Metric geometry in homogeneous spaces of the unitary group of a C^*-algebra. Part II. Geodesics joining fixed endpoints.[*]

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Abstract

This article focuses on the study of the metric geometry of homogeneous spaces $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ (the unitary group of a C^*-algebra $\mathcal{A}$ modulo the unitary group of a C^*-subalgebra $\mathcal{B}$) where the invariant Finsler metric in $\mathcal{P}$ is induced by the quotient norm of $\mathcal{A}/\mathcal{B}$. Under the assumption that $\mathcal{B}$ is of *compact type*, i.e. when the unitary group is relatively compact in the strong operator topology, this work presents local and global versions of Hopf-Rinow-like theorems: given points $\rho_0, \rho_1 \in \mathcal{P}$, there exists a minimal uniparametric group curve joining ρ_0 and ρ_1.

Key words: uniparametric group curves, minimal curves, von Neumann algebras, strong operator topology, Hopf-Rinow theorem, unitary group.

1 Introduction

This paper continues the work in [10], namely the study of the geometry of *generalized flags* (homogeneous spaces corresponding to pairs of $\mathcal{B} \subset \mathcal{A}$ of C^*-algebras). Below there is a brief description of these spaces (a more complete introduction can be found in [10]). The main results are theorems [II-1,II-2], which are local and global versions of Hopf-Rinow-like theorems: given points $\rho_0, \rho_1 \in \mathcal{P}$, there exists a minimal uniparametric group curve joining ρ_0 and ρ_1. The staments appear later in this introduction after some notation and definitions.

Generalized flags are Banach manifolds $\mathcal{P}$ in which the unitary group $U(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ acts transitively, say on the left, and let us denote the action of $g \in U(\mathcal{A})$ on $\rho \in \mathcal{P}$ by $L_g\rho$. The isotropy $\{g \in U(\mathcal{A}) \ / \ L_g\rho = \rho\}$ will be required to be the unitary group $U(\mathcal{B})$ of a C^*-subalgebra $\mathcal{B} \subset \mathcal{A}$. Thus the homogeneous spaces that studied, correspond to the 'relative C^*-algebra category', i.e. pairs $(\mathcal{A}$,$\mathcal{B})$, $\mathcal{B} \subset \mathcal{A}$, of C^*-algebras. Among the examples of generalized flags there are the Grassmannian of a general C^*-algebra [4,6], the finite flags of a general C^*-algebra [5] and the spaces of spectral measures [2]. All these spaces have a canonical Banach manifold structure as quotients of the unitary group of the algebra [2].

The spaces are endowed with the Finsler quotient metric, i.e. consider each tangent space $(T\mathcal{P})_\rho$ as the Banach quotient $(T\mathcal{P})_\rho = T_1U(\mathcal{A})/T_1U(\mathcal{B}) = \mathcal{A}^{\rm ant}/\mathcal{B}^{\rm ant}$, where $\mathcal{A}^{\rm ant}$ and $\mathcal{B}^{\rm ant}$ denote the antisymmetric parts of the algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. So the Finsler norm in $\mathcal{P}$ is defined by $||X|| = \inf_{b\in\mathcal{B}^{\rm ant}} |Z+b|$, where Z projects to X in the quotient. Let us denote by $|\cdot|$ the norm in the C^*-algebra $\mathcal{A}$.

In part I of this paper [10] uniparametric group curves which are minimizing are presented. Namely, the following two theorems were proven:

Theorem I-1 *Let $\mathcal{P}$ be a generalized flag. Consider $\rho \in \mathcal{P}$ and $X \in (T\mathcal{P})_\rho$. Suppose that there exists $Z \in \mathcal{A}^{ant}$ which is a 'minimal' lift of X i.e. $|Z| = ||X||_\rho$. Then the* uniparametric group curve $\gamma(t)$ *defined by $\gamma(t) = L_{e^{tZ}}\rho_0$ has minimal length in the class of all curves in $\mathcal{P}$ joining $\gamma(0)$ to $\gamma(t)$ for each t with $|t| \leq \frac{\pi}{2|Z|}$.*

Such minimal lifts always exists in von Neumann algebras:

Theorem I-2 *Let $\mathcal{A}$ be a W^*-algebra, and let $\mathcal{P}$ be a generalized flag of the unitary group of $\mathcal{A}$. Let $X \in (T\mathcal{P})_\rho$. Then there is a lift Z of X which satisfies $|Z| = ||X||_\rho$, and therefore the uniparametric group curve $\gamma(t) = L_{e^{tZ}}\rho$ has minimal length in $\mathcal{P}$ among curves joining $\gamma(0)$ and $\gamma(t)$, for each t with $|t| \leq \frac{\pi}{2|Z|}$.*

Remark: Theorem I-1 in fact gives a characterization of the uniparametric group curves which are minimizing, and in the proof of theorem I-2 it can be checked that the algebra $\mathcal{A}$ can be taken arbitrary; it is just the subalgebra $\mathcal{B}$ that needs to be a W^*-algebra in order for the conclusion to follow.

In part I then, it was solved the initial value problem: to find geodesics with given initial conditions.

The present paper, studies the boundary value problem: to find geodesics joining given endpoints. As it was mentioned above, two Hopf-Rinow type theorems are proven for the generalized flag $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$. This is done in the case when the subalgebra $\mathcal{B}$ is of *compact type* as defined below.

Definition 1.1 *A von Neumann algebra $\mathcal{B}$ is said to be of* compact type *when the unitary group of $\mathcal{B}$, $U(\mathcal{B})$ is compact in the strong operator topology.*

From now on in this work, the word 'strong' (topology, closure, etc) will always refer to the strong operator topology.

Also, for the corresponding generalized flags let us define the following,

Definition 1.2 *A generalized flag $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ will be called* isotropy compact*, if $\mathcal{B}$ is of compact type*

A description of both compact type algebras, and isotropy compact generalized flags, will appear in section 4.2. The main theorems are the following:

Theorem II-1 (Local Hopf-Rinow) *Let $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ be an isotropy compact generalized flag, and let $\rho_0 \in \mathcal{P}$. There exists an open neighborhood $\mathcal{V}_{\rho_0}$ of ρ_0 such that, if $\rho_1 \in \mathcal{V}_{\rho_0}$, there exists a minimal uniparametric group curve joining ρ_0 to ρ_1.*

Theorem II-2 (Global Hopf-Rinow) *Let $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ be an isotropy compact generalized flag, where $\mathcal{A}$ is a von Neumann algebra. Let ρ_0 and ρ_1 be points in $\mathcal{P}$. Then, there exists $Z \in \mathcal{A}^{ant}$, such that the uniparametric group curve $\gamma(t) = L_{e^{tZ}}\rho_0$ for $t \in [0,1]$ joins ρ_0 to ρ_1, and its length is minimal among piecewise smooth curves joining ρ_0 to ρ_1.*

Notice that in theorem II-1, the algebra $\mathcal{A}$ may be any C^*-algebra, whereas in theorem II-2, the algebra $\mathcal{A}$ is required that $\mathcal{A}$ is a von Neumann algebra.

