it commutes with the spectral projections P_n of A^2. Then

$$(P_n A)^2 = \lambda_n P_n.$$

Thus if we regard $P_n A$ as an operator in $R(P_n)$, it is of the form

$$P_n A = \sqrt{\lambda_n} P_n^+ - \sqrt{\lambda_n} P_n^-,$$

with $P_n^+ + P_n^- = P_n$, $P_n^+ P_n^- = 0$. Tnen

$$A = \sum_{n\geq 1} P_n A = \sum_{n\geq 1} \sqrt{\lambda_n} P_n^+ - \sum_{n\geq 1} \sqrt{\lambda_n} P_n^-.$$

□

With the current notations we have:

Proposition 3.4. *The following are equivalent*

1. $E \in \mathcal{Q}_d$.
2. $E_{12}E_{12}^*$ *is diagonalizable in* $R(E)$.
3. $P_{R(E)} - P_{N(E)}$ *is diagonalizable in* $\mathcal{H}$.
4. X *is diagonalizable in* $\mathcal{L}$.

Proof. In matrix form

$$EE^* = \begin{pmatrix} 1 & E_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E_{12}^* & 0 \end{pmatrix} = \begin{pmatrix} 1 + E_{12}E_{12}^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus apparently EE^* is diagonalizable if and only if $E_{12}E_{12}^*$ is diagonalizable.

Denote $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$. Using formula (2),

$$EE^* = P_1(P_1 - P_0)^{-2}P_1.$$

Using the (three space) decomposition $\mathcal{H} = \mathcal{H}_{10} \oplus \mathcal{H}_{01} \oplus (\mathcal{L} \times \mathcal{L})$,

$$(P_1 - P_0)^2 = 1 \oplus 1 \oplus \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix}$$

and thus

$$EE^* = 1 \oplus 0 \oplus \begin{pmatrix} S^{-2} & 0 \\ 0 & 0 \end{pmatrix}.$$

Apparently EE^* is diagonalizable if and only if S^{-2} is diagonalizable in $\mathcal{L}$, which is equivalent both to S and X being diagonalizable in $\mathcal{L}$. If S^2 is diagonalizable, then clearly $(P_1 - P_0)^2$ and $P_1 - P_0$ are diagonalizable in $\mathcal{H}$.

Conversely, if $(P_1 - P_0)^2$ is diagonalizable, the matrix

$$\begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix}$$

is diagonalizable. Any eigenvector (ξ_n, η_n) of this matrix with eigenvalue s_n consists of a pair of eigenvectors of S^2 with the same eigenvalue. On the other hand, any pair of s_n-eigenvectors of S^2 is an eigenvector of this matrix. We must show that the linear span of the set of eigenvectors of S^2 is dense in $\mathcal{L}$. Suppose that ξ_0 is orthogonal to all the eigenvectors of S^2. Then the pair (ξ_0, ξ_0) is orthogonal to all pairs of eigenvectors of S^2, i.e. all eigenvectors ot the matrix. Then $\xi_0 = 0$. Thus S^2 and S are diagonalizable. □

Using Lemma (2.1), one can characterize the connected components of $\mathcal{Q}_d$ (with the relative topology given by the norm of $\mathcal{B}(\mathcal{H})$). Recall the elementary fact that two orthogonal projections lie in the same connected component of $\mathcal{P}$ (or are unitarilly equivalent) if and only if they have the same rank and nullity.

Proposition 3.5. *Let $E, F \in \mathcal{Q}_d$. Then they lie in the same connected component if and only if*

$$\dim(R(E)) = \dim(R(F)) \text{ and } \dim(N(E)) = \dim(N(F)).$$

Proof. Using Lemma (2.1), we may reduce to the case $R(E) = R(F)$. Indeed, the dimension conditions above occur if and only if $P_{R(E)}$ and $P_{R(F)}$ lie in the same connected component of $\mathcal{P}$.

Then

$$E = \begin{pmatrix} 1 & E_{12} \\ 0 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & F_{12} \\ 0 & 0 \end{pmatrix}$$

in the same decomposition. Let

$$E(t) = \begin{pmatrix} 1 & tE_{12} \\ 0 & 0 \end{pmatrix}.$$

Clearly $t \mapsto E(t)$ is a continuous path with values in $\mathcal{Q}_d$ ($E_{12}(t)E_{12}^*(t) = t^2 E_{12}E_{12}^*$ is diagonalizable), which connects E to $P_{R(E)}$. There is a similar path $F(t)$ connecting F to $P_{R(F)} = P_{R(E)}$. Thus E and F lie in the same connected component of $\mathcal{Q}_d$. □

The following is a straightforward consequence of the Theorem of Weyl and von Neuman:

Proposition 3.6. *$\mathcal{Q}_d$ is dense in $\mathcal{Q}$.*

Proof. Pick $E \in \mathcal{Q}$. Using the three space decomposition, we can suppose that E is of the form

$$1 \oplus 0 \oplus \begin{pmatrix} 1 & -S^{-1}C \\ 0 & 0 \end{pmatrix}.$$

Note that $-S^{-1}C$ is selfadjoint (S and C commute). Then, by the Theorem of Weyl and von Neumann, for any $\epsilon > 0$ there exists a selfadjoint operator B_ϵ acting in $\mathcal{L}$, which is diagonalizable, such that $\| - S^{-1}C - B_\epsilon\| < \epsilon$. Let E_ϵ be

$$E_\epsilon = 1 \oplus 0 \oplus \begin{pmatrix} 1 & B_\epsilon \\ 0 & 0 \end{pmatrix}.$$

Apparently, $\|E - E_\epsilon\| = \| - S^{-1}C - B_\epsilon\| < \epsilon$. Clearly $E_\epsilon \in \mathcal{Q}_d$: B_ϵ^2 is diagonalizable. □

4 Idempotents with compact off diagonal entry

In this section we study the set

$$\mathcal{Q}_k = \{E \in \mathcal{Q} : E_{12} \text{ is compact } \}$$

of idempotents with compact off-diagonal entry, or shortly, off-diagonal compact idempotents.

Proposition 4.1. *Let $E \in \mathcal{Q}$. The following are equivalent:*

1. $E \in \mathcal{Q}_k$.
2. $E - E^*$ *is compact.*
3. $P_{R(E)} + P_{N(E)} - 1$ *is compact.*
4. C *is compact in* $\mathcal{L}$.
5. $P_{R(E)}P_{N(E)}$ *is compact.*

Proof. In matrix form

$$E - E^* = \begin{pmatrix} 0 & E_{12} \\ -E_{12}^* & 0 \end{pmatrix}.$$

Apparently $E - E^*$ is compact if and only if E_{12} is compact. As before, denote $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$. Using the formulas (1),

$$P_1 - P_0 - 1 = E(E + E^* - 1)^{-1} + (1 - E)(1 - E - E^*)^{-1} - 1 = (E - E^*)\{E + E^* - 1\}^{-1},$$

it follows that $E - E^*$ is compact if and only if $P_1 + P_0 - 1$ is compact.

In the three space decomposition

$$E - E^* = 0 \oplus 0 \oplus \begin{pmatrix} 0 & -S^{-1}C \\ -S^{-1}C & 0 \end{pmatrix}.$$

Thus it is compact if and only if C is compact (recall that S in invertible in $\mathcal{L}$).

