Metric geometry in homogeneous spaces of the unitary group of a C^*-algebra. Part I. Minimal curves.*

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Dedicated to Horacio Porta on his 65th birthday.

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Abstract

We study the metric geometry of homogeneous spaces $\mathcal{P}$ of the unitary group of a C^*-algebra $\mathcal{A}$ modulo the unitary group of a C^*-subalgebra $\mathcal{B}$, where the invariant Finsler metric is induced by the quotient norm of $\mathcal{A}/\mathcal{B}$.

The main results include the following. In the von Neumann algebra context, for each tangent vector X at a point $\rho \in \mathcal{P}$, there is a geodesic $\gamma(t)$, $\dot{\gamma}(0) = X$, which is obtained by the action on ρ of a 1-parameter group in the unitary group of $\mathcal{A}$. This geodesic is minimizing up to length $\pi/2$.

Key words: Finsler metric; homogeneous spaces; minimal curves; unitary group; von Neumann algebra.

1 Introduction

1.1 The category of unitary homogeneous spaces

The aim of this work is to study the metric geometry of homogeneous spaces of the 'elliptic type', *in a C^*-algebra context.*

Let us elaborate on the emphasized phrase: we assume that the homogeneous spaces in question are Banach manifolds $\mathcal{P}$ in which the unitary group $\mathcal{U}$ of a C^*-algebra $\mathcal{A}$ acts transitively, say on the left; we denote the action of $g \in \mathcal{U}$ on $\rho \in \mathcal{P}$ by $L_g\rho$. The isotropy $\mathcal{I}_\rho = \{g \in \mathcal{U} \ / \ L_g\rho = \rho\}$ will be required to be the unitary group of a C^*-subalgebra $\mathcal{B} \subset \mathcal{A}$. Thus the homogeneous spaces we study correspond to the 'relative C^*-algebra category', i.e. pairs $(\mathcal{A},\mathcal{B})$, $\mathcal{B} \subset \mathcal{A}$, of C^*-algebras.

The standard way of endowing a homogeneous space with a metric is to consider a (usually Riemannian) bi-invariant metric on the group and then pushing it down to the quotient $\mathcal{P}$. But C^*-algebras come equipped with an essentially unique norm (i.e. the operator norm in some representation of $\mathcal{A}$ into the space of operators in a Hilbert space $\mathcal{H}$) which is bi-invariant although certainly non-Riemannian. This norm is quite non-regular, in two senses: first, it is not differentiable but merely continuous. Also, it is not strictly convex; there are many open sets of affine subspaces contained in the unit sphere, which hampers many constructions in the calculus of variations. This lack of regularity is actually a blessing in disguise: it forces purely geometric constructions (i.e. metric geometry *à la* Gromov), instead of playing around with tensors and differential equations which sometimes obscure the underlying geometry.

There remains the question of how to push down the metric to $\mathcal{P}$. The natural way to do it is to consider each tangent space $(T\mathcal{P})_\rho$ as the Banach quotient $(T\mathcal{P})_\rho = (T\mathcal{U})_1/(T\mathcal{I}_\rho)_1 = \mathcal{A}^{\text{ant}}/\mathcal{B}^{\text{ant}}$, where $\mathcal{A}^{\text{ant}}$ and $\mathcal{B}^{\text{ant}}$ denote the antisymmetric parts of the algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. So we define the Finsler norm in $\mathcal{P}$ by $||X|| = \inf_{b\in\mathcal{B}^{\text{ant}}} |Z + b|$, where Z projects to X in the quotient. We denote by $|\cdot|$ the norm in the C^*-algebra $\mathcal{A}$.

In this paper we will mainly endow $(T\mathcal{P})_\rho$ with the quotient metric, but the results have easy partial generalizations to metrics that are topologically equivalent to the quotient metric (an important instance of which are 'connection metrics', defined in the next section).