The proof of a 'classical' (say, finite-dimensional, Riemannian) Hopf-Rinow theorem [13,17] essentially consists on translating the Dirichlet problem to a Cauchy problem. Instead of finding a geodesic joining two given points p and q, one finds a tangent vector $X \in T_pM$ that 'points' towards q. The geodesic that begins with initial condition X should hopefully be a minimal curve joining p to q. Upon careful examination of the classical proof, finding such a director vector X requires two steps: the first one is the use of geometrical reasoning to intuit how to find such an X. This leads to a sequence X_n of vectors that point closer and closer to q, which leads to the second step, analytical in nature, and concerns the convergence of X_n to a 'best' vector X pointing to q. This second step is often overlooked because of its triviality on the finite dimensional case.

The proof and the structure of this paper reflect these steps. In section 3 the geometrical ideas are presented, reflecting the relative geometry of the pair $(\mathcal{A}, \mathcal{B})$. This leads to a candidate for a director vector, which is the solution of a problem of convergence to a minimum (presented in section 4). The hypothesis that $U(\mathcal{B})$ is relatively compact in the strong topology, is used only in order to solve the convergence problem mentioned above. However, the authors conjecture that the Hopf-Rinow theorem is true for generalized flags $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ where $\mathcal{B}$ is supposed to be an arbitrary von Neumann algebra instead of being of compact type.

Finally, the paper is organized as follows: in section 2 the notation and preliminaries are introduced, and it includes a review of the geometry of the unitary group of a C^*-algebra. Section 3 deals with the geometry: where to point a geodesic in order to reach a given point, and in section 4 the convergence problem is solved. The paper concludes with remarks and open problems in section 5.

2 Notation and preliminaries

Let M be a Banach manifold. A *Finsler structure* on M is a continuous selection of norms $||.||_m$ on each tangent space TM_m.

Remark: The usual definition of a Finsler structure includes differentiability and strict convexity of the norm; this notion is too restrictive for the cases presented in this work. See section 9 of [11] for a discussion of C^0 calculus of variations. Let us remark that the lack of differentiability prompts us to apply direct *metric* (as opposed to topological) methods. Recall that for any Finsler structure, the length of a curve $w(t)$ defined for $a \leq t \leq b$ is given by,

$$\ell(w) = \int_a^b ||\dot{w}(t)||_{w(t)}\, dt.$$

The distance d in M is given as follows: let R_{ρ_0,ρ_1} be the set of piecewise smooth paths w ($w : [0,1] \to M$) which join $w(0) = \rho_0$ to $w(1) = \rho_1$. Consider,

$$d(\rho_0, \rho_1) = \inf\{\ell(w) \,|\, w \in R_{\rho_0,\rho_1}\}.$$

Definition 2.1 *A curve w is said to be* minimal *in M if its length is the distance between its endpoints.*

In this work a *generalized flag* means the following data:

- A C^∞ Banach manifold $\mathcal{P}$.
- A unital C^*-algebra $\mathcal{A}$ whose unitary group $U(\mathcal{A})$ acts transitively and smoothly on $\mathcal{P}$ on the left. Let us indicate by $L_g\rho$, $g \in U(\mathcal{A})$, $\rho \in \mathcal{P}$, the action of $U(\mathcal{A})$ on $\mathcal{P}$.
- The isotropy group at $\rho \in \mathcal{P}$ given by, $\{g \in U(\mathcal{A}) \mid L_g\rho = \rho\}$, is the unitary group $U(\mathcal{B}_\rho)$ of a C^*-subalgebra $\mathcal{B}_\rho$ of $\mathcal{A}$.
- The derivative $(T\Pi_\rho)_1 : (TU(\mathcal{A}))_1 = \mathcal{A}^{\mathrm{ant}} \to (T\mathcal{P})_\rho$ of the natural mapping $\Pi_\rho : U(\mathcal{A}) \to \mathcal{P}$ given by $\Pi_\rho(g) = L_g(\rho)$ is surjective (hence open by the Banach Open Mapping Theorem).
- The Finsler structure in $\mathcal{P}$ is given by $||X||_\rho = \inf\{|Z + b| \ : \ b \in \mathcal{B}_\rho^{\mathrm{ant}}\}$, where $(T\Pi_\rho)_1(Z) = X$, i.e. for $\rho \in \mathcal{P}$, the norm $||X||_\rho$ is the Banach quotient norm of X in $(TU(\mathcal{A}))_1/(TU(\mathcal{B}_\rho))_1 = \mathcal{A}^{\mathrm{ant}}/\mathcal{B}_\rho^{\mathrm{ant}}$. Observe that this Finsler structure is invariant under the action of $U(\mathcal{A})$.

Definition 2.2 *A curve $\gamma : I \to \mathcal{P}$ of the form $\gamma(t) = L_{e^{tZ}}\rho$ for $Z \in \mathcal{A}^{ant}$ and $t \in I = [a,b] \subset \mathbb{R}$ is called a* uniparametric group curve.

Definition 2.3 *A $Z \in \mathcal{A}^{ant}$ is said to be a* lift *of $X \in (T\mathcal{P})_\rho$, if $(T\Pi_\rho)_1(Z) = X$.*

Observe that if Z is a lift of $X \in (T\mathcal{P})_\rho$, then the uniparametric group curve $\gamma(t) = L_{e^{tZ}}\rho$ satisfies $\gamma(0) = \rho$ and $\dot\gamma(0) = X$.

Definition 2.4 *At $Z \in \mathcal{A}^{ant}$ is a* minimal lift *of $X \in (T\mathcal{P})_\rho$, if $(T\Pi_\rho)_1(Z) = X$, and $|Z| = \inf\{|Z + b| \ : \ b \in \mathcal{B}_\rho^{ant}\}$.*

Thus, if Z is a minimal lift of X, then $|Z| = ||X||_\rho$.

2.1 A review of the metric geometry of the unitary group of a C^*-algebra

Let us review some results for completeness (see also [3]).

Let $\mathcal{A}$ be a C^*-algebra. The unitary group $U(\mathcal{A})$ is provided with the bi-invariant Finsler metric given by

$$||X|| = |X|, \text{ for } X \in (TU(\mathcal{A}))_1 = \mathcal{A}^{\text{ant}} \subset \mathcal{A}$$

Notice that this Finsler structure is well defined and bi-invariant because left or right multiplication by a unitary element is an isometry of the algebra.

Theorem 2.1 *For a non-zero tangent vector $X \in (TU(\mathcal{A}))_1$, the curve*

$$\gamma(t) = e^{tX}, \ 0 \le t \le L$$

has minimal length among all curves joining its end points for $L \le \pi/|X|$.

Proof: It is enough to show the result for X of unit length. Choose a faithful representation of $\mathcal{A}$ into a Hilbert space $\mathcal{H}$ such that there is a unit vector $\xi \in \mathcal{H}$ which is a norming eigenvector of X^2, i.e. $X^2\xi = -\xi$. Next consider the unit sphere $\mathcal{S}$ of the Hilbert space $\mathcal{H}$ provided with the induced Riemannian metric. Define $F : U(\mathcal{A}) \to \mathcal{S}$ given by

$$F(u) = u\xi, \text{ for } u \in U(\mathcal{A}).$$

Observe that F is a length reducing map. In fact, F is equivariant for the left action of $U(\mathcal{A})$ on itself and the natural action of $U(\mathcal{A})$ on $\mathcal{H}$, and since these actions are isometries of the manifolds involved, it is enough to show the length reducing property of F, at the identity of $U(\mathcal{A})$. But, $(TF)_1(X) = X\xi$, and therefore $||(TF)_1(X)|| \leq ||X||$.