Finally, note that in this decomposition,

$$P_1 P_0 = 0 \oplus 0 \oplus \begin{pmatrix} C^2 & CS \\ 0 & 0 \end{pmatrix},$$

which is compact in $\mathcal{H}$ if and only if C is compact in $\mathcal{L}$. □

In particular, $E \in \mathcal{Q}_k$ if and only if $E^* \in \mathcal{Q}_k$.

Remark 4.2. If $E \in \mathcal{Q}_k$ is non orthognal, since the operator $C = cos(X)$ has non trivial kernel, it follows that

$$X = \sum_{n \geq 1} x_n P_n,$$

with x_n a strictly increasing sequence converging to $\pi/2$, and P_n pairwise ortohogonal of finite rank, with $\sum_{n \geq 1} P_n = 1_{\mathcal{L}}$.

Note that $\mathcal{Q}_k \subset \mathcal{Q}_d$.

Proposition 4.3. *Let $E, F \in \mathcal{Q}_k$. Then E and F lie in the same connected component of $\mathcal{Q}_k$ if and only if*

$$\dim(R(E)) = \dim(R(F)) \text{ and } \dim(N(E)) = \dim(N(F)).$$

Proof. Using the same argument as in the analogous result in the previous section, based on Lemma 2.1, we can suppose that E and F are of the form

$$E = \begin{pmatrix} 1 & E_{12} \\ 0 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & F_{12} \\ 0 & 0 \end{pmatrix}$$

in the same decomposition (i.e. $R(E) = R(F)$). Both idempotents can be connected within $\mathcal{Q}_k$ by means of the line segment

$$E(t) = \begin{pmatrix} 1 & tE_{12} + (1-t)F_{12} \\ 0 & 0 \end{pmatrix}.$$

□

We shall see that $\mathcal{Q}_k$ is a differentiable submanifold of $\mathcal{B}(\mathcal{H})$. It lies inside $\mathcal{Q}$, which is a complemented submanifold of $\mathcal{B}(\mathcal{H})$ [9]. However, $\mathcal{Q}_k$ is not necessarily a *complemented* submanifold. These fact is based on the following result:

Lemma 4.4. *Fix an orthogonal projection P in $\mathcal{B}(\mathcal{H})$. Then the set*

$$\mathcal{P}_P = \{Q \in \mathcal{P} : [Q, P] \text{ is compact } \}$$

is a closed C^∞ submanifold of $\mathcal{B}(\mathcal{H})$.

Proof. Apparently $\mathcal{P}_p$ is a closed subset of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{B}_P$ be

$$\mathcal{B}_P = \{A \in \mathcal{B}(\mathcal{H}) : [A, P] \text{ is compact } \}.$$

Then $\mathcal{B}_P$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Indeed, if

$$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$$

is the quotient map onto de Calkin algebra ($\mathcal{K}(\mathcal{H})$ is the ideal of compact operators), then

$$\mathcal{B}_P = \pi^{-1}(\{a \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) : [a, \pi(P)] = 0\}).$$

Then $\mathcal{B}_P$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, being the pre-image of a C^*-algebra by a $*$-homomorphism. The space $\mathcal{P}_P$ is the space of selfadjoint projections of $\mathcal{B}_P$. In [9] it was proven the the space of selfadjoint projections of an arbitrary C^*-algebra is a complemented submanifold of the algebra. Thus $\mathcal{P}_P$ is a submanifold of $\mathcal{B}(\mathcal{H})$, which may not be complemented, since $\mathcal{B}_p$ may not be a complemented subalgebra of $\mathcal{B}(\mathcal{H})$. □

Remark 4.5. $\mathcal{B}_P$ is complemented in $\mathcal{B}(\mathcal{H})$ only if P has finite or cofinite rank, in which case $\mathcal{B}_p = \mathcal{B}(\mathcal{H})$. Indeed, if we fix $P \in \mathcal{P}$ and write the elements of $\mathcal{B}(\mathcal{H})$ as 2×2 matrices in terms of P, a simple comuputation shows that

$$\mathcal{B}_P = \{A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : A_{12}, A_{21} \text{ are compact}\}.$$

Note that the subspace

$$\mathcal{S}_{12} = \{B = \begin{pmatrix} 0 & B_{12} \\ 0 & 0 \end{pmatrix} : B_{12} \text{ is compact}\}$$

is apparently complemented in $\mathcal{B}_p$. Thus, if $\mathcal{B}_P$ were complemented in $\mathcal{B}(\mathcal{H})$, then also $\mathcal{S}_{12}$ would be complemented in $\mathcal{B}(\mathcal{H})$: $\mathcal{S}_{12} \oplus \mathcal{R} = \mathcal{B}(\mathcal{H})$. Pick any operator $T \in \mathcal{B}(N(P), R(P))$, consider T'

$$T' = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}.$$

Then there exist unique $R' \in \mathcal{R}$ and $S \in \mathcal{S}_{12}$ such that $T' = S + R'$. Apparently, R' is of the form

$$R' = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix},$$

for some $R \in \mathcal{B}(N(P), R(P))$. This would imply that the space of compact operators in $\mathcal{B}(N(P), R(P))$ would be complemented in $\mathcal{B}(N(P), R(P))$, which means that either $N(P)$ or $R(P)$ is finite dimensional.

Let us recall the following fact concerning the geometry of $\mathcal{P}$ [9]:

Remark 4.6. Let $P, Q \in \mathcal{P}$ such that $\|P - Q\| < 1$. Then there exists a unique selfadjoint operator X which astisfies:

1. $e^{iX} P e^{-iX} = Q$.
2. $\|X\| < \pi/2$.
3. X is P-codiagonal: $PXP = (1-P)X(1-P) = 0$.
4. X is a C^∞ map in the arguments P, Q.

This operator X provides the exponent of the unique (minimal) geodesic of $\mathcal{P}$ joining P and Q, according to the linear connection and the Finsler metric in $\mathcal{P}$, introduced by Corach, Porta and Recht in [9]. The geodesic is

$$\delta(t) = e^{itX} P e^{-itX}.$$

Theorem 4.7. *$\mathcal{Q}_k$ is a closed differentiable manifold of $\mathcal{Q}$ (and therefore also of $\mathcal{B}(\mathcal{H})$).*

Proof. It is apparent $\mathcal{Q}_k$ is closed in $\mathcal{Q}$, for instance using the characterization that $E \in \mathcal{Q}$ belongs to $\mathcal{Q}_k$ if and only if $E - E^* \in \mathcal{K}(\mathcal{H})$ (which is closed in norm).

Fix $E_0 \in \mathcal{Q}_k$, let us construct a local chart for E_0. Denote by $P_1 = P_{R(E_0)}$ and $P_0 = P_{N(E_0)}$. It is a known fact that two orthogonal projections P, Q such that $\|P - Q\| < 1$ are unitarily equivalent, with a unitary operator $U = U(P, Q)$ which is a smooth (and explicit) formula in terms of P and Q. By (1), the map $E \mapsto P_{R(E)}$ is continuous (in fact smooth). Thus the set

$$\mathcal{V}_{E_0} = \{E \in \mathcal{Q}_k : \|P_{R(E)} - P_1\| < r_{E_0} \leq 1\}$$

is an open neighbourhood of E_0 in $\mathcal{Q}_k$. Moreover, there exists a smooth map

$$\mu : \{Q \in \mathcal{P} : \|Q - P_1\| < 1\} \to \mathcal{U}(\mathcal{H}),$$

such that $\mu(E)P_1\mu(E)^* = P_{R(E)}$, and $\mu(E_0) = 1$ (μ is the unitary operator mentioned above). By the facts collected in Remark 4.6 above, $\mu(E) = e^{iX(E)}$, where $X(E)$ is a selfadjoint operator with $\|X(E)\| < \pi/2$, which is codiagonal with respect to P_1. Moreover, the map $E \mapsto X(E)$ defined in $\mathcal{V}_{E_0}$ is smooth.