1.2 Generalized flags

In the finite dimensional, multiplicity-free case, assuming that $\mathcal{A} = M_n(\mathbb{C})$, the inclusion of the unitary group of the C^*-subalgebra $\mathcal{B}$ can be presented as a diagonal inclusion in $U(n)$ of the form

$$\begin{pmatrix} U(k_1) & & & \\ & U(k_2) & & \\ & & \ddots & \\ & & & U(k_m) \end{pmatrix},$$

as it is well known. The homogeneous space is therefore a flag manifold.

For this reason the homogeneous spaces in this paper will be called *generalized flags*. For the rest this paper we shall automatically assume that the generalized flag in question is endowed with the quotient metric in each tangent space.

A partial list of examples in the infinite dimensional case is:

- The Grassmannian of a general C^*-algebra [3,6].
- Finite flags of a general C^*-algebra [4].
- Spaces of spectral measures [2].

Each of the previous items can be thought of as a special case of a space of spectral measures. These examples retain some of the flavour of the flag manifolds.

All these spaces have a canonical Banach manifold structure as quotients of the unitary group of the algebra [2].

1.3 The results

The main results in this article refer to *minimal curves* in $\mathcal{P}$, i.e. curves with minimal length joining fixed endpoints. We will call these curves *geodesics*. The length of a curve $\rho(t)$, $0 \leq t \leq 1$, is given by $\ell(\rho) = \int_0^1 ||\dot{\rho}(t)||_{\rho(t)}\, dt$, where $||X||_{\rho(t)}$ denotes the Finsler norm of the tangent vector X at the point $\rho(t) \in \mathcal{P}$.

> In this article, we study the problem: given an initial velocity vector $X \in (T\mathcal{P})_\rho$, find geodesics γ satisfying $\dot{\gamma}(0) = X$.

Let $X \in (T\mathcal{P})_\rho$ be a tangent vector to a generalized flag. We say that a vector $Z \in (T\mathcal{U})_1 = \mathcal{A}^{ant}$ is a *lift* of X if $\frac{d}{dt}|_{t=0} L_{e^{tZ}}\rho = X$, i.e. Z projects to X in the quotient.

Theorem I *Let $\mathcal{P}$ be a generalized flag. Consider $\rho \in \mathcal{P}$ and $X \in (T\mathcal{P})_\rho$. Suppose that there exists $Z \in \mathcal{A}^{ant}$ which is a 'minimal' lift of X i.e. $|Z| = ||X||_\rho$. Then the* uniparametric group curve *$\gamma(t)$ defined by $\gamma(t) = L_{e^{tZ}}\rho_0$ has minimal length in the class of all curves in $\mathcal{P}$ joining $\gamma(0)$ to $\gamma(t)$ for each t with $|t| \leq \frac{\pi}{2|Z|}$.*

In the case where $\mathcal{A}$ is a von Neumann algebra (W^*-algebra), the existence of such a minimal lift is guaranteed:

Theorem II *Let $\mathcal{A}$ be a W^*-algebra, and let $\mathcal{P}$ be a generalized flag of the unitary group of $\mathcal{A}$. Let $X \in (T\mathcal{P})_\rho$. Then there is a lift Z of X which satisfies $|Z| = ||X||_\rho$, and therefore the uniparametric group curve $\gamma(t) = L_{e^{tZ}}\rho$ has minimal length in $\mathcal{P}$ among curves joining $\gamma(0)$ and $\gamma(t)$, for each t with $|t| \leq \frac{\pi}{2|Z|}$.*

Thus, for von Neumann algebras, for every 'direction' $X \in (T\mathcal{P})_\rho$ ($||X||_\rho = 1$) there is a uniparametric group curve $\gamma(t) = L_{e^{tZ}}\rho$, $\dot\gamma(0) = X$, which is a minimal curve up to length $\pi/2$. Note that, due to the lack of strict convexity of the norm, there might be other geodesics with the same initial velocity vectors.

In **Part II** of this paper (see [10]) we study the problem of finding minimal curves connecting given endpoints. We prove a global Hopf-Rinow theorem:

> **Theorem** *Let $\mathcal{P}$ be a generalized flag. Given points $p, q \in \mathcal{P}$, there exists a minimal uniparametric group curve $\gamma(t) = L_{e^{tZ}}p$ joining p to q.*

for certain types of von Neumann algebras.