Next observe that $\delta(t) = F(\gamma(t)) = e^{tX}\xi$ is a geodesic in the Riemannian manifold $\mathcal{S}$. Indeed, the acceleration vector $\ddot{\delta}(t) = e^{tX}X^2\xi = -\delta(t)$ is normal to $\mathcal{S}$. Finally, the velocity vector $\dot{\delta}(t) = e^{tX}X\xi$ has unit length. Indeed, $||\dot{\delta}(t)|| = ||e^{tX}X\xi|| = ||X\xi||$ since e^{tX} is unitary. But

$$||X\xi||^2 = \langle X\xi \, , \, X\xi\rangle = \langle -X^2\xi \, , \, \xi\rangle = 1.$$

Now the proof concludes with an argument already used in part I of this paper, ([10]). It goes like this: given a curve $\phi(t)$ joining 1 to $\gamma(L)$ in $U(\mathcal{A})$, define $\psi(t) = F(\phi(t))$. Then,

$$\text{length}_0^L(\phi(t)) \geq \text{length}_0^L(\psi(t)) \geq \text{length}_0^L(\delta(t)) = \text{length}_0^L(\gamma(t))$$

The first inequality is due to the fact that F is length reducing. The second inequality is true because δ is a geodesic in $\mathcal{S}$ of length no greater that π. The last equality follows from the fact that γ and δ have constant speed 1.

□

The next lemmas show that the unitary group of a von Neumann algebra endowed with the algebra norm is a 'Blaschke manifold', i.e. the injectivity radius of the exponential map coincides with its diameter. These lemmas will be used in section 4 (see also [9] for more details).

Lemma 2.2 *The exponential map is a Banach diffeomorphism between the set* $\{Z \in \mathcal{A}^{ant}, \, |Z| < \pi\}$ *and the set* $\{u \in U(\mathcal{A}) \, , \, |1-u| < 2\}$, *where both sets are provided with the norm topology.*

Proof: This lemma is an easy consequence of the analytic functional calculus in a C^*-algebra. In fact, to construct the *logarithm* it suffices to observe that the spectrum of a unitary element whose distance to 1 is less than 2, does not contain -1.

□

Lemma 2.3 *Let* $Z, Z' \in \mathcal{A}^{ant}$ *such that* $|Z| < |Z'| < \pi$. *Then*

$$|1 - e^Z| < |1 - e^{Z'}| < 2\,.$$

Proof: Observe first that $1 - e^Z$ and $1 - e^{Z'}$ are both normal operators, so their norms are their spectral norms. Finally the spectral mapping theorem (see [7]) can be used to compare the spectra of $1 - e^Z$ and $1 - e^{Z'}$.

□

The following lemma is an immediate consequence of lemma 2.2 above.

Lemma 2.4 *Let $u \in U(\mathcal{A})$ such that $|1 - u| = 2$. Then for any $Z \in \mathcal{A}^{ant}$ such that $e^Z = u$, $|Z| \geq \pi$.*

Lemma 2.5 (Hopf-Rinow for the unitary group) *Let $u_0, u_1 \in U(\mathcal{A})$, where $\mathcal{A}$ is a von Neumann algebra. Then there exists a uniparametric group curve $\gamma(t) = e^{tZ}u_0$ joining u_0 to u_1 that minimizes length among all curves joining u_0 to u_1. Furthermore, the length of γ is less than or equal to π.*

Proof: Since $\mathcal{A}$ is a von Neumann algebra, there exists a positive element B of norm less than or equal to 2π such that $e^{iB} = -u_1u_0^{-1}$ (see [14]). The element $B' = \pi - B$ is symmetric, it has norm less than or equal to π and it satisfies $e^{iB'} = u_1u_0^{-1}$. Thus if $Z = iB'$, the curve $\gamma(t) = e^{tZ}u_0$ joins u_0 to u_1 and it has length $\ell(\gamma) = |Z| \leq \pi$. The curve γ is then minimal by theorem 2.1.

□

Remark: In reference to lemma 2.5 above, the more general case where $\mathcal{A}$ is not a von Neumann algebra is more delicate, just because the unitary group need not be connected. Moreover, in the connected component of the identity of $U(\mathcal{A})$ there may be elements which are not exponentials of antisymmetric ones [14, p. 287].

3 Geometry

In this section geometrical methods are used to reduce the Hopf-Rinow problem (i.e. given points $\rho_0, \rho_1 \in \mathcal{P}$, there exists a minimal uniparametric group curve joining ρ_0 and ρ_1) to the analytical problem of finding a 'minimal director'. The reduction has two parts: the first one is to prove a local Hopf-Rinow theorem (theorem II-1), and the second part uses the first, to make it a global result (theorem II-2).

3.1 Local Hopf-Rinow

Let us first recall Theorem I-1 of [10].

Theorem I-1 *Let $\mathcal{P}$ be a generalized flag. Consider $\rho \in \mathcal{P}$ and $X \in (T\mathcal{P})_\rho$. Suppose that there exists $Z \in \mathcal{A}^{ant}$ which is a 'minimal' lift of X i.e. $|Z| = ||X||_\rho$. Then the* uniparametric group curve *$\gamma(t)$ defined by $\gamma(t) = L_{e^{tZ}}\rho_0$ has minimal length in the class of all curves in $\mathcal{P}$ joining $\gamma(0)$ to $\gamma(t)$ for each t with $|t| \leq \frac{\pi}{2|Z|}$.*

This theorem suggests that in order to find a minimal curve joining arbitrary $\rho_0, \rho_1 \in \mathcal{P}$, one should find a vector $Z \in \mathcal{A}^{\text{ant}}$ satisfying:

1. $L_{e^Z}\rho_0 = \rho_1$,
2. The vector Z is minimal in its class, i.e. $|Z| \leq |(Z + b)|$ for all $b \in \mathcal{B}_{\rho_0}$.
3. $|Z| \leq \pi/2$.

A vector Z satisfying condition 1 will be called a *director*, and recall that if Z satisfies condition 2 it is called a *minimal lift* of $X = (T\Pi_{\rho_0})_1(Z)$. If $Z \in \mathcal{A}^{\text{ant}}$ satisfies 1-3 above, then theorem I-1 guarantees that the curve $F(\rho_1)(t) = L_{e^{tZ}}\rho_0, t \in [0,1]$, is a minimal curve joining ρ_0 and ρ_1. Alas, such a Z might not exist, since the diameter of these homogeneous spaces might be bigger than $\pi/2$ (see sections 6 and 7 of [10]). One needs to deal with the fact that $|Z|$ might be bigger than $\pi/2$ (done in section 3.2); for now let us concentrate on finding Z satisfying 1 and 2 above.

The main difficulty in finding such a Z is the reconciliation of being a director, which is a 'global' condition on $\mathcal{P}$, with being a minimal lift, which is an infinitesimal condition in $T_{\rho_0}\mathcal{P}$. The resolution of that difficulty is presented in theorem 3.2.

Given $\rho_0 \in \mathcal{P}$ fixed, let $F(\rho_1)$ be the set

$$F(\rho_1) = \{Z \in \mathcal{A}^{\text{ant}} \ / \ L_{e^Z}\rho_0 = \rho_1\}\,.$$

This set $F(\rho_1)$ (see Figure 1) can be considered as the space of uniparametric group curves joining ρ_0 and ρ_1. Note that $F(\rho_1)$ can also be understood from $U(\mathcal{A})$:

$$F(\rho_1) = \{Z \in \mathcal{A}^{\text{ant}} \ / \ e^Z \in \Pi_{\rho_0}^{-1}(\rho_1)\}\,,$$

recalling that $\Pi = \Pi_{\rho_0} : U(\mathcal{A}) \to \mathcal{P}$ is given by $\Pi_{\rho_0}(u) = L_u\rho_0$.

The next lemma shows that the set $F(\rho_1)$ is not empty.

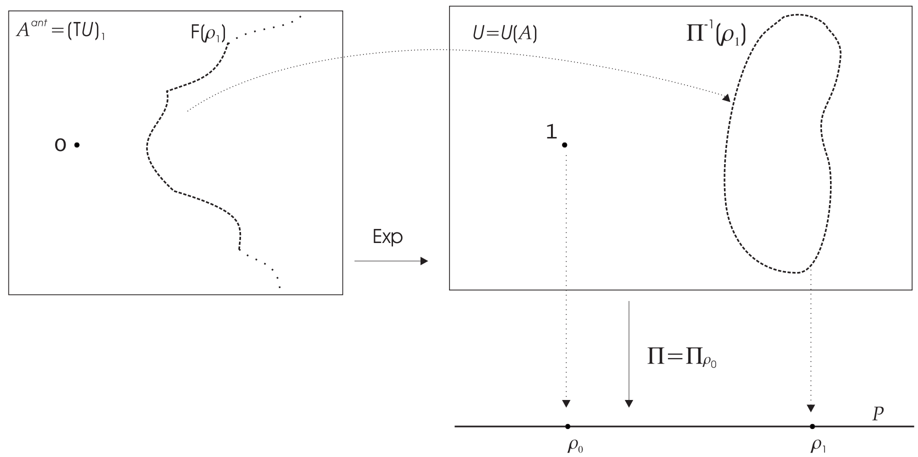

Figure 1: The set $F(\rho_1)$

Lemma 3.1 *Given $\rho_0, \rho_1 \in \mathcal{P}$, there exists $Z \in \mathcal{A}^{ant}$ such that $L_{e^Z}\rho_0 = \rho_1$. Moreover, the norm of Z can be taken to be less than or equal to π.*

Proof: By hypothesis, the action of $U(\mathcal{A})$ in $\mathcal{P}$ is transitive. Therefore there exists $u \in U(\mathcal{A})$ such that $L_u\rho_0 = \rho_1$. Now the proof follows from lemma 2.5.

□

Now let us relate directors with minimal lifts. A director Z is said to be *minimal director* if it minimizes the norm among all director vectors.

Theorem 3.2 *Consider ρ_1 and ρ_0 in $\mathcal{P}$. If Z_1 is a minimal director, then Z_1 is a minimal lift.*

A picture presenting this situation can be seen in Figure 2. The authors think that this theorem is of independent interest, giving a 'tangency' relation between the non-linear set $F(\rho_1)$ and the affine set $Z+\mathcal{B}_{\rho_0}$, and it tells 'where to point' in order to transform the Dirichlet problem into a Cauchy problem.

The proof of theorem 3.2 is a consequence of the following lemmas. The first one asserts that the condition of being a minimal director is stable along a geodesic.

Lemma 3.3 *Consider any pair ρ_1, ρ_0 in $\mathcal{P}$. Suppose that $Z_1 \in F(\rho_1)$ and that $|Z_1| = \inf\{|Z|;\ \text{with}\ Z \in F(\rho_1)\}$. For any $\mu \in \mathbb{R}$ define $Z_\mu = \mu Z_1 \in \mathcal{A}^{ant}$ and $\rho_\mu = L_{e^{Z_\mu}}\rho_0$. Then for any $\mu \in (0,1)$, $Z_\mu \in F(\rho_\mu)$ and $|Z_\mu| = \inf\{|Z|;\ \text{with}\ Z \in F(\rho_\mu)\}$.*

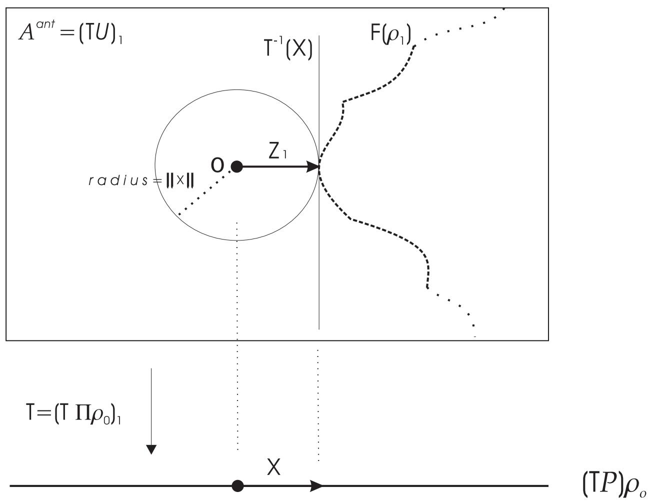

Figure 2: Minimal directors are minimal lifts

Proof: The fact that $Z_\mu \in F(\rho_\mu)$ follows straight forward, for $\rho_\mu = L_{e^{Z_\mu}}\rho_0$. The basic idea of the proof is to lift everything to the unitary group $U(\mathcal{A})$, illustrated in figure 3. Recall that uniparametric groups e^{tZ} are actually geodesics of the unitary group, which are minimizing up to length π (theorem 2.1). For any $Z \in \mathcal{A}^{\text{ant}}$ and $t \in [0,1]$ let's denote by $\gamma_Z(t) = e^{tZ}$ the path taken ($t \in [0,1]$) from the uniparametric group curve directed by Z. Thus if $Z_1 \in F(\rho_1)$, the condition $|Z_1| = \inf\{|Z|;\ \text{ with }\ Z \in F(\rho_1)\}$, is equivalent to $\gamma_{Z_1}(t) = e^{tZ_1}$ being the shortest path joining $1 \in U(\mathcal{A})$ with the fiber $\Pi_{\rho_0}^{-1}(\rho_1)$.

To show $|Z_\mu| = \inf\{|Z|;\ \text{ with }\ Z \in F(\rho_\mu)\}$ suppose, on the contrary, that for some $\mu \in (0,1)$, $|Z_\mu| > |W|$, for some $W \in F(\rho_\mu)$. Then $\ell(\gamma_{Z_\mu}) = |Z_\mu| > |W| = \ell(\gamma_W)$. Consider the curve $\delta(t) = e^{Z_\mu(1-t)+tZ_1}$ for $t \in [0,1]$ which joins e^{Z_μ} to e^{Z_1}, consider the curve $\sigma(t) = \delta(t)\, e^{-Z_\mu}e^W$ for $t \in [0,1]$ which joins e^W to $e^{(1-\mu)Z_1}e^W$. Observe that the length of σ equals the length of δ, for they differ by the unitary factor $e^{-Z_\mu}e^W$ (a traslation of $\delta(t)$). Hence $\ell(\sigma) = \ell(\delta) = (1-\mu)|Z_1|$. Observe also that the end-point $e^{(1-\mu)Z_1}e^W$ of σ is in $\Pi_{\rho_0}^{-1}(\rho_1)$. Recall that the length of γ_{Z_1} is $\ell(\gamma_{Z_1}) = |Z_1|$. Now consider the curve $\hat\gamma$ constructed by the concatenation of γ_W followed with σ. Observe that $\hat\gamma$ joins $1 \in U(\mathcal{A})$ with the fiber $\Pi_{\rho_0}^{-1}(\rho_1)$. Notice that,

$$\begin{aligned}\ell(\hat\gamma) &= \ell(\gamma_W) + \ell(\delta) = |W| + (1-\mu)|Z_1| \\ &< |Z_\mu| + (1-\mu)|Z_1| = \mu|Z_1| + (1-\mu)|Z_1| = |Z_1|\,.\end{aligned}$$

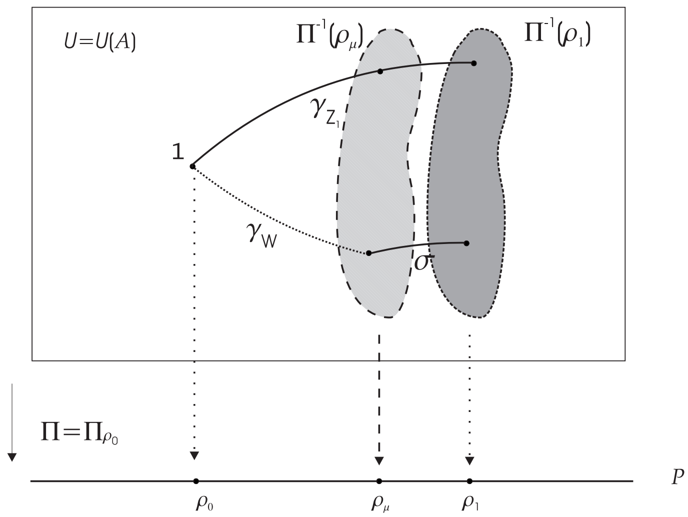

Figure 3: The curves γ_{Z_1} and $\hat{\gamma}$ (the concatenation of γ_W with σ)

Hence $\hat{\gamma}$ is strictly shorter than γ_{Z_1}. This contradicts the observation above, that γ_{Z_1} is the shortest path joining $1 \in U(\mathcal{A})$ with the fiber $\Pi_{\rho_0}^{-1}(\rho_1)$.

$\square$

The next lemma just says that $e^{X+Y} = e^X e^Y$ up to the order that we need,

Lemma 3.4 *Let* $X,\ y \in \mathcal{A}^{ant}$ *then* $\log\left(e^X e^Y\right) = X + Y + R_2(X, Y)$ *where*

$$\lim_{\lambda \to 0} \frac{R_2(\lambda X, \lambda Y)}{\lambda} = 0.$$

Proof: The series of the analytic function $\log\left(e^X e^Y\right)$ in the variables, $X,\ Y$ has linear part $X + Y$, and the remainder term $R_2(X, Y)$ is analytic with all its terms of degree at least 2 in the variables $X,\ Y$ (see [20, p. 119]).

$\square$

Proof of Theorem 3.2:

For any $\mu \in (0, 1)$ consider $Z_\mu = \mu Z_1$, and $\rho_\mu = L_{e^{Z_\mu}} \rho_0$. Now from lemma 3.3 $|Z_\mu| = \inf\{|Z|;\ \text{ with } \ Z \in F(\rho_\mu)\}$.

Let $b \in \mathcal{B}_{\rho_0}$. Notice that $\log\left(e^{Z_1} e^b\right) \in F(\rho_1)$. Similarly, for any $\mu \in (0, 1)$, the point $e^{Z_\mu} e^b$ is in the set $e^{F(\rho_\mu)}$. The hypothesis and lemma 3.4 imply,

$$|Z_1| \leq \left|\log\left(e^{Z_1} e^b\right)\right| = \left|Z_1 + b + R_2(Z_1, b)\right|.$$

By lemma 3.3, if $\mu \in (0,1)$,

$$|Z_\mu| \le |\log\left(e^{Z_\mu}e^{\mu b}\right)| = |Z_\mu + \mu b + R_2(Z_\mu, \mu b)|\,.$$

which implies that,

$$\begin{aligned} \mu|Z_1| &\le |\mu Z_1 + \mu b + R_2(\mu Z_1, \mu b)| \\ &\le |\mu Z_1 + \mu b| + |R_2(\mu Z_1, \mu b)|\,. \end{aligned}$$

Dividing by μ,

$$|Z_1| \le |Z_1 + b| + \left|\frac{R_2(\mu Z_1, \mu b)}{\mu}\right|\,.$$

Now taking the limit $\mu \to 0$, and using lemma 3.4, one gets $|Z_1| \le |Z_1 + b|$, as desired.

□

In section 4 it is shown that minimal directors always exists when the subalgebra $\mathcal{B}$ is a von Neumann algebra of compact type. Under that assumption, one can prove,

Theorem II-1 (Local Hopf-Rinow) *Let $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ be an isotropy compact generalized flag, and let $\rho_0 \in \mathcal{P}$. There exists an open neighborhood $\mathcal{V}_{\rho_0}$ of ρ_0 such that, if $\rho_1 \in \mathcal{V}_{\rho_0}$, there exists a minimal uniparametric group curve joining ρ_0 to ρ_1.*

Proof: The set $\mathcal{V}_{\rho_0}$ is actually quite big: let $\mathcal{V}_{\rho_0} = \{L_u\rho_0 \;/\; u = e^X, |X| < \pi/2\}$. For any $\rho_1 \in \mathcal{V}_{\rho_0}$, the set $F(\rho_1)$ by definition contains some Z with $|Z| < \pi/2$. Thus theorems I-1 and 3.2 guarantee that a minimal director Z is the initial velocity vector of a minimizing uniparametric group curve joining ρ_0 to ρ_1.

□

3.2 Relative minimal curves

In this section let us deal with the problem that minimal directors Z might have $|Z| > \pi/2$. In this case theorem I-1 does not guarantee that the curve $\gamma(t) = L_{e^{tZ}}\rho_0$ is minimizing. It will be shown that the curve $\gamma(t)$ is actually a minimal curve joining ρ_0 and ρ_1, thus proving the global Hopf-Rinow theorem.

Proposition 3.5 *Let $\Gamma(t) = e^{tZ}$, $\rho_0 = \Pi_{\rho_0}(1)$, $\rho_1 = \Pi_{\rho_0}(e^Z) = L_{e^Z}\rho_0$. If Z is a minimal director, then $\Gamma : [0,1] \to U(\mathcal{A})$ minimizes length among those curves joining $\Pi_{\rho_0}^{-1}(\rho_0)$ to $\Pi_{\rho_0}^{-1}(\rho_1)$.*

Proof: Assume there exists a curve $\sigma : [0,1] \to U(\mathcal{A})$ joining $\Pi_{\rho_0}^{-1}(\rho_0)$ to $\Pi_{\rho_0}^{-1}(\rho_1)$ with $\ell(\sigma) < \ell(\Gamma)$. Let $u_0 = \sigma(0) \in \Pi_{\rho_0}^{-1}(\rho_0)$, $u_1 = \sigma(1) \in \Pi_{\rho_0}^{-1}(\rho_1)$. Without loss of generality, via a translation, one can assume that $u_0 = 1 \in U(\mathcal{A})$. By theorem 2.5, there exists a uniparametric group $\widehat{\Gamma}(t) = e^{tW}$ joining $1 = \sigma(0) = \widehat{\Gamma}(0)$ to $u_1 = \sigma(1) = \widehat{\Gamma}(1)$ with

$$|W| = \ell(\widehat{\Gamma}) \leq \ell(\sigma) < \ell(\Gamma) = |Z|,$$

which contradicts the fact that Z is a minimal director.

□

The property that the curve Γ minimizes length among curves joining the fibers $\Pi_{\rho_0}^{-1}(\rho_0)$ and $\Pi_{\rho_0}^{-1}(\rho_1)$ is *almost* equivalent to say that $\gamma := \Pi_{\rho_0} \circ \Gamma$ minimizes length among curves joining ρ_0 and ρ_1 ([1]). The 'almost' part being due in this case to the difficulty of lifting curves on the fibration $U(\mathcal{A}) \to \mathcal{P}$ while preserving their lengths. There are enough curves that one can lift, to bypass this difficulty.

Definition 3.1 *A continuous curve $\alpha : [0,1] \to \mathcal{P}$ will be called a* broken geodesic *if there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for each $i = 1, \ldots, n$, the restriction of α to the interval $[t_{i-1}, t_i]$ is of the form $L_{e^{r_i(t) Z_{i-1}}}\alpha(t_{i-1})$, where $r_i(t)$ is an affine function satisfying $r_i(t_{i-1}) = 0$ and $r_i(t_i) = 1$, and $Z_i \in \mathcal{A}^{ant}$ is a minimal lift (of some tangent vector) at the point $\alpha(t_{i-1})$.*

Note that, by definition, a broken geodesic α admits a lift $\tilde{\alpha}$ defined by $\tilde{\alpha}(0) = 1$ and $\tilde{\alpha}(t) = e^{r_i(t)Z_{i-1}}\tilde{\alpha}(t_{i-1})$ for $t \in [t_{i-1}, t_i]$. The Z_i are required to be minimal lifts, it follows that $\ell(\alpha) = \ell(\tilde{\alpha})$. In this definition the pieces of the broken geodesic are not required to be minimizing. Still, one has the following lemmas.

Lemma 3.6 *Let $\mathcal{P}$ be an isotropy compact generalized flag, and let $\sigma : [0,1] \to \mathcal{P}$ be a continuous curve. Then there exists a broken geodesic α such that $\alpha(0) = \sigma(0), \alpha(1) = \sigma(1)$ and $\ell(\alpha) \leq \ell(\sigma)$.*

Proof: Select a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\sigma(t_i) \in \mathcal{V}_{\sigma(t_{i-1})}$, where $\mathcal{V}_{\sigma(t_{i-1})}$ as in the proof of the Local Hopf-Rinow theorem II-1, which then provides us with minimal uniparametric group curves α_i joining $\sigma(t_{i-1})$ to $\sigma(t_i)$, and thus satisfying $\ell(\alpha_i) \leq \ell(\sigma|_{[t_{i-1},t_i]})$. The concatenation of all the α_i's gives us the desired broken geodesic.

$\square$

Theorem 3.7 *Let $\rho_0, \rho_1 \in \mathcal{P}$. Let $\Gamma(t) = e^{tZ}$, where Z is a minimal director, and $L_{e^Z}\rho_0 = \rho_1$. Then $\gamma(t) = L_{e^{tZ}}\rho_0$ minimizes length among curves joining ρ_0 to ρ_1.*

Proof: Assume on the contrary that there exists a curve $\sigma(t)$ joining ρ_0 to ρ_1 with $\ell(\sigma) < \ell(\gamma)$. By lemma 3.6, there exists a broken geodesic α joining ρ_0 and ρ_1 such that $\ell(\alpha) \leq \ell(\sigma) < \ell(\gamma) = \ell(\Gamma)$. The lift $\tilde{\alpha}$ constructed as above is a curve joining $\Pi_{\rho_0}^{-1}(\rho_0)$ with $\Pi_{\rho_0}^{-1}(\rho_1)$ of length strictly less than the length of Γ, contradicting proposition 3.5.

$\square$

Thus, theorem 3.7 reduces the global Hopf-Rinow problem to the existence of a minimal director.

4 Analysis

Let $\rho_0, \rho_1 \in \mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ be fixed. In this section it is shown the existence of minimal directors, i.e. a vector Z of minimal norm among those such that $L_{e^Z}\rho_0 = \rho_1$, in the case $\mathcal{B}$ is of the compact type. In subsection 4.1, the proof of the main theorem is given, whereas in subsection 4.2 the definition of von Neumann algebras $\mathcal{B}$ of compact type is reviewed, and some examples are presented. In addition, some examples of isotropy compact generalized flags $U(\mathcal{A})/U(\mathcal{B})$ are given.

4.1 Proof of the Global Hopf-Rinow theorem

Theorem II-2 (Global Hopf-Rinow) *Let $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ be an isotropy compact generalized flag, where $\mathcal{A}$ is a von Neumann algebra. Let ρ_0 and ρ_1 be points in $\mathcal{P}$. Then, there exists $Z \in \mathcal{A}^{ant}$, such that the uniparametric group curve $\gamma(t) = L_{e^{tZ}}\rho_0$ for $t \in [0,1]$ joins ρ_0 to ρ_1, and its length is minimal among piecewise smooth curves joining ρ_0 to ρ_1.*

The following theorem solves the step still pending in the discution of the previous section.

Theorem 4.1 *Let $\mathcal{A}$ be a von Neumann algebra, and $\mathcal{P}$ a generalized flag of $U(\mathcal{A})$ such that the isotropy algebra $\mathcal{B}$ at $\rho_0 \in \mathcal{P}$ is of compact type. Then for any $\rho_1 \in \mathcal{P}$ there exists a minimal director for ρ_1.*

Now it is straight forward that theorems 3.7 and 4.1 prove the Global Hopf-Rinow theorem (II-2).

Proof of theorem 4.1:

Denote by $\alpha = \inf\{|1-u|\,,\, u \in \Pi_{\rho_0}^{-1}(\rho_1)\}$.

First let us show that there exists $v \in \Pi_{\rho_0}^{-1}(\rho_1)$ such that $|1-v| = \alpha$. Let $u_n \in \Pi_{\rho_0}^{-1}(\rho_1)$ for $n = 1, 2, 3, \ldots$, be such that $|1-u_n| < \alpha + \frac{1}{n}$. Next choose v in the set $L = \cap_{n=1}^{\infty} \overline{C}_n \subset \Pi_{\rho_0}^{-1}(\rho_1)$ where $C_n = \{u_k \,|\, k \geq n\}$ and the closure is understood in the strong topology. Observe that L is non-empty because of the compactness hypothesis of the unitary group $U(\mathcal{B})$ and the continuity in the strong topology of left multiplication (observe that $\Pi_{\rho_0}^{-1}(\rho_1) = g\,U(\mathcal{B})$ for any fixed $g \in \Pi_{\rho_0}^{-1}(\rho_1)$).

To show that $|1-v| = \alpha$, let $\epsilon > 0$ and $\xi \in \mathcal{H}$ with $||\xi|| = 1$, be such that $||(1-v)\xi|| > |1-v| - \epsilon$. Now let $n \in \mathbb{N}$ be such that $||u_n\xi - v\xi|| < \epsilon$. Then,

$$\begin{aligned} |1-v| &< ||(1-v)\xi|| + \epsilon = ||1 - u_n\xi + u_n\xi - v\xi|| + \epsilon \\ &\leq ||1 - u_n\xi|| + ||u_n\xi - v\xi|| + \epsilon < \alpha + 1/n + 2\epsilon\,. \end{aligned}$$

Since n and ϵ are arbitrary this shows that $|1-v| \leq \alpha$, and therefore $|1-v| = \alpha$. To conclude the proof let us consider two cases:

If $|1-v| = 2$, then by lemma 2.4 for *any* $Z \in \mathcal{A}^{\text{ant}}$ such that $e^Z = v$, $|Z| \geq \pi$. Since there exists $Z \in \mathcal{A}^{\text{ant}}$, $|Z| \leq \pi$ such that $e^Z = v$, any such Z satisfies $\pi = |Z| = \inf\{|W|\,,\, W \in F(\rho_1)\}$.

Next, if $|1-v| < 2$, then by lemma 2.2 there exists a unique $Z \in \mathcal{A}^{\text{ant}}$, $|Z| < \pi$ such that $e^Z = v$. Now observe that $|Z| = \inf\{|W|\,,\, W \in F(\rho_1)\}$. Indeed, if there is $Z' \in F(\rho_1)$ with $|Z'| < |Z|$, then $v' = e^{Z'} \in \pi^{-1}(\rho_1)$ and by lemma 1, $|1-v'| < |1-v|$, which contradicts the choice of v.

□

4.2 Von Neumann algebras of compact type

Recall that in the Introduction, von Neumann algebras of *compact type* were defined as follows: the von Neumann algebra $\mathcal{A}$ is said to be of *compact type* if $U(\mathcal{A})$ is compact in the strong (operator) topology.

Remark: In this definition it is enough to require that the strong closure of the unitary group $U(\mathcal{A})$ of $\mathcal{A}$ is compact in the strong topology. This is true because $U(\mathcal{A})$ turns out to be closed in the strong topology. In fact let $\{u_n\}_{n\in I}$ be a net in $U(\mathcal{A})$ with strong limit $g \in \mathcal{A}$. The net $\{u_n^*\}_{n\in I}$ has a subnet $\{u_k^*\}_{k\in K}$ which converges to some $h \in \mathcal{A}$. Since multiplication is strongly continuos on bounded sets of $\mathcal{A}$, the constant net $\{1 = u_k\, u_k^*\}_{k\in K}$ converges to $1 = g\,h$. This shows that g is invertible, and since g is an isometry then g is unitary.

In order to give examples of von Neumann algebras of compact type, let us first present a general result about products of von Neumann algebras. Recall first the definition of a product of von Neumann algebras. Let $\{\mathcal{H}_i\}_{i\in I}$, be a family of Hilbert spaces and let $\mathcal{H}$ be their Hilbert sum, i.e. the collection of all families $\{\xi_i\}_{i\in I}$ such that $\xi_i \in \mathcal{H}_i$ and $\displaystyle\sum_{i\in I} ||\xi_i||^2 < \infty$. For $i \in I$, let $\mathcal{A}_i$ be a von Neumann algebra of operators in $\mathcal{H}_i$. For every family $\{a_i\}_{i\in I}$ such that $a_i \in \mathcal{A}_i$ and $\sup\{|a_i|,\ i \in I\} < \infty$ define the operator $\{\xi_i\} \mapsto \{a_i(\xi_i)\}$ in $\mathcal{H}$. The collection of all these is a von Neumann algebra of operators in $\mathcal{H}$ which is called the product of $\{\mathcal{A}_i\}_{i\in I}$ and here it is denoted by $\times_{i\in I}\mathcal{A}_i$. Observe that $\times_{i\in I}\mathcal{A}_i$ is a subset of the cartesian product $\prod_{i\in I}\mathcal{A}_i$. Let us denote by $\mathcal{T}$ the cartesian product topology of the strong topologies of the $\mathcal{A}_i$'s on $\prod_{i\in I}\mathcal{A}_i$

Theorem 4.2 *In the above notation, let $C \subset \mathcal{A} = \times_{i\in I}\mathcal{A}_i$ be a (norm) bounded set. Then on C the strong topology of $\mathcal{A}$ coincides with the restriction of $\mathcal{T}$ to C.*

Proof: For $\tilde{a}$ in $\mathcal{A}$ fixed, let us define two types of sets.

Type 1. Finite intersections of sets of the form

$$V_{\tilde{a}}(\xi,\epsilon) = \left\{ a \in \mathcal{A} \;\middle|\; ||(a-\tilde{a})\xi||^2 = \sum_{i\in I} ||(a_i - \tilde{a}_i)\xi_i||^2 < \epsilon \right\}$$

where $\xi \in \mathcal{H}$ and $\epsilon > 0$.

Type 2. Sets of the form

$$W_{\tilde{a}}(\{\eta_i\}_{i\in F}, \epsilon) = \left\{a \in \mathcal{A} \,\middle|\, ||(a_i - \tilde{a}_i)\xi_i||^2 < \epsilon,\ i \in F\right\}$$

where $F \subset I$ is finite and $\eta_i \in \mathcal{H}_i$ for $i \in F$.

Of course, the sets of type 1, form a base of neighborhoods of $\tilde{a}$ in the strong topology of $\mathcal{A}$ and the sets of type 2 form a base of neighborhoods of $\tilde{a}$ in the topology $\mathcal{T}$.

It is clear that the strong topology is finer than $\mathcal{T}$. Conversely let us show that on the bounded set C the topology $\mathcal{T}$ is finer than the strong topology. Let $V_{\tilde{a}}(\xi, \epsilon)$ be given. There is a finite set $F \subset I$ such that

$$M \sum_{i \notin I} ||\xi_i||^2 < \epsilon \qquad (1)$$

where $M > ||a_i - \tilde{a}_i||^2$, for $i \in I$ and all $a \in C$. Now we claim that if $|F|$ = number of elements of F then

$$W_{\tilde{a}}\left(\{\xi_i\}_{i\in F}, \frac{\epsilon}{2\,|F|}\right) \subset V_{\tilde{a}}(\xi, \epsilon),$$

and this will conclude the proof. In fact

$$\sum_{i\in I} ||(a_i - \tilde{a}_i)\xi_i||^2 = \sum_{i\in F} ||(a_i - \tilde{a}_i)\xi_i||^2 + \sum_{i\in I\setminus F} ||(a_i - \tilde{a}_i)\xi_i||^2 < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

because in the first sum $||(a_i - \tilde{a}_i)\xi_i||^2 < \frac{\epsilon}{2\,|F|}$ for $i \in F$, and for the second sum, use inequality (1).

□

Theorem 4.3 *Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of von Neumann algebras of compact type. Then their product $\mathcal{A} = \times_{i\in I}\mathcal{A}_i$ is also of compact type.*

Proof: In fact the unitary group $U(\mathcal{A})$ is bounded in $\mathcal{A}$ and it is the cartesian product of the unitary groups $U(\mathcal{A}_i)$ of the algebras $\mathcal{A}_i$ for $i \in I$. Therefore, by theorem 4.2, the strong topology in $U(\mathcal{A})$ coincides with the cartesian product topology of the strong topologies of the $U(\mathcal{A}_i)$'s. The theorem follows from Tychonoff's Theorem.

□

To obtain examples of von Neumann algebras of compact type one may consider strongly closed subalgebras of arbitrary products of von Neumann algebras of compact type. In particular any product of finite dimensional C^*-algebras is a von Neumann algebra of compact type.

As a consequence one gets examples of isotropy compact generalized flags $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ for particular algebras $\mathcal{A}$ and $\mathcal{B}$.

Example 1.

Let $\mathcal{H}$ be a Hilbert space and let $E = \{\epsilon_k \mid k = 1, 2 \ldots\}$ be an orthonormal basis of $\mathcal{H}$. Let $\mathcal{A} = L(\mathcal{H})$ and $\mathcal{B} \subset L(\mathcal{H})$ be the algebra of all operators which are diagonal with respect to the basis E, i.e. $b \in \mathcal{B}$ satisfies $\langle b(\epsilon_i), \epsilon_j \rangle = 0$ for $i \neq j$. Then $\mathcal{B}$ is of compact type. In fact the unitary group $U(\mathcal{B})$ with the strong topology is isomorphic to the infinite dimensional torus $\prod_{k=1}^{\infty} S_k$ (with the product topology) where each S_k is a copy of the unit circle of the complex plane. The homogeneous space $\mathcal{P} = U(L(\mathcal{H}))/U(\mathcal{B})$ may be considered an infinite dimensional version of the flag manifold $U(n)/\underbrace{U(1) \times \ldots \times U(1)}_{n \text{ factors}}$.

This example may be easily generalized by considering a Hilbert sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$ where each $\mathcal{H}_i$ is a finite dimensional subspace of $\mathcal{H}$, and defining $\mathcal{B}$ as the subalgebra of $\mathcal{A}$ consisting of all 'block diagonal' operators with respect to the above decomposition.

Example 2.

Let $\mathcal{A}$ be a *uniformly hyperfinite (UHF)* C^*-algebra (see, [15, Ch. 12]), presented as the direct limit of the sequence

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots$$

where $\mathcal{A}_i$ is isomorphic to the matrix algebra $M_{n_i}(\mathbb{C})$, for example the Fermion algebra (see [12]). Interesting examples of isotropy compact generalized flags can be gotten by considering $U(\mathcal{A})/U(\mathcal{B})$ where $\mathcal{B}$ is any of the $\mathcal{A}_i$'s.

Notice that one has canonical projections $U(\mathcal{A})/U(\mathcal{A}_i) \to U(\mathcal{A})/U(\mathcal{A}_j)$ for $i < j$. These projections reduce distances in the natural Finsler structures.

5 Concluding remarks

The basic problems of the Calculus of Variations for curve length in our context are: given a generalized flag $\mathcal{P}$,

1. Initial value problem: Does there always exists a minimizing uniparametric group curve γ with a given initial velocity vector $\gamma'(0) = v \in T\mathcal{P}$?

2. Local Hopf-Rinow theorem for $\mathcal{P}$: Let $\rho_0 \in \mathcal{P}$. Is there an open neighborhood $\mathcal{V}_{\rho_0}$ of ρ_0 such that, if $\rho_1 \in \mathcal{V}_{\rho_0}$, there exists a minimal uniparametric group curve joining ρ_0 to ρ_1?

3. Global Hopf-Rinow theorem for $\mathcal{P}$: Given ρ_0, ρ_1 in the same connected component of $\mathcal{P}$, is there a minimizing uniparametric group curve joining ρ_0 to ρ_1?

In [10] and in the present paper, the authors have studied the classical problems above: the initial value problem 1 (in [10]) and the boundary value problems 2 and 3 (in this paper). The geometric aspects of these problems have been solved under convenient hypothesis of analytical nature on the subalgebra $\mathcal{B}$. It has been required for $\mathcal{B}$ to be a von Neumann algebra for the initial value problem, and $\mathcal{B}$ has to be of compact type for the boundary value problems.

The authors suspect that problems 1, 2 and 3 above, have a positive answer for more general conditions on the pair $(\mathcal{A},\mathcal{B})$ of C^*-algebras. Notice that for problem 1, it is enough to suppose $\mathcal{B}$ to be a von Neumann algebra (see [10]), and probably this hypothesis is enough for the boundary value problems.

The following specific instances of generalized flags may be testing grounds for these conjectures:

- Let $\tilde{\mathcal{P}}$ be the space of 3-flags in an arbitrary C^*-algebra $\mathcal{A}$, that is $\tilde{\mathcal{P}}$ is the set of triples of orthogonal projectors $\vec{p} = (p_1, p_2, p_3)$ such that $p_i p_j = 0$ if $i \neq j$, and $p_1 + p_2 + p_3 = 1$. The unitary group of $\mathcal{A}$ acts on $\tilde{\mathcal{P}}$ by conjugation; let $\mathcal{P}$ be the orbit through some $\vec{p} \in \tilde{\mathcal{P}}$. Thus $\mathcal{P}$ may be identified with $U(\mathcal{A})/U(\mathcal{B}_{\vec{p}})$ where $\mathcal{B}_{\vec{p}} = \{b \in \mathcal{A} \ / \ bp_i = p_i b,\ i = 1, 2, 3\}$. Since $\mathcal{B}_{\vec{p}}$ is not necessarily of compact type nor is it necessarily a von Neumann algebra, the answer to either 1, 2 or 3 above is unknown to the authors.

- Let $\mathcal{P}$ be just as before, but with $\mathcal{A} = \mathcal{L}(\mathcal{H})$. Then $\mathcal{A}$ is a von Neumann algebra, and so is $\mathcal{B}_{\vec{p}}$. Thus 1 above has a positive answer because of [10], but 2 and 3 are still open.

- The examples above could have been stated in a simpler case. Namely consider pairs $\vec{p} = (p_1, p_2)$ of orthogonal projections (p_1, $p_2 = 1 - p_1$). In this case $\tilde{\mathcal{P}}$ is just the Grassmann manifold of a C^*-algebra. In this example (perhaps due to the fact that $\mathcal{P}$ is a symmetric space) both the Cauchy and Dirichlet problems have been solved in [18]. However, if $\mathcal{A}$ is not a von Neumann algebra, then the structure of each connected component of $\mathcal{P}$ is more complicated, as it is shown in [19, pp. 421-422]. Thus problem 3 above is still open even in this case.

Some other related lines of research are:

- Many generalized flags $\mathcal{P}$ can be realized as submanifolds of the algebra $\mathcal{A}$ (but one has to be careful with the topology, see [2]). For example, in the case of finite systems of projectors $\vec{p} = (p_1, \ldots, p_n)$ —n-flags— an orbit is diffeomorphic to the orbit in $\mathcal{A}$ of the point $a = \lambda_1 p_1 + \cdots + \lambda_n p_n$, where $\lambda_i \neq \lambda_j$ if $i \neq j$. The general problem here is to relate the intrinsic (as homogenous space of $U(\mathcal{A})$) and extrinsic (as submanifolds of $\mathcal{A}$) geometry. For instance, in [18] a Hopf-Rinow Theorem in the Grassmannian $Gr(\mathcal{A})$ of a C^*-algebra is stated as follows: two reflections ρ_0 and ρ_1 in $Gr(\mathcal{A})$ whose distance in $\mathcal{A}$ is less than 1, can be joined by a minimal uniparametric group curve.

- Referring to Example 2 of section 4.2 about UHF algebras, notice that the diameters $\operatorname{diam}(U(\mathcal{A})/U(\mathcal{A}_i))$ form a decreasing sequence, bounded below by $\pi/2$. The authors believe that the limit

$$\lim_{i\to\infty} \operatorname{diam}\left(\frac{U(\mathcal{A})}{U(\mathcal{A}_i)}\right)$$

is an interesting metric invariant of the algebra $\mathcal{A}$.

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