Note that

$$P_{R(E)} + P_{N(E)} - 1 = \mu(E)\{P_1 + \mu(E)^* P_{N(E)}\mu(E) - 1\}\mu(E)^*$$

is compact, thus $P_1 + \mu(E)^* P_{N(E)}\mu(E) - 1$ is compact, or equivalently,

$$\mu(E)^* P_{N(E)}\mu(E)P_1 \text{ is compact.}$$

We can further shrink r_{E_0} in the definition of $\mathcal{V}_{E_0}$ (which would make $\mu(E)$ closer to 1 and $P_{N(E)}$ closer to P_0), in order that $\mu(E)^* P_{N(E)}\mu(E)$ lies in a coordinate neighbourhood $\mathcal{W}_{P_0}$ of P_0 in the manifold $\mathcal{P}_{P_0}$ [9],

$$\varphi_{P_0} : \mathcal{W}_{P_0} \to \mathcal{Z}_{P_0} = \{Z \in \mathcal{B}_{P_0} : Z^* = Z \text{ is } P_0 - \text{codiagonal}, \|Z\| < \pi/2\}.$$

Then we can define

$$\theta_{E_0} : \mathcal{V}_{E_0} \to \{X \in \mathcal{B}(\mathcal{H}) : X^* = X, \|X\| < \pi/2, X \text{ is } P_0 - \text{codiagonal}\} \times \mathcal{Z}_{P_0},$$

$$\theta_{E_0}(E) = (X(E), \varphi_{P_0}(\mu(E)^* P_{N(E)}\mu(E))).$$

Clearly θ is a smooth map whose inverse is $\theta_{E_0}^{-1}(X, Z) = F$, where F is determined by

$$P_{R(F)} = e^{iX} P_1 e^{-iX} \text{ and } P_{N(F)} = e^{iX}(\varphi_{P_0}^{-1}(e^{iZ} P_0 e^{-iZ}))e^{-iX}.$$

□

Let $Gl_\infty(\mathcal{H})$ be the Linear Fredholm group of $\mathcal{H}$, namely,

$$Gl_\infty(\mathcal{H}) = \{G \in \mathcal{B}(\mathcal{H}) : G \text{ is invertible and } G - 1 \text{ is compact}\}.$$

This group is an analytic Banach Lie group, whose Banach lie algebra identifies with the ideal $\mathcal{K}(\mathcal{H})$ of compact operators. Note that $Gl_\infty(\mathcal{H})$ acts in $\mathcal{Q}_k$. If $G = 1 + K \in Gl_\infty(\mathcal{H})$ with $G^{-1} = 1 + K'$, for $K, K' \in \mathcal{K}(\mathcal{H})$, then

$$GEG^{-1} - (GEG^{-1})^* = (1 + K)E(1 + K') - (1 + K'^*)E^*(1 + K^*) = E - E^* + K'',$$

for some $K'' \in \mathcal{K}(\mathcal{H})$. Thus $GEG^{-1} - (GEG^{-1})^*$ is compact.

Proposition 4.8. *Let $E \in \mathcal{Q}$. Then $E \in \mathcal{Q}_k$ if and only if there exists $G \in Gl_\infty(\mathcal{H})$ such that $E = GP_{R(E)}G^{-1}$.*

Proof. Clearly the selfadjoint projection $P_{R(E)} \in \mathcal{Q}_k$, thus for any $G \in Gl_\infty(\mathcal{H})$, $GP_{R(E)}G^{-1} \in \mathcal{Q}_k$.

Conversely, suppose that $E \in \mathcal{Q}_k$. In the three space decompostion induced by E, consider the operator

$$G = 1 \oplus 1 \oplus \begin{pmatrix} 1 & S^{-1}C \\ 0 & 1 \end{pmatrix}.$$

Apparently G is invertible, is of the form 1 plus compact, and satisfies $GP_{R(E)} = EG$. □

Let us characterize the orbits of this action. First note that the group $Gl_\infty(\mathcal{H})$ is connected (it is an exponential group: any $G \in Gl_\infty(\mathcal{H})$ is of the form $G = e^K$, for some compact operator K, by a straightforward argument using the holomorphic functional calculus in the Banach algebra $\mathcal{B}(\mathcal{H})$). Therefore any pair of elements E, F in the same orbit must lie in the same connected component: $\dim(N(E) = \dim(N(F))$, $\dim(R(E)) = \dim(R(F))$.

Let $P, Q \in \mathcal{P}$. Recall [15] that a projection Q belongs to the *restricted Grassmannian* $G_{res}(P)$ induced by P if

$$PQ|_{R(Q)} : R(Q) \to R(P)$$

is a Fredholm operator. The index of this operator parametrizes the connected components of $G_{res}(P)$: two projections Q, Q' in $G_{res}(P)$ belong to the same component if and only if they have the same index. In [8], A.L. Carey and D.E. Evans proved that the components coincide with the orbits of the action of the *unitary* Fredholm group $\mathcal{U}_\infty(\mathcal{H})$,

$$\mathcal{U}_\infty(\mathcal{H}) = \{U \in \mathcal{B}(\mathcal{H}) : U \text{ is unitary and } U - 1 \text{ is compact}\}.$$

Namely, Q, Q' in $G_{res}(P)$ have the same index if and only if there exists $U \in \mathcal{U}_\infty(\mathcal{H})$ such that $Q' = UQU^*$. In order to characterize the $Gl_\infty(\mathcal{H})$ orbits of elements $E \in \mathcal{Q}_k$, the following elementary fact will be useful:

Lemma 4.9. *Let G in $Gl_\infty(\mathcal{H})$. Then the unitary part U in the polar decomposition of G,*

$$G = U|G|,$$

belongs to $\mathcal{U}_\infty(\mathcal{H})$.

Proof. Since $G = 1 + K$, $|G|^2 = G^*G = 1 + K^*K + K + K^*$ is of the form 1 plus compact, and selfadoint. By the diagonalization theorem of compact selfadjoint operators, it follows that $|G| \in Gl_\infty(\mathcal{H})$. Then

$$U = G|G|^{-1} \in Gl_\infty(\mathcal{H}).$$

□

Proposition 4.10. *Let $E, F \in \mathcal{Q}_k$. Then they lie in the same orbit of the action of $Gl_\infty(\mathcal{H})$ if and only if $P_{R(F)}$ belongs to the connected component of $P_{R(E)}$ in $G_{res}(P_{R(E)})$, i.e. the zero index component of $G_{res}(P_{R(E)})$. Or equivalently*

$$P_{R(E)}P_{R(F)}|_{R(F)} : R(F) \to R(E)$$

is a zero-index Fredholm operator.

Proof. Suppose that E and F lie in the same $G_\infty(\mathcal{H})$ orbit. By the above Proposition, this implies that there exists $G \in G_\infty(\mathcal{H})$ such that $GP_{R(E)}G^{-1} = P_{R(F)}$. It is well known (and an elementary fact, see for instance [9]), that this implies that the unitary part U in the polar decompositon of G also satisfies $UP_{R(E)}U^* = P_{R(F)}$. Therefore, by the above Lemma and remarks on the structure of the connected components of the restricted Grassmannian, it follows that $P_{R(F)}$ belongs to the zero index component of $G_{res}(P_{R(E)})$.

Conversely, suppose $UP_{R(E)}U^* = P_{R(F)}$ for some $U \in U_\infty(\mathcal{H})$. By Proposition (4.8), there exist $G, G' \in Gl_\infty(\mathcal{H})$ such that

$$E = GP_{R(E)}G^{-1} \text{ and } F = G'P_{R(F)}G'^{-1}.$$

Then

$$F = G'U^*G^{-1}E(G'U^*G^{-1})^{-1},$$

with $G'U^*G^{-1} \in Gl_\infty(\mathcal{H})$. □

Using this results, one obtains that

Theorem 4.11. *The orbits of the action of $Gl_\infty(\mathcal{H})$ on $\mathcal{Q}_k$ coincide with the connected components of $\mathcal{Q}_k$.*

Proof. Fix $E \in \mathcal{Q}_k$. We claim that the set

$$\{F \in \mathcal{Q}_k : P_{R(E)}P_{R(F)}|_{R(F)} \in \mathcal{B}(R(F), R(E)) \text{ is a zero index Fredholm operator}\},$$

is an open subset of $\mathcal{Q}_k$. Note that by the above Proposition, this set coincides with the $Gl_\infty(\mathcal{H})$-orbit of E. Indeed, by the first of the formulas in 1, the map

$$\mathcal{Q}_k \to \mathcal{P} \times \mathcal{P} \ , \quad F \mapsto (P_{R(E)}, P_{R(F)})$$

is continuous. Thus it suffices to show that the set

$$\{(P,Q) \in \mathcal{P} \times \mathcal{P} : PQ|_{R(Q)} : R(Q) \to R(P) \text{ is a zero index Fredholm operator}\}$$

is open in $\mathcal{P} \times \mathcal{P}$. The proof of this fact is fairly straightforward ([3]). We include a proof of this fact in the Section treating Fredholm idempotents (Section 5).

Therefore the $Gl_\infty(\mathcal{H})$-orbits $\mathcal{O}_E$ of elements E in $\mathcal{Q}_k$ are open. Therefore they are also closed:

$$\mathcal{Q}_k \setminus \mathcal{O}_E = \cup_{\mathcal{O}_F \neq \mathcal{O}_E} \mathcal{O}_F$$

is open in $\mathcal{Q}_k$. It follows that the orbits coincide with the connected components. □

In other words, if $E, F \in \mathcal{Q}_k$, the condition

$$P_{R(E)}P_{R(F)}|_{R(F)} : R(F) \to R(E) \ \text{ is a zero index Fredholm operator}$$

is equivalent to

$$\dim(R(E)) = \dim(R(F)) \ \text{ and } \quad \dim(N(E)) = \dim(N(F)).$$

5 Idempotents in generic position

In this section we study the set $\mathcal{Q}_g$,

$$\mathcal{Q}_g = \{E \in \mathcal{Q} : R(E) \text{ and } N(E) \text{ are in generic position}\}.$$

This means that $R(E) \cap N(E)^\perp = N(E) \cap R(E)^\perp = \{0\}$. Given $E \in \mathcal{Q}_g$, putting $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$, in [3] it was proven that these comditions imply that there exists a unique (minimal) geodesic in $\mathcal{P}$ joining P_1 and P_0:

$$P_0 = e^{iZ} P_1 e^{-iZ}$$

for a uniquely determined selfadjoint operator Z which is P_1 and P_0 codiagonal and satisfies $\|Z\| \leq \pi/2$. In terms of the operator X acting in $\mathcal{L}$ (in Halmos' model), $C = \cos(X)$, $S = \sin(X)$, e^{iZ} and Z are given by

$$e^{iZ} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix} \text{ and } Z = \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}.$$

Chandler Davis in [10] proved that to any decomposition $A = P_1 - P_0$ of an operator as a difference of projections in generic position, there corresponds a unique symmetry $V = V(P_1, P_0)$, $V^* = V = V^{-1}$, which anti-commutes with A: $VA = -AV$. Explicitly

$$P_1 = \frac{1}{2}\{1 + A + V(1 - A^2)^{1/2}\} \text{ and } P_0 = \frac{1}{2}\{1 - A + V(1 - A^2)^{1/2}\}.$$

Note that this symmetry V satisfies $VP_1V = P_0$ and therefore

$$VEV = 1 - E.$$

The symmetry V and the unique geodesic joining P_1 and P_0 are related by the formula [4]

$$V = e^{iZ}(2P_1 - 1) = (2P_0 - 1)e^{-iZ}.$$

Proposition 5.1. *Let $E \in \mathcal{Q}$. The following are equivalent:*

1. *$E \in \mathcal{Q}_g$.*
2. *$N(E + E^* - 2) = N(E + E^*) = \{0\}$.*
3. *E_{12} has trivial nullspace and dense range.*
4. *There exists a unique minimal geodesic of $\mathcal{P}$ joining P_1 and P_0.*

Proof. As usual $P_1 = P_{R(E)}$ and $P_0 = P_{N(E))}$. As remarked above, $\mathcal{H}_{10} = N(P_1 - P_0 - 1)$ and $\mathcal{H}_{01} = N(P_1 - P_0 + 1)$. Note that

$$P_1 - P_0 - 1 = (E + E^* - 1)^{-1} - 1 = (E + E^* - 1)^{-1}\{2 - E - E^*\},$$

And thus $\mathcal{H}_{10} = N(E + E^* - 2)$. Similarly $\mathcal{H}_{01} = N(E + E^*)$. This proves that the first two conditions are equivalent.

In matrix form

$$E + E^* - 2 = \begin{pmatrix} 0 & E_{12} \\ E_{12}^* & -2 \end{pmatrix}.$$

Then $(\xi_1, \xi_2) \in N(E + E^* - 2)$ if and only if $E_{12}\xi_2 = 0$ and $E_{12}^*\xi_1 = 2\xi_2$. Then

$$E_{12}E_{12}^*\xi_1 = 2E_{12}\xi_2 = 0,$$

which implies $E_{12}^*\xi_1 = 0$, and thus also $\xi_2 = 0$. Conversely, clearly a pair $(\xi_1, \xi_2) \in N(E_{12}^*) \oplus \{0\}$ lies in the nullspace of $E + E^* - 2$. Then

$$N(E + E^* - 2) = N(E_{12}^*) \oplus \{0\}.$$

Similarly

$$N(E + E^*) = \{0\} \oplus N(E_{12}).$$

Thus $E \in \mathcal{Q}_g$ if and only if $N(E_{12}) = N(E_{12}^*) = \{0\}$, i.e. E_{12} has trivial nullspace and dense range.

The equivalence with the last condition was stated above. □

In particular, $E \in \mathcal{Q}_g$ if and only if $E^* \in \mathcal{Q}_g$

Note that if $E \in \mathcal{Q}_g$, the unitary part in the polar decomposition of $E_{12} : N(E) \to R(E)$ is an onto isometry between $N(E)$ and $R(E)$.

Theorem 5.2. *$\mathcal{Q}_g$ is arcwise connected.*

Proof. The last sentence above implies that if $E \in \mathcal{Q}_g$, both $N(E)$ and $R(E)$ are infinite dimensional, thus any pair $E, F \in \mathcal{Q}_g$ belong to the same connected component in $\mathcal{Q}$. Thus we may use again Lemma 2.1, and reduce to the case when $R(E) = R(F)$. Also $\mathcal{H} = \mathcal{H}_0$ can be replaced by the space $\mathcal{L} \times \mathcal{L}$. In matrix form

$$E = \begin{pmatrix} 1 & E_{12} \\ 0 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & F_{12} \\ 0 & 0 \end{pmatrix}.$$

Let $E_{12} = U_E|E_{12}|$ and $F_{12} = U_F|F_{12}|$, where U_E and U_F are unitary operators in $\mathcal{L}$. Since the unitary group of $\mathcal{L}$ is connected, there are continuous paths $U_E(t)$ and $U_F(t)$ of unitaries in $\mathcal{L}$ connecting $U_E(0) = U_E$ with $U_E(1) = 1$ and $U_F(0) = U_F$ with $U_F(1) = 1$. The continuous path

$$\begin{pmatrix} 1 & U_E(t)|E_{12}| \\ 0 & 0 \end{pmatrix}$$

connects E with

$$\begin{pmatrix} 1 & |E_{12}| \\ 0 & 0 \end{pmatrix}$$

inside $\mathcal{Q}_g$. Similarly for F. Thus it remains to see that

$$\begin{pmatrix} 1 & |E_{12}| \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & |F_{12}| \\ 0 & 0 \end{pmatrix}$$

can be connected inside $\mathcal{Q}_g$. Or equivalently, that two positive operators $|E_{12}|, |F_{12}|$ with trivial nullspace (and therefore dense range) can be connected with a continuous path of positive operators with trivial nullspace. It is easy to see that the set of positive operators with trivial nullspace is convex, and the proof follows. □

6 Fredholm idempotents

In this section we study the set $\mathcal{Q}_f$ of Fredholm idempotents,

$$\mathcal{Q}_f = \{E \in \mathcal{Q} : (P_{R(E)}, P_{N(E)}) \text{ is a Fredholm pair}\}.$$

In other words, $E \in \mathcal{Q}_f$ if [5], [1] if and only if

$$P_{N(E)}P_{R(E)}|_{R(E)} : R(E) \to N(E)$$

is a Fredholm operator. The index of this operator (usually called the index of the pair), which we shall call here $i(E)$, the index of E, is

$$i(E) = i(P_{R(E)}, P_{N(E)}) = \dim(R(E) \cap N(E)^{\perp}) - \dim(N(E) \cap R(E)^{\perp}).$$

By the computations in the previous section, this index is also

$$i(E) = \dim(N(E + E^* - 2)) - \dim(N(E + E^*)).$$

These pairs can also be described as those such that $P_{N(E)}$ belongs to the restricted Grassmannian $G_{res}(P_{R(E)})$ (as in Section 3).

Remark 6.1. In [5] it was proven that (P, Q) is a Fredholm pair if and only if ± 1 are isolated (or absent) in the spectrum of $P - Q$, and have finite multiplicity.

The following characterization follows:

Proposition 6.2. *Let $E \in \mathcal{Q}$. The following are aquivalent:*

1. *$E \in \mathcal{Q}_f$.*
2. *$0, 2$ are isolated in the spectrum of $E + E^*$, and have finite multiplicity.*
3. *$E_{12} : R(E)^{\perp} \to R(E)$ is a Fredholm operator.*

In this case, $i(E) = index(E_{12})$.

Proof. The equivalence of the first two conditions follows form the above remark and the computations in the previous section. Recall also that (in terms of the decomposition $\mathcal{H} = R(E) \oplus R(E)^{\perp}$)

$$N(E + E^* - 2) = N(E_{12}^*) \oplus \{0\} \text{ and } N(E + E^*) = \{0\} \oplus N(E_{12}).$$

Thus $E \in \mathcal{Q}_f$ if and only if $N(E_{12})$ and $N(E_{12})^*$ are finite dimensional and $0, 2$ are isolated in the spectrum of $E + E^*$. Let us examine this latter condition. It is equivalent to ± 1 being isolated in the spectrum of $P_{R(E)} - P_{N(E)}$, or equivalently, that 1 is isolated in the spectrum of $(P_{R(E)} - P_{N(E)})^2$. In matrix form

$$(P_{R(E)} - P_{N(E)})^2 = (E + E^* - 1)^2 = \begin{pmatrix} 1 & E_{12} \\ E_{12}^* & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 + E_{12}E_{12}^* & 0 \\ 0 & 1 + E_{12}^*E_{12} \end{pmatrix}.$$

Then 1 is isolated in the spectrum of $(P_{R(E)} - P_{N(E)})^2$ if and only if 0 is isolated in the spectrum of $E_{12}E_{12}^*$ (the other follows). This is equivalent to the fact that E_{12} has closed range. It follows that $E \in \mathcal{Q}_f$ if and only if $E_{12} : R(E)^{\perp} \to R(E)$ is a Fredholm operator. Apparently

$$index(E_{12}) = \dim(N(E_{12}^*)) - \dim(N(E_{12})) = \dim(N(E + E^* - 2)) - \dim(N(E + E^*)) = i(E).$$

□

In particular, $E \in \mathcal{Q}_f$ if and only if $E^* \in \mathcal{Q}_f$. Also note that in this case, $i(E) = i(E^*)$.

Theorem 6.3. *Let $E, F \in \mathcal{Q}_f$. Then they lie in the same connected component of $\mathcal{Q}_f$ if and only if*

$$i(E) = i(F).$$

Proof. First note the fact that $E \in \mathcal{Q}_f$ implies that both $R(E)$ and $N(E)$ are infinite dimensional (E_{12} is a Fredholm operator between these spaces). It follows that E and F lie in the same connected component in $\mathcal{Q}$. Lemma 2.1 applies again here, and we may suppose that $R(E) = R(F)$. It follows that E_{12}, F_{12} are Fredholm operators in $\mathcal{B}(R(E)^\perp, R(E))$. It is a well known fact that they lie in the same connected component of the set of Fredholm operators between $R(E)^\perp$ and R(E) if and only if they have the same index. A continuous path $E_{12}(t)$ between E_{12} and F_{12} would provide a continuous path between E and F inside $\mathcal{Q}_f$:

$$E(t) = \begin{pmatrix} 1 & E_{12}(t) \\ 0 & 0 \end{pmatrix}.$$

□

Proposition 6.4. *$\mathcal{Q}_f$ is open in $\mathcal{Q}$.*

Proof. By the continuity of the range projection map $F \mapsto P_{R(F)}$ in $\mathcal{Q}$, given a fixed $E \in \mathcal{Q}_f$, there exists a positive radius $d = d_E$ such that if $F \in \mathcal{Q}$ satisfies $\|F - E\| < d$ then $\|P_{R(F)} - P_{R(E)}\| < 1$. Then there exists a unitary operator $\mu(F)$ in $\mathcal{H}$ (a continuous map in the parameter F, with $\mu(E) = 1$) such that $\mu(F)P_{R(E)}\mu^*(F) = P_{R(F)}$. Thus $\mu^*(F)F\mu(F)$ and E have the same range. In matrix form in terms of $\mathcal{H} = R(E) \oplus R(E)^\perp$,

$$\mu^*(F)F\mu(F) = \begin{pmatrix} 1 & F'_{12} \\ 0 & 0 \end{pmatrix} \text{ and } E = \begin{pmatrix} 1 & E_{12} \\ 0 & 0 \end{pmatrix}.$$

Note that if one shrinks $d = d_E$, then $\|\mu^*(F)F\mu(F) - E\| = \|F'_{12} - E_{12}\|$ tends to zero. Since the set of Fredholm operators between $R(E)^\perp$ and $R(E)$ is open, it follows that there exists d such that $\|F - E\| < d$ implies F'_{12} is a Fredholm operator in $\mathcal{B}(R(E)^\perp, R(R))$. Note that $\mu(E)$ maps $R(E)$ onto $R(F)$ (and thus also their orthogonal supplements). It follows that $\|E - F\| < d$ implies that

$$\mu(E)F'_{12}\mu^*(E) = P_{R(F)}FP_{R(F)^\perp} = F_{12}$$

is a Fredholm operator between $R(F)^\perp$ and $R(F)$, i.e. $F \in \mathcal{Q}_f$. □

7 Three symmetries in $\mathcal{Q}$

Given $E \in \mathcal{Q}$, there are several symmetries induced by E. Among these, we shall focus on the following. The first was considered by Corach, Porta and Recht in [9]:

Consider the polar decomposition

$$2E - 1 = \rho_E|2E - 1|.$$

Then ρ_E is a selfadjoint unitary operator (a symmetry), which satisfies $\rho_E|2E-1| = |2E-1|^{-1}\rho_E$. In particular this implies that $\rho_E(2E - 1) = (2E^* - 1)\rho_E$, or equivalently,

$$\rho_E E \rho_E = E^*.$$

The second symmetry is obtained from the polar decomposition of $P_{R(E)} - P_{N(E)}$. Since this operator is invertible and selfadjoint, the unitary part s_E in the (commuting) factorization

$$P_{R(E)} - P_{N(E)} = s_E |P_{R(E)} - P_{N(E)}| = |P_{R(E)} - P_{N(E)}| s_E$$

is a selfadjoint unitary operator.

Proposition 7.1. *With the above notations,*

$$s_E E s_E = E^*.$$

Proof. Recall that $P_{R(E)} - P_{N(E)} = (E + E^* - 1)^{-1}$. In matrix form, as seen above

$$(E + E^+ - 1)^2 = \begin{pmatrix} 1 + E_{12}E_{12}^* & 0 \\ 0 & 1 + E_{12}^*E_{12} \end{pmatrix},$$

and thus

$$s_E = (E + E^* - 1)|E + E^* - 1|^{-1} = \begin{pmatrix} (1 + E_{12}E_{12}^*)^{-1/2} & E_{12}(1 + E_{12}^*E_{12})^{-1/2} \\ E_{12}^*(1 + E_{12}E_{12}^*)^{-1/2} & -(1 + E_{12}^*E_{12})^{-1/2} \end{pmatrix}.$$

After straightforward computations

$$s_E E s_E = \begin{pmatrix} 1 & 0 \\ E_{12}^* & 0 \end{pmatrix} = E^*.$$

□

Remark 7.2. Both symmetries ρ_E and s_E conjugate E with E^*. They can be computed in the three space decomposition. Namely, recall that $S \geq 0$, and then

$$(P_1 - P_0)^2 = 1 \oplus 1 \oplus \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix} \text{ so that } |P_1 - P_0| = 1 \oplus 1 \oplus \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}.$$

Thus

$$s_E = (P_1 - P_0)|P_1 - P_0|^{-1} = 1 \oplus -1 \oplus \begin{pmatrix} S & -C \\ -C & -S \end{pmatrix}.$$

For the computation of ρ_E, put $\Gamma = S^{-1}C$ (the cotangent of X). Note that

$$2E - 1 = 1 \oplus -1 \oplus \begin{pmatrix} 1 & -\Gamma \\ 0 & -1 \end{pmatrix} \text{ and } |2E - 1|^2 = 1 \oplus 1 \oplus \begin{pmatrix} 1 & -\Gamma \\ -\Gamma & 1 + \Gamma^2 \end{pmatrix}.$$

Straightforward computations show that the square root of this operator is

$$|2E - 1| = 1 \oplus 1 \oplus \begin{pmatrix} 2(4 + \Gamma^2)^{-1/2} & -\Gamma(4 + \Gamma^2)^{-1/2} \\ -\Gamma(4 + \Gamma^2)^{-1/2} & (\Gamma^2 + 2)(4 + \Gamma^2)^{-1/2} \end{pmatrix},$$

and thus

$$\rho_E = |2E - 1|(2E - 1) = 1 \oplus -1 \oplus \begin{pmatrix} 2(4 + \Gamma^2)^{-1/2} & -\Gamma(4 + \Gamma^2)^{-1/2} \\ -\Gamma(4 + \Gamma^2)^{-1/2} & -2(4 + \Gamma^2)^{-1/2} \end{pmatrix}.$$

The fact that both s_E and ρ_E intertwine E and E^* imply that the products

$$\rho_E s_E \text{ and } s_E \rho_E$$

commute with E and E^*.

The third symmetry was introduced in Section 4. It is the symmetry $V = V_E$, obtained by Davis [10], which is defined only for $E \in \mathcal{Q}_g$, and satisfies

$$V_E E V_E = 1 - E.$$

Note that this symmetry could not be defined in the other classes of $\mathcal{Q}$, which are not invariant for the map $E \mapsto 1 - E$. In terms of C and S in Halmos' model,

$$V = \begin{pmatrix} C & S \\ S & -C \end{pmatrix}.$$

The symmetry V has the following geometric characterization:

Theorem 7.3. *Let $E \in \mathcal{Q}_g$. Then the projection $\frac{1}{2}(1+V)$ onto the 1 eigenspace of V, is the middlepoint of the unique geodesic joinming $P_{R(E)}$ and $P_{N(E)}$*

Proof. As before, put $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$. Let $\delta(t) = e^{itZ} P_1 e^{-itZ}$ be the unique geodesic joining P_1 and P_0. Recall from Section 4 that $V = e^{iZ}(2P_1 - 1)$. Since Z anti-commutes with V, one has that

$$V = e^{\frac{i}{2}Z}(2P_1 - 1)e^{-\frac{i}{2}Z},$$

and thus

$$\frac{1}{2}(1+V) = e^{\frac{i}{2}Z} P_1 e^{-\frac{i}{2}Z} = \delta(\frac{1}{2}).$$

□

Remark 7.4. Suppose that $E \in \mathcal{Q}_k$. In Proposition 4.10 it was shown that $E = G P_{R(E)} G^{-1}$ for some $G \in Gl_\infty(\mathcal{H})$. In the polar decomposition of $2E - 1 = \rho_E |2e - 1|$ above, Corach, Porta and Recht [9] noted that

$$2E - 1 = \rho_E |2E - 1| = |2E - 1|^{-1} \rho_E.$$

Thus $2E - 1 = |2E - 1|^{-1/2} \rho_E |2E - 1|^{1/2}$, and therefore

$$E = |2E - 1|^{-1/2} \frac{1}{2}\{\rho_E + 1\} |2E - 1|^{1/2},$$

where $\frac{1}{2}\{\rho_E + 1\}$ is the orthogonal projection onto the 1-eigenspace of the symmetry ρ_E. Note that $|2E - 1| \in Gl_\infty(\mathcal{H})$. Indeed, in the three space decompositon of $|2E - 1|$, $\Gamma = S^{-1}C$ is a compact operator in $\mathcal{L}$. Then also $|2E-1|^{1/2} \in Gl_\infty(\mathcal{H})$. It follows that $P_{R(E)}$ and $\frac{1}{2}\{\rho_E + 1\}$ are orthogonal projections for which there exists $G_0 \in Gl_\infty(\mathcal{H})$ such that $G_0 P_{R(E)} G_0^{-1} = \frac{1}{2}\{\rho_E + 1\}$. Then, the unitary U_0 in the polar decompositioon of G_0 verifies

$$U_0 P_{R(E)} U_0^* = \frac{1}{2}\{\rho_E + 1\},$$

and by Lemma 4.9, $U_0 \in \mathcal{U}_\infty(\mathcal{H})$.

8 Cyclic idempotents

In this section we study the set $\mathcal{Q}_c$ of cyclic idempotents

$$\mathcal{Q}_c = \{E \in \mathcal{Q} : P_{R(E)} - P_{N(E)} \text{ is a cyclic operator in } \mathcal{H}\}.$$

In other words, the commutative C^*-algebra $C^*(P_{R(E)} - P_{N(E)})$ has a cyclic vector. Apparently, this implies that the C^*-algebra $C^*(P_{R(E)}, P_{N(E)}) = C^*(E)$ generated by the two projections (or equivalently by E) has a cyclic vector in $\mathcal{H}$. It is clearly a weaker condition.

The equality $P_{R(E)} - P_{N(E)} = (E + E^* - 1)^{-1}$ clearly implies the following:

Proposition 8.1. *$E \in \mathcal{Q}_c$ if and only if $E + E^*$ (or equivalently $E + E^* - 1$) is a cyclic operator in $\mathcal{H}$.*

Also it is apparent that for any unitary operator U, $E \in \mathcal{Q}_c$ implies that $UEU^* \in \mathcal{Q}_c$. In particular, $E^* \in \mathcal{Q}_c$.

Remark 8.2. In the three space decomposition $\mathcal{H} = \mathcal{H}_{10} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_0$, recall that

$$\mathcal{H}_{10} = N(E + E^* - 2) \quad \text{and} \quad \mathcal{H}_{01} = N(E + E^*).$$

If $E \in \mathcal{Q}_c$, this implies that

$$\dim \mathcal{H}_{10} \leq 1 \quad \text{and} \quad \dim \mathcal{H}_{01} \leq 1.$$

Indeed, the fact that $E + E^*$ is cyclic implies that any eigenvalue must have multiplicity less or equal than 1.

In terms of the Halmos' model:

Theorem 8.3. *$E \in \mathcal{Q}_c$ if and only if*

$$\dim \mathcal{H}_{10} \leq 1\ , \quad \dim \mathcal{H}_{01} \leq 1$$

and the operator Z acting in the generic part $\mathcal{H}_0$,

$$Z = \begin{pmatrix} 0 & -iX \\ iX & 0 \end{pmatrix}$$

is cyclic in $\mathcal{H}_0$.

This operator Z is the exponent of the unique geodesic joining the generic parts of $P_{R(E)}$ and $P_{N(E)}$.

Proof. As usual, denote $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$. Suppose first that $E \in \mathcal{Q}_c$. As seen above this implies the bounds for the dimensions of $\mathcal{H}_{10}$ and $\mathcal{H}_{01}$. Let A_0 be the generic part of $P_1 - P_0$. Identifying $\mathcal{H}_0$ and $\mathcal{L} \times \mathcal{L}$, we have

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} = \begin{pmatrix} S^2 & -CS \\ -CS & -S^2 \end{pmatrix}.$$

The symmetry defined by Davis, induced by this decomposition of A_0 is

$$V = \begin{pmatrix} C & S \\ S & -C \end{pmatrix}.$$

Clearly, the assumption that $A = P_1 - P_0$ is cyclic in $\mathcal{H}$ implies that A_0 is cyclic in $\mathcal{H}_0$. Consider

$$B_0 = VA_0.$$

Clearly B_0 also anti-commutes with V. In [4] it was shown that if A_0 is cyclic, then one can find a cyclic vector ξ_0 such that $V\xi_0 = \xi_0$. Then

$$B_0^n \xi_0 = (VA_0)^n \xi_0 = (-1)^n A_0 V \xi_0 = (-1)^n A_0^n \xi_0.$$

It follows that B_0 is also cyclic (with the same cyclic vector ξ_0). Note that in matrix form

$$B_0 = VA_0 = \begin{pmatrix} C & S \\ S & -C \end{pmatrix} \begin{pmatrix} S^2 & -CS \\ -CS & -S^2 \end{pmatrix} = \begin{pmatrix} 0 & -S \\ S & 0 \end{pmatrix}.$$

It follows that iB_0 is selfadjoint and cyclic. We claim that $iB_0 = \sin(Z)$ and that Z is also cyclic (with the same cyclic vector ξ_0). Indeed, the first claim follows from a straightforward matrix computation. In our case, S is invertible in $\mathcal{L}$. Clearly Z is an analytic function in terms of iB_0, $Z = f(iB_0)$, with $f(0) = 0$. In particular, any vector in $\mathcal{H}_0$ which is orthogonal to $Z^n\xi_0$, for all $n \geq 1$, is also orthogonal to $(iB_0)^n \xi_0$ for all $n \geq 1$, and thus trivial. Then Z is cyclic with cyclic vector ξ_0.

The fact that $e^{iZ} P_1 e^{-iZ} = P_0$ was shown in Section 4.

Conversely, assuming $\dim \mathcal{H}_{10} \leq 1$ and $\dim \mathcal{H}_{01} \leq 1$, it remains to prove that A_0 is cyclic in $\mathcal{H}_0$. The same argument above shows that if Z cyclic with cyclic vector ξ_0, then $\sin(Z) = iB_0$ is cyclic, and therefore $A_0 = VB_0$, by the same computation above. □

With respect to the off-diagonal entry E_{12}, we have sufficient conditions:

Proposition 8.4. *Let $E \in \mathcal{Q}$ such that $N(E_{12}) = \{0\}$, $N(E_{12}E_{12}^* - 1) = \{0\}$, and $E_{12}E_{12}^*$ is cyclic in $R(E)$, with cyclic vector $\xi_1 \in R(E)$. Then $E \in \mathcal{Q}_c$, with $\xi_0 = \xi_1 + E_{12}^*\xi_1$ cyclic for $E + E^* - 1$.*

Proof. First let us compute the powers of $E + E^* - 1$. After straightforward computations, if $n = 2k$ is even,

$$(E + E^* - 1)^n = \begin{pmatrix} (1 + E_{12}E_{12}^*)^k & 0 \\ 0 & (1 + E_{12}^*E_{12})^k \end{pmatrix}.$$

If $n = 2k + 1$ is odd

$$(E + E^* - 1)^n = \begin{pmatrix} (1 + E_{12}E_{12}^*)^k & (1 + E_{12}E_{12}^*)^k E_{12} \\ (1 + E_{12}^*E_{12})^k E_{12}^* & (1 + E_{12}^*E_{12})^k \end{pmatrix}.$$

Let $\eta = \eta_1 + \eta_2 \in \mathcal{H}$, $\eta_1 \in R(E)$, $\eta_2 \in R(E)^\perp$, such that $\eta \perp (E + E^* - 1)(\xi_1 + E_{12}^*\xi_)1$ for all $n \geq 0$. Then if $n = 2k$

$$\langle \eta_1, (1 + E_{12}E_{12}^*)^k \xi_1 \rangle + \langle \eta_2, (1 + E_{12}^*E_{12})^k E_{12}^* \xi_1 \rangle = 0 \tag{3}$$

for all $k \geq 0$. If $n = 2j + 1$,

$$\langle \eta_1, (1 + E_{12}E_{12}^*)^j \xi_1 + (1 + E_{12}E_{12}^*)^j E_{12}E_{12}^* \xi_1 \rangle + \langle \eta_2, 2(1 + E_{12}^*E_{12})^j E_{12}^* \xi_1 \rangle = 0$$

for all $j \geq 0$. This term equals

$$\langle \eta_1, (1 + E_{12}E_{12}^*)^{j+1}\xi_1\rangle + 2\langle \eta_2, (1 + E_{12}^*E_{12})^j E_{12}^*\xi_1\rangle = 0. \tag{4}$$

Putting $j = k \geq 0$, multiplying equation (3) by 2 and substracting from it equation (4), one obtains

$$\langle \eta_1, (1 - E_{12}E_{12}^*)(1 + E_{12}E_{12}^*)^k\xi_1\rangle = 0$$

Apparently, the fact that the set of vectors $\{(E_{12}E_{12}^*)^k\xi_1 : k \geq 0\}$ spans a dense subspace of $R(E)$, implies that also the set $\{(1 + E_{12}E_{12}^*)^k\xi_1 : k \geq 0\}$ spans a dense subspace of $R(E)$. By hypothesis, $1 - E_{12}E_{12}^*$ has dense range in $R(E)$, it follows that the set

$$\{(1 - E_{12}E_{12}^*)(1 + E_{12}E_{12}^*)^k\xi_1 : k \geq 0\}$$

spans a dense subset of $R(E)$. It follows that $\eta_1 = 0$. Similarly, putting $j+1 = k$ for $j \geq 0$, and substracting equation (3) from equation (4), one obtains

$$0 = \langle \eta_2, (1 - E_{12}^*E_{12})(1 + E_{12}^*E_{12})^j E_{12}^*\xi_1\rangle = \langle \eta_2, E_{12}^*(1 - E_{12}E_{12}^*)(1 + E_{12}E_{12}^*)^j\xi_1\rangle$$

for all $j \geq 0$. The hypothesis $N(E_{12}) = \{0\}$ implies that $E_{12}^* : R(E) \to R(E)^\perp$ has dense range. Thus similarly as above, $\eta_2 = 0$, and therefore $\xi_1 + E_{12}^*\xi_1$ is a cyclic vector for $E + E^* - 1$ in $\mathcal{H}$. □

Remark 8.5. Analogously, one can prove that if $N(E_{12}^*) = N(1 - E_{12}^*E_{12}) = \{0\}$ and $E_{12}^*E_{12}$ is cyclic in $R(E)^\perp$ with cyclic vector ξ_2, then $E + E^* - 1$ is cyclic in $\mathcal{H}$, with cyclic vector $\xi_2 + E_{12}\xi_2$.

Remark 8.6.

1. In the above Proposition, the condition $N(E_{12}) = \{0\}$ could be replaced by the condition $\mathcal{H}_{01} = \{0\}$. Indeed, recall from Section 4 that

$$\mathcal{H}_{10} = N(E + E^*) = \{0\} \oplus N(E_{12}).$$

   Also note that if E is cyclic, one has $\dim \mathcal{H}_{01} \leq 1$, so that E_{12} is not far from having trivial nullspace. However it appears not to be a necessary condition.

2. Something similar happens with the other condition, $N(E_{12}E_{12}^* - 1) = \{0\}$. If one asks that $E_{12}E_{12}^*$ be cyclic in $R(E)$, then all eventual eigenvalues must have multiplicity at most 1, i.e. $\dim N(E_{12}E_{12}^* - 1) \leq 1$.

With reference to this last condition, let us point out that in Halmos' model for the generic part of E, this last condition is automatically fulfilled:

Lemma 8.7. *Let E_0 be the generic part of E acting in $\mathcal{H}_0 = \mathcal{L} \times \mathcal{L}$:*

$$E_0 = \begin{pmatrix} 1 & -S^{-1}C \\ 0 & 0 \end{pmatrix}.$$

Then $N((S^{-1}C)^2 - 1) = \{0\}$.

Proof. Suppose that there exists a vector $\xi \in \mathcal{L}$ such that $(S^{-1}C)^2\xi = \xi$. Since $S^{-1}C$ is a positive operator in $\mathcal{L}$ (C and S commute), this implies that $S^{-1}C\xi = \xi$. Recall that there exist $0 \le X \le \pi/2$ such that $C = \cos(X)$ and $S = \sin(X)$. The fact that S is invertible implies further that $0 < r \le X \le \pi/2$. Therefore the continuous function $cotg : [r, \pi/2] \to [0, cotg(r)]$, $cotg(t) = \frac{\cos(t)}{\sin(t)}$ has a continuous inverse $cotg^{-1}$. Note that $cotg(X) = S^{-1}C$ and thus $cotg^{-1}(S^{-1}C) = X$. The function $cotg^{-1}$ is a uniform limit of polynomials in the interval $[0, cotg(r)]$,

$$cotg^{-1}(t) = \lim_{n\to\infty} p_n(t).$$

Since $S^{-1}C\xi = \xi$, it follows that $p_n(S^{-1}C)\xi = p_n(1)\xi$. Taking limits,

$$X\xi = cotg^{-1}(X)\xi = \lim_{n\to\infty} p_n(X)\xi = \lim_{n\to\infty} p_n(1)X\xi = cotg^{-1}(1)\xi = \frac{\pi}{4}\xi.$$

Therefore $S\xi = C\xi = \frac{1}{\sqrt{2}}\xi$. Consider the vector $\bar{\xi} = (\xi, 0) \in \mathcal{L} \times \mathcal{L}$. Then

$$P_{R(E_0)}\bar{\xi} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \bar{\xi}$$

and

$$P_{N(E_0)}\bar{\xi} = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \bar{\xi},$$

i.e. $\xi = 0$, a contradiction. □

The folowing result holds:

Corollary 8.8. *Suppose that $E \in \mathcal{Q}_g$ (the set of idempotents in generic position). With the above notations, if X (or equivalently, CS^{-1}) is cyclic in $\mathcal{L}$, then $E \in \mathcal{Q}_c$.*

Proof. By Lemma 8.7, in this case the sufficient conditions in Proposition 8.4 applied to the Halmos model reduce to CS^{-1} being cyclic in $\mathcal{L}$. By the computation in Lemma 8.7, $CS^{-1} = cotg(X)$ is cyclic in $\mathcal{L}$ if and only if X is cyclic in $\mathcal{L}$ □

Remark 8.9. This result implies that the conditions in Proposition 8.4 are not necessary for E to belong to $\mathcal{Q}_c$. Indeed, the class $\mathcal{Q}_c$ is unitarily invariant. Whereas for an arbitrary idempotent E in generic position (which is unitarilly equivalent to a Halmos model), the off diagonal entry E_{12} (with trivial nullspace and dense range) need not verify $N(E_{12}E_{12}^* - 1) = \{0\}$. In other words, this last condition is not unitarilly invariant.

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