1.4 Intrinsic and extrinsic geometry

The main idea in the proofs of the aforementioned theorems is the following. First, given a tangent vector $X \in (T\mathcal{P})_\rho$, find 'length reducing maps' $F_X : \mathcal{P} \to \mathcal{S}$, where $\mathcal{S}$ is the sphere of radius $1/2$ of a certain Hilbert space $\mathcal{H}$ in which $\mathcal{A}$ is represented. The geodesics of $\mathcal{S}$ are the usual well-known great circles parametrized by arc-length, and we find that for certain lifts Z of X, the image of $\gamma(t) = L_{e^{tZ}}\rho$ under F_X is a great circle in the sphere isometric to γ. Since the maps are length reducing, it follows that γ minimizes length between its endpoints for lengths up to $\pi/2$.

The length reducing maps we shall use, factor through the Grassmannian $\mathrm{Gr}(\mathcal{H})$, of the C^*-algebra $\mathrm{End}(\mathcal{H})$ (the bounded operators of the Hilbert space $\mathcal{H}$) in which the given C^*-algebra $\mathcal{A}$ is represented: $\mathcal{P} \xrightarrow{\mathrm{F}} \mathrm{Gr}(\mathcal{H}) \xrightarrow{m_\xi} \mathcal{S}$. Each of these maps will reduce the length of curves, and the curves whose length is preserved are minimal curves.

Note that these constructions are *extrinsic*, in principle depending on a representation of $\mathcal{A}$ into the algebra of bounded operators of a Hilbert space, and also depending on the map $\mathrm{F} : \mathcal{P} \to \mathrm{Gr}(\mathcal{H})$ which is defined in terms of the representation. Sections 3 and 4 dwell upon this extrinsic metric results. Then, in sections 5 and 6 we use the GNS theory of representations of C^*-algebras to construct representations based on intrinsic data.

A simple particular case of this idea was used in [16].

1.5 Some precedents

The metric properties of homogeneous spaces of groups of invertible elements in a C^*-algebra, have been useful in operator theory:

- Segal's inequality turns out to be equivalent to a metric property (expansivity) of the exponential mapping of the canonical connection in the homogeneous space of positive invertible elements in a C^*-algebra(see [5]).
- Also in the homogeneous space of positive invertible elements, for geodesics γ and δ, the Finsler distance function between $\gamma(t)$ and $\delta(t)$

turns out to be a convex function of the parameter t. This fact leads to interesting operator inequalities (see [7]).

- Given a C^*-algebra $\mathcal{A}$, a C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ and a conditional expectation $\Phi : \mathcal{A} \to \mathcal{B}$, it is shown in [17] the following 'relative polar decomposition theorem': Every invertible element $g \in \mathcal{A}$ can be uniquely expressed in the form $g = b\,c\,u$ where u is unitary, b is a positive invertible element in $\mathcal{B}$ and c is the exponential of a symmetric element in the kernel of Φ.

We remark that these examples are of 'hyperbolic nature' i.e. the spaces involved have 'non-positive curvature'. In this paper the authors endeavour to begin the study of the elliptic case.

2 Preliminaries and notation

2.1 Generalized flags

In this work a *generalized flag* means the following data:

- A C^∞ Banach manifold $\mathcal{P}$.
- A unital C^*-algebra $\mathcal{A}$ whose unitary group $\mathcal{U}$ acts transitively and smoothly on $\mathcal{P}$ on the left. We indicate by $L_g\rho$, $g \in \mathcal{U}$, $\rho \in \mathcal{P}$, the action of $\mathcal{U}$ on $\mathcal{P}$.
- The isotropy group $\mathcal{I}_\rho$ at $\rho \in \mathcal{P}$ given by, $\mathcal{I}_\rho = \{g \in \mathcal{U} \mid L_g\rho = \rho\}$, is the unitary group $\mathcal{U}_\rho$ of a C^*-subalgebra $\mathcal{B}_\rho$ of $\mathcal{A}$.
- The derivative $(T\Pi_\rho)_1 : (T\mathcal{U})_1 = \mathcal{A}^{\mathrm{ant}} \to (T\mathcal{P})_\rho$ of the natural mapping $\Pi_\rho : \mathcal{U} \to \mathcal{P}$ given by $\Pi_\rho(g) = L_g(\rho)$ is surjective (hence open by the Banach Open Mapping Theorem).
- The Finsler structure in $\mathcal{P}$ is given by $||X||_\rho = \inf\{|Z + b| \ : \ b \in \mathcal{B}_\rho^{\mathrm{ant}}\}$, where $(T\Pi_\rho)_1(Z) = X$, i.e. for $\rho \in \mathcal{P}$, the norm $||X||_\rho$ is the Banach quotient norm of X in $(T\mathcal{U})_1/(T\mathcal{I}_\rho)_1 = \mathcal{A}^{\mathrm{ant}}/\mathcal{B}_\rho^{\mathrm{ant}}$. Observe that this Finsler structure is invariant under the action of $\mathcal{U}$.

Definition 2.1 *A curve $\gamma : I \to \mathcal{P}$ of the form $\gamma(t) = L_{e^{tZ}}\rho$ for $Z \in \mathcal{A}^{ant}$ and $t \in I = [a, b] \subset \mathbb{R}$ is called a* uniparametric group curve*.*

Definition 2.2 *We say that $Z \in \mathcal{A}^{ant}$ is a* lift *of $X \in (T\mathcal{P})_\rho$, if $(T\Pi_\rho)_1(Z) = X$.*

Observe that if Z is a lift of $X \in (T\mathcal{P})_\rho$, then the uniparametric group curve $\gamma(t) = L_{e^{tZ}}\rho$ satisfies $\gamma(0) = \rho$ and $\dot\gamma(0) = X$.

2.2 Metric structures on generalized flags

Let M be a Banach manifold. A *Finsler structure* on M is a continuous selection of norms $||.||_m$ on each tangent space TM_m.

Remark: The usual definition of a Finsler structure includes differentiability and strict convexity of the norm; this notion is too restrictive in the cases we are dealing with. See section 9 of [11] for a discussion of C^0 calculus of variations. Let us remark that the lack of differentiability prompts us to apply direct *metric* (as opposed to topological) methods. We will call *regular* Finsler structures those which are differentiable and strictly convex.

Recall that for any Finsler structure, the length of a curve $w(t)$ defined for $a \leq t \leq b$ is given by,

$$\ell(w) = \int_a^b ||\dot{w}(t)||_{w(t)}\, dt.$$

The distance d in M is given as follows: we let R_{ρ_0,ρ_1} be the set of piecewise smooth paths w ($w : [0,1] \to M$) which join $w(0) = \rho_0$ to $w(1) = \rho_1$. We set,

$$d(\rho_0, \rho_1) = \inf\{\ell(w) \,|\, w \in R_{\rho_0,\rho_1}\}.$$

Definition 2.3 *We say that a curve w is* minimal *in M if its length is the distance between its endpoints.*

Given a Finsler metric on a space E and a submersion $\pi : E \to B$ (e.g. the projection to a quotient), the most natural metric on B is this quotient metric. These metrics have been considered in the finite-dimensional regular case in [1]. There, the geodesics on the base are projections of certain 'horizontal' geodesics in E.

Another example of an invariant Finsler structure arises by using an invariant connection (reductive structure) on the bundle $\mathcal{U} \to \mathcal{P}$ to push down the operator norm on $\mathcal{U}$: Consider $\mathcal{P}$ to be a homogeneous reductive space (see [14]) over $\mathcal{U}$. We call K the 1-form (reductive structure) in $\mathcal{P}$. For each $\rho \in \mathcal{P}$ and $X \in (T\mathcal{P})_\rho$ we have $K_\rho(X) \in \mathcal{A}^{\mathrm{ant}}$. The reductive Finsler structure on $\mathcal{P}$ is defined at each ρ by the reductive norm, $|| \cdot ||_K$, given as follows: for $X \in (T\mathcal{P})_\rho$,

$$||X||_K = |K_\rho(X)|.$$

Reductive structures arise in the C^*-algebra context in the presence of a *conditional expectation* in the algebra, and thus their study can be considered as the geometry associated to conditional expectations (see for example [17]).

It is clear that, for any reductive structure K, this norm satisfies $||X||_K \geq ||X||$. We will present some results concerning norms other than the quotient norm in section 8.

2.3 Grassmann representations and isotropic reflections

We are interested in equivariant maps of $\mathcal{P}$ into Grassmann manifolds. This will be accomplished with the use of adapted representations, similar in spirit to the "class 1 representations" [12], which give equivariant embeddings of homogeneous spaces into spheres.

Definition 2.4 *Let $\rho_0 \in \mathcal{P}$ be fixed and let r_0 be a unitary reflection of a Hilbert space $\mathcal{H}$, i.e. ${r_0}^* = r_0 : \mathcal{H} \to \mathcal{H}$ and ${r_0}^2 = 1$. We say that a representation of $\mathcal{A}$ into End($\mathcal{H}$) is a* Grassmann representation *at ρ_0 with respect to r_0, if for each element $g \in \mathcal{U}_{\rho_0}$ its image (or representative) $\tilde{g} \in \mathrm{End}(\mathcal{H})$ is in the commutant of r_0 in $\mathrm{End}(\mathcal{H})$, i.e. $\tilde{g}\, r_0 = r_0\, \tilde{g}$. We shall call this reflection r_0 an* isotropic reflection *adapted to ρ_0.*

Remark: In this work a representation of a C^*-algebra in a Hilbert space may not be faithful.

Let $\mathrm{Gr}(\mathcal{H})$ denote the Grassmann manifold of $\mathcal{H}$ which is just the set of unitary reflections of $\mathcal{H}$ (as in [6]). The Grassmannian $\mathrm{Gr}(\mathcal{H})$ is a homogeneous space under the unitary group of the C^*-algebra $\mathrm{End}(\mathcal{H})$. By means of the representation assumed above, we can think $\mathrm{Gr}(\mathcal{H})$ as a homogeneous space under the unitary group $\mathcal{U}$ of $\mathcal{A}$. For any Grassmann representation at ρ_0 with respect to r_0, we consider the mapping $\mathrm{F} : \mathcal{P} \to \mathrm{Gr}(\mathcal{H})$, given as follows: take any $g \in \mathcal{U}$ which satisfies $\rho = L_g \rho_0$, then

$$\mathrm{F}(\rho) = \tilde{g}\, r_0\, \tilde{g}^{-1}.$$

F is well defined because of the hypothesis about the commutant of r_0. In fact, suppose that $g_i \in \mathcal{U}_{\rho_0}$ for $i = 1, 2$ satisfy $\rho = L_{g_i} \rho_0$. We must check that

$$\tilde{g_1}\, r_0\, \tilde{g_1}^{-1} = \tilde{g_2}\, r_0\, \tilde{g_2}^{-1}$$

and this is equivalent to the identity,

$$\tilde{g_2}^{-1}\, \tilde{g_1}\, r_0 = r_0\, \tilde{g_2}^{-1} \tilde{g_1}.$$

But this is immediate from the assumption about the commutant of r_0, for $g_2^{-1} g_1 \in \mathcal{U}_{\rho_0}$, and $\tilde{g_2}^{-1}\, \tilde{g_1} = \widetilde{g_2^{-1}} g_1$.

Lemma 2.1 *Each mapping F is compatible with the action of $\mathcal{U}$ on the corresponding homogeneous spaces, i.e. for all $g \in \mathcal{U}$ and $\rho \in \mathcal{P}$, $F(L_g \rho) = \tilde{g}\, F(\rho)\, \tilde{g}^{-1}$.*

Proof: In fact, if $\mathrm{F}(L_g \rho) = \tilde{g_1}\, r_0\, \tilde{g_1}^{-1}$ where $g_1 \in \mathcal{U}$ satisfies $L_g \rho = L_{g_1} \rho_0$, then $\rho = L_{g^{-1} g_1} \rho_0$ and $\mathrm{F}(\rho) = \widetilde{g^{-1} g_1}\, r_0\, (\widetilde{g^{-1} g_1})^{-1}$. Then $\tilde{g}\, \mathrm{F}(\rho)\, \tilde{g}^{-1} = \tilde{g_1}\, r_0\, \tilde{g_1}^{-1} = \mathrm{F}(L_g \rho)$, as claimed.

□

2.4 Some examples

Let us give a couple of examples which illustrate the spirit of the construction.

Full flags

Let $\mathcal{F}$ be the homogenous space of full flags in $\mathbb{C}^n$, that is, $\mathcal{F}$ is the set of all $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ where the $p_i \in End(\mathbb{C}^n)$ are mutually orthogonal, rank 1 projections (so that $p_1 + p_2 + \cdots + p_n = 1$). The unitary group $U(n) \subset End(\mathbb{C}^n)$ acts naturally and transitively on $\mathcal{F}$.

Equivariant mappings σ from $\mathcal{F}$ into Grassmann manifolds can be obtained as follows: given a subset $\mathcal{S}$ of $\{1, \ldots, n\}$, we define the function $\sigma_{\mathcal{S}}$ from $\mathcal{F}$ into a Grassmann manifold by

$$\sigma_{\mathcal{S}}(p_1, \ldots, p_n) = \frac{1}{2}\left(1 + \sum_{j \in \mathcal{S}} p_j\right).$$

In this example the representation of $End(\mathbb{C}^n)$ is the identity. Given $\mathbf{p}_0 \in \mathcal{F}$ its image $\rho_0 = \sigma_{\mathcal{S}}(\mathbf{p}_0)$ is an isotropic reflection adapted to $\mathbf{p}_0$.

Spectral measures

The previous example may be generalized as follows. Consider the set $\mathcal{M}$ of spectral measures μ defined on an algebra of sets in a measure space Ω, with values in the set of projections of a C^*-algebra $\mathcal{A}$. The natural action of the unitary group $\mathcal{U}$ of $\mathcal{A}$ is given by $(L_g\mu)(M) = g\mu(M)g^{-1}$ for M measurable in Ω (see [2]). Orbits of this action are homogenous spaces of the type we are considering. Fixing an element M of the measurable algebra of Ω, we have a map $\mathcal{Q}_M : \mathcal{M} \to Gr(\mathcal{A})$, given by $\mathcal{Q}_M(\mu) = \mu(M)$. Note that in the previous example $\Omega = \{1, \ldots, n\}$.

3 Length reducing maps

We present sufficient conditions to construct length reducing maps from generalized flags to Grassmann manifolds of the form $Gr(\mathcal{H}) = Gr(End(\mathcal{H}))$ where $\mathcal{H}$ is some Hilbert space.

Recall that the Finsler structure of the homogeneous space $Gr(\mathcal{H})$ is obtained as follows. For $X \in (T\,Gr(\mathcal{H}))_r$ we have,

$$||X||_r = \frac{1}{2}|X|,$$

where X is identified with an element of $End(\mathcal{H})$ (see [6]).

Proposition 3.1 *Let $\mathcal{P}$ be a generalized flag. For any Grassmann representation of $\mathcal{A}$ at ρ_0 with respect to r_0, the corresponding mapping F reduces length, i.e. $||\,(TF)_\rho\,(X)||_{F(\rho)} \leq ||X||$, for all $\rho \in \mathcal{P}$, $X \in (T\mathcal{P})_\rho$.*

Proof: By the equivariance of F with respect to the (isometric) actions of the unitary group (see lemma 2.1), and the invariance of the Finsler structure, it is enough to verify the result in $\rho = \rho_0$. Let $X \in (T\mathcal{P})_{\rho_0}$. Consider any curve

$\gamma(t)$ by ρ_0 which verifies $\dot\gamma(0)=X$; for example we consider $\gamma(t)=L_{e^{tZ}}\rho_0$, where Z is some lift of X. This curve $\gamma(t)$ is transformed into

$$\mathrm{F}(\gamma(t))=\widetilde{e^{tZ}}\mathrm{F}(\rho_0)\widetilde{e^{-tZ}}=e^{t\widetilde{Z}}\,r_0\,e^{-t\widetilde{Z}},$$

hence,

$$(T\mathrm{F})_{\rho_0}(X)=\left.\frac{d}{dt}\right|_{t=0}\mathrm{F}(\gamma(t))=\left[\widetilde{Z},r_0\right].$$

Consider the orthogonal decomposition induced by r_0, $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$, with $\mathcal{H}_+=\{x\mid r_0(x)=x\}$ and $\mathcal{H}_-=\{x\mid r_0(x)=-x\}$. With respect to this decomposition, we write

$$r_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \widetilde{Z}=\begin{pmatrix}a_{11}&a_{12}\\-a_{12}^*&a_{22}\end{pmatrix},$$

where a_{11} y a_{22} are antisymmetric maps $a_{11}:\mathcal{H}_+\to\mathcal{H}_+$ and $a_{22}:\mathcal{H}_-\to\mathcal{H}_-$ respectively. Then we have,

$$(T\mathrm{F})_{\rho_0}(X)=\left[\widetilde{Z},r_0\right]=-2\begin{pmatrix}0&a_{12}\\a_{12}^*&0\end{pmatrix}.$$

Since $||X||_{\rho_0}=\inf\{|Z|\ :\ Z \text{ is a lift of } X\}$, for any $\varepsilon>0$, we can choose Z so that $|Z|<||X||_{\rho_0}+\varepsilon$. Now we have, $||X||_{\rho_0}+\varepsilon>|Z|\geq\left|\left|\widetilde{Z}\right|\right|$ and $||(T\mathrm{F})_{\rho_0}(X)||_{r_0}=\frac{1}{2}\left|\left|\left[\widetilde{Z},r_0\right]\right|\right|$. To complete the proof, we just have to use the lemma 3.2 below, which asserts that:

$$\left|\left|\begin{pmatrix}a_{11}&a_{12}\\-a_{12}^*&a_{22}\end{pmatrix}\right|\right|\geq\left|\left|\begin{pmatrix}0&a_{12}\\a_{12}^*&0\end{pmatrix}\right|\right|,$$

for then we get $||X||_{\rho_0}+\varepsilon>||(T\mathrm{F})_{\rho_0}(X)||_{r_0}$ which implies $||X||_{\rho_0}\geq||(T\mathrm{F})_{\rho_0}(X)||_{r_0}$ because $\varepsilon>0$ can be chosen arbitrarily small.

□

Lemma 3.2 *Let $H=S_0+S_1$ be an orthogonal decomposition of a Hilbert space H. Consider bounded operators X and Y on H which, with respect to this orthogonal decomposition, are represented by matrices of operators as:*

$$X=\begin{pmatrix}a&b\\-b^*&c\end{pmatrix},\quad \textit{and}\quad Y=\begin{pmatrix}0&b\\b^*&0\end{pmatrix}.$$

Then $||X||\geq||b||=||Y||$, where $||\cdot||$ indicates the usual norm for operators.

Proof: Recall $||Y||^2=||YY^*||$, but

$$||YY^*||=\left|\left|\begin{pmatrix}bb^*&0\\0&b^*b\end{pmatrix}\right|\right|=||bb^*||=||b||^2.$$

Hence, $||Y||=||b||$. Now we show that $||X||\geq||b||$. For any $\xi\in H_0$ with $||\xi||=1$,

$$||b^*\xi||\leq\left|\left|\begin{pmatrix}a\xi\\-b^*\xi\end{pmatrix}\right|\right|=\left|\left|\begin{pