ON THE MULTIPLICATIVE PRODUCTS OF THE n-DIMENSIONAL DISTRIBUTIONAL HANKEL TRANSFORMS

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I. abstract. In this note, we prove several multiplicative products of the n-dimensional Hankel transform. In fact the following formulae:

$$\mathcal{H}\left\{\delta^{(k)}(u(x))\right\}.\mathcal{H}\left\{\delta^{(\ell)}(u(x))\right\} = D\mathcal{H}\left\{\delta^{\left(k+\ell+\frac{n-2}{2}\right)}(u(x))\right\}, \qquad \text{(III,1)}$$

$$\mathcal{H}\left\{\delta^{(k)}(P)\right\}.\mathcal{H}\left\{\delta^{(\ell)}(P)\right\} = \mathcal{CH}\left\{\delta^{\left(k+\ell+\frac{n-2}{2}\right)}(P)\right\}, \qquad \text{(IV,9)}$$

$$\mathcal{H}\left\{\delta^{(k)}(|x|^2)\right\}.\mathcal{H}\left\{\delta^{(\ell)}(|x|^2)\right\} = \mathcal{CH}\left\{\delta^{\left(k+\ell+\frac{n-2}{2}\right)}(|x|^2)\right\}, \qquad \text{(V,I)}$$

We observe that the above results were inspirated in a non-edited paper due to Manuel Aguirre Téllez (cfr. [8]).

II. Introduction.

We begin with some definitions. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional Euclidean space $I\!R^n$. Consider a non-degenerate quadratic form in n variables of the form

$$P = P(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \ , \qquad (\mathrm{II}, 1)$$

where $n = p + q$.

We define the two following distributions, as follows

$$P_+^\lambda = \begin{cases} P^\lambda & \text{if } P > 0, \\ 0 & \text{if } P \le 0; \end{cases} \qquad (\mathrm{II}, 2)$$

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and

$$P_-^\lambda = \begin{cases} 0 & \text{if } P > 0, \\ (-P)^\lambda & \text{if } P \leq 0. \end{cases} \qquad \text{(II, 3)}$$

$\mathcal{H}$ denotes the distributional Hankel transform. Let $\phi(t)$ be defined in $I\!R^+ : \{t, t > 0\}$. By the Hankel transform of the function $\phi(t)$ we mean the function $g(s)$, $0 \leq s < \infty$, defined by the formula

$$g(s) = \mathcal{H}\{\phi(t)\} = \int_0^\infty \phi(t) J_\nu(xt)\sqrt{xt}dt \ , \qquad \text{(II, 4)}$$

or, equivalently,

$$g(s) = (\mathcal{H}\{\phi(t)\}) = \frac{1}{2}\int_0^\infty \phi(t) t^{\frac{n-2}{2}} R_{\frac{n-2}{2}}(\sqrt{st})dt \ , \qquad \text{(II, 5)}$$

$$R_m(x) = \frac{J_m(x)}{x^m} \ , \qquad \text{(II, 6)}$$

and $J_m(x)$ is the well-known Bessel function defined by the formula

$$J_m(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \left(\frac{x}{2}\right)^{m+2\nu}}{\nu!\,\Gamma(m+\nu+1)} \ . \qquad \text{(II, 7)}$$

It is well known (cfr.[1], p. 240) that if $\phi(t)$ satisfies adequate conditions, for example if $\phi(t)$ belongs to $S_{I\!R^+}$, the following formula is valid:

$$\phi(t) = (\mathcal{H}\{g(s)\}) = \frac{1}{2}\int_0^\infty g(s) s^{\frac{n-2}{2}} R_{\frac{n-2}{2}}(\sqrt{st})ds \ . \qquad \text{(II, 8)}$$

Let $S_{I\!R^+}$ designate the space of functions $f \in S$ defined in the positive half line $I\!R^+ = \{t, t > 0\}$. By $S'_{I\!R^+}$ we designate the dual of $S_{I\!R^+}$.

Let $U(t) \in S'_{I\!R^+}$. The Hankel transform of $U(t)$ will be, by definition, the distribution $v(s) \in S'_{I\!R^+}$, defined by the formula

$$\langle \mathcal{H}\{U(t), \phi(s)\}\rangle = \langle U(t), (\mathcal{H}\{\phi(s)\})\rangle \ , \qquad \text{(II, 9)}$$

for every $\phi \in S_{I\!R^+}$.

There are other definitions of the Hankel transform of distributions (cfr. [2]) but, here we use the definition which appears in [3], Appendix I, p. 64, especially, Theorem 26, p. 72. In fact, we have that

$$\widetilde{\mathcal{H}(\widetilde{T})} = \{T\}^\wedge \ , \qquad \text{(II, 10)}$$

here $\widetilde{T}$ is the image of T belongs to $S'^{\natural}_{I\!R^n}$ in S', defined by the formula

$$\langle \widetilde{T}, \phi(t) \rangle = \langle T, \phi(r^2) \rangle \ , \qquad (\mathrm{II}, 11)$$

for every $\phi \in S_{I\!R^+}$.

We designate $S^{\natural}_{I\!R^n}$ the family of functions $f(x)$ belongs to $S_{I\!R^n}$ and, further, invariable by rotations. Moreover, $S'^{\natural}_{I\!R^n}$ designates the dual of $S^{\natural}_{I\!R^n}$.

Following strictly the definitions of [8], we shall define the k-th derivative of Dirac delta in $u(x_1, x_2, \ldots, n_n)$.

Let ϕ_t denote a distribution of one variable t. Let $u \in C^\infty(I\!R^n)$ be such that $(n-1)$-dimensional manifold $u(x_1, x_2, \ldots, x_n) = 0$ has no critical point. By $\phi_{u(x)}$ (cfr. [9], page 102) we designate the distribution defined on $I\!R^n$ by

$$\langle \phi_{u(x)}, \varphi(x) \rangle = \langle \phi_t, \psi(t) \rangle \ , \qquad (\mathrm{II}, 12)$$

where

$$\psi(t) = \int_{u(x)=t} \varphi(x) w_u(x, dx) \ , \qquad (\mathrm{II}, 13)$$

and $\varphi \in C_0^\infty(I\!R^n)$ is the set of infinitely differentiable functions with compact support and w_u is a $(n-1)$-dimensional exterior differential form on u defined as follows:

$$du \wedge dw = dx_1 \wedge \ldots \wedge dx_n \ , \qquad (\mathrm{II}, 14)$$

and the orientation of the manifold $u(x) = t$ is such that $w_u(x, dx) > 0$.

On the other hand (cfr. [10], p. 230, form. (6)), we have

$$\Big(\delta^{(k)}(G(x_1, \ldots, x_n), \varphi(x_1, \ldots, x_n)\Big) = (-1)^k \int_{G(x)=0} w_k(\varphi) \ , \qquad (\mathrm{II}, 15)$$

$k = 0, 1, \ldots$; where $x = (x_1, \ldots, x_n)$, $G(x_1, \ldots, x_n)$ is such an infinitely differentiable function that

$$\operatorname{grad} G = \left(\frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n} \right) \neq 0 \ , \qquad (\mathrm{II}, 16)$$

$$w_k(\varphi) = \frac{\partial^k}{\partial u_1^k} \left\{ D \binom{x}{u} \varphi, (u_1, \ldots, u_n) \right\} du_1 \ldots du_n \ , \qquad (\mathrm{II}, 17)$$

$$w_0 = \varphi . w \ , \qquad (\mathrm{II}, 18)$$

$$
\begin{aligned}
u_1 &= G(x_1, \ldots, x_n), \\
u_2 &= x_2, \\
&\vdots \\
u_n &= x_n,
\end{aligned}
\tag{II,19}
$$

and

$$
D\binom{x}{u} = \left[D\binom{u}{x}^{-1} \right]^{-1} = \frac{1}{\frac{\partial G}{\partial x_1}} \ , \tag{II,20}
$$

with

$$
\frac{\partial G}{\partial x_1} > 0 \ . \tag{II,21}
$$

Otherwise, from [10], p. 211, form.. (8),

$$
\delta^{(k)} \langle (G(x)), \varphi \rangle = (-1)^k \int_G f_{u_1}^{(k)}(0, u_2, \ldots, u_n) du_2, \ldots, du_n \ , \tag{II,22}
$$

where

$$
f(u_1, u_2, \ldots, u_n) = \varphi_1(u_1, \ldots, u_n) D\binom{x}{u} \ , \tag{II,23}
$$

$$
\varphi_1(u_1, u_2, \ldots, u_n) = \varphi(x_1, x_2, \ldots, x_n) \ , \tag{II,24}
$$

and $D\binom{x}{u}$ is defined by (II,20).

III. The multiplicative product of the Hankel transform of the k-th derivative of the Dirac delta in $u(x_1, x_2, \ldots, x_n)$.

In this paragraph we shall obtain the following formula

$$
\mathcal{H}\left\{ \delta^{(k)}(u(x)) \right\} . \mathcal{H}\left\{ \delta^{(\ell)}(u(x)) \right\} = D\mathcal{H}\left\{ \delta^{(k+\ell+\frac{n-2}{2})}(u(x)) \right\} \ , \tag{III,1}
$$

where D is a constant given by (III,8).

Taking into account the formula (26), p. 4 of [8], we know that

$$
\mathcal{H}\left\{ \delta^{(k)}(u(x)) \right\} = \mathcal{H}\left\{ \delta^{(k)}(u(x_1, x_2, \ldots, x_n)) \right\} = B_{k,n}(u(y_1, y_2, \ldots, y_n))^{\frac{n-2}{2}+k} \ , \tag{III,2}
$$

where $\delta^{(k)}(u(x_1,\ldots,x_n))$ is given by the formula (II,22) and

$$B_{k,n} = \frac{1}{2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right)} \ , \qquad \text{(III, 3)}$$

$u(y_1,y_2,\ldots,y_n) \in C^\infty(I\!R^n)$ be such that the $(n-1)$-dimensional manifold $u(y_1,\ldots,y_n) = 0$ has no critical points.

Then, we have

$$\begin{aligned}\mathcal{H}\left\{\delta^{(k)}(u(x))\right\}.\mathcal{H}\left\{\delta^{(\ell)}(u(x))\right\} &= B_{k,n}(u(y))^{\frac{n-2}{2}+k}.B_{\ell,n}(u(y))^{\frac{n-2}{2}+\ell} \\ &= B_{k,n}.B_{\ell,n}(u(y))^{\frac{n-2}{2}+k+\frac{n-2}{2}+\ell} \ .\end{aligned} \qquad \text{(III, 4)}$$

The passage from the second to the third equality of (III,4) is licit. Indeed, we can multiplicate $[u(y)]^\lambda.[u(y)]^\mu$ first, as locally integrable functions for $\mathrm{Re}\,\lambda > 0$, $\mathrm{Re}\,\mu > 0$, and then, by analytical continuation, for every $\lambda,\mu \in C\!\!\!\!I$.

Otherwise, we know (cfr. form. (III,2)) that

$$\mathcal{H}\left\{\delta^{(k+\ell+\frac{n-2}{2})}(u(x))\right\} = B_{k+\ell+\frac{n-2}{2},n}(u(y))^{\frac{n-2}{2}+k+\ell+\frac{n-2}{2}} \ . \qquad \text{(III, 5)}$$

From (III,4) and (III,5), we have

$$\mathcal{H}\left\{\delta^{(k+\ell+\frac{n-2}{2})}(u(x))\right\} = \frac{B_{k+\ell+\frac{n-2}{2},n}}{B_{k,n}.B_{\ell,n}}\mathcal{H}\left\{\delta^{(k)}(u(x))\right\}.\mathcal{H}\left\{\delta^{(\ell)}(u(x))\right\} \ . \qquad \text{(III, 6)}$$

That is,

$$\mathcal{H}\left\{\delta^{k+\ell+\frac{n-2}{2})}(u(x))\right\} = D\mathcal{H}\left\{\delta^{(k)}(u(x))\right\}.\mathcal{H}\left\{\delta^{(\ell)}(u(x))\right\} \ , \qquad \text{(III, 7)}$$

here

$$D = \frac{B_{k+\ell+\frac{n-2}{2},n}}{B_{k,n}.B_{\ell,n}} \ , \qquad \text{(III, 8)}$$

or, equivalently,

$$D = \frac{\Gamma\left(\frac{n}{2}+k\right)\Gamma\left(\frac{n}{2}+\ell\right)}{2^{\frac{n}{2}-2}\Gamma(n+k+\ell-1)} \ . \qquad \text{(III, 9)}$$

Otherwise, by remembering the Pochhammer symbol, defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = (-1)^n\frac{\Gamma(1-a)}{\Gamma(1-a-n)} \ , \qquad \text{(III, 10)}$$

where a is an arbitrary complex and

$$\begin{aligned}(a)_0 &= 1 \ , \\ (a)_n &= a(a+1)\cdots(a+n-1) \ ,\end{aligned} \qquad \text{(III, 11)}$$

$n = 1, 2, \ldots$; we can express the constant D (form. (III,9)) in the equivalently form

$$D = \frac{\Gamma\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)_k \Gamma\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)_\ell}{2^{\frac{n}{2}-2}\left(k+\ell-1\right)_n \Gamma(k+\ell-1)} \ . \qquad \text{(III, 12)}$$

Finally, we note that, taking into account the theorem of identity for Hankel transforms and the formula (III,2), the following formula is valid:

$$\delta^{(k)}\left[u(x_1, x_2, \ldots, x_n)\right] = B_{k,n}\mathcal{H}\left\{\left[u(x_1, x_2, \ldots, x_n)\right]^{\frac{n-2}{2}+k}\right\} \ , \qquad \text{(III, 13)}$$

where $B_{k,n}$ is the constant given by (III,3).

IV. The multiplicative product of the Hankel transform of the k-th derivative of the Dirac delta in $P(x)$.

In this paragraph we obtain the multiplicative product of the Hankel transform of the k-th derivative of the Dirac delta in $P(x)$.

$$\mathcal{H}\left\{\delta^{(k)}(P)\right\}.\mathcal{H}\left\{\delta^{(\ell)}(P)\right\} = C\mathcal{H}\left\{\delta^{\left(k+\ell+\frac{n-2}{2}\right)}(P)\right\} \ , \qquad \text{(IV, 1)}$$

where C is the constant given by (IV,9).

We know (cfr. [4], form. (36), p. 279) that

$$\mathcal{H}\left\{\delta^{(k)}(P)\right\} = \frac{1}{2^{2k+\frac{n}{2}}\ \Gamma\left(\frac{n}{2}+k\right)} Q_+^{\frac{n-2}{2}+k} \ , \qquad \text{(IV, 2)}$$

here

$$Q = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2 \ , \qquad \text{(IV, 3)}$$

where $p + q = n$.

Also, we note by

$$Q_+ = \begin{cases} Q & \text{if } Q > 0 \ , \\ 0 & \text{if } Q \leq 0 \ . \end{cases} \qquad \text{(IV, 4)}$$

Therefore, we obtain the following equation

$$\mathcal{H}\left\{\delta^{(k)}(P)\right\}.\mathcal{H}\left\{\delta^{(\ell)}(P)\right\} = \frac{Q_+^{\frac{n-2}{2}+k}}{2^{2k+\frac{n}{2}}\ \Gamma\left(\frac{n}{2}+k\right)} \cdot \frac{Q_+^{\frac{n-2}{2}+\ell}}{2^{2\ell\frac{n}{2}}\ \Gamma\left(\frac{n}{2}+\ell\right)} \ . \qquad \text{(IV, 5)}$$

By taking into account the Theorem 2, p. 23 of [5], we can write, equivalently, the formula (IV,5) as

$$\mathcal{H}\left\{\delta^{(k)}(P)\right\}.\mathcal{H}\left\{\delta^{(\ell)}(P)\right\} = \frac{Q_+^{\frac{n-2}{2}+k+\frac{n-2}{2}+\ell}}{2^{2k+\frac{n}{2}+2\ell+\frac{n}{2}}\ \Gamma\left(\frac{n}{2}+k\right)\Gamma\left(\frac{n}{2}+\ell\right)}\ . \qquad (\mathrm{IV},6)$$

Otherwise, we have

$$\mathcal{H}\left\{\delta^{\left(k+\ell+\frac{n-2}{2}\right)}(P)\right\} = \frac{Q_+^{\frac{n-2}{2}+k+\ell+\frac{n-2}{2}}}{2^{2\left(k+\ell+\frac{n-2}{2}\right)+\frac{n}{2}}\ \Gamma\left(\frac{n}{2}+k+\ell+\frac{n-2}{2}\right)}\ . \qquad (\mathrm{IV},7)$$

From (IV,6) and (IV,7), we arrive at the following formula

$$\mathcal{H}\left\{\delta^{(k)}(P)\right\}.\mathcal{H}\left\{\delta^{(\ell)}(P)\right\} = C\mathcal{H}\left\{\delta^{\left(k+\ell+\frac{n-2}{2}\right)}(P)\right\}\ , \qquad (\mathrm{IV},8)$$

where C is the constant given by

$$C = \frac{2^{\frac{n}{2}-2}\ \Gamma(n+k+\ell-1)}{\Gamma\left(\frac{n}{2}+k\right)\ \Gamma\left(\frac{n}{2}+\ell\right)}\ . \qquad (\mathrm{IV},9)$$

V. The multiplicative product of the Hankel transform of the k-th derivative of the Dirac delta of $|x|^2$.

We know that $P = P(x) = |x|^2$ if $p = n$ and $q = 0$ in the formula (II,1), so the formula (IV,8) arrives at

$$\mathcal{H}\left\{\delta^{(k)}(|x|^2)\right\}.\mathcal{H}\left\{\delta^{(\ell)}(|x|^2)\right\} = C\mathcal{H}\left\{\delta^{\left(k+\ell+\frac{n-2}{2}\right)}(|x|^2)\right\}\ , \qquad (\mathrm{V},1)$$

where C is the constant given by (IV,9).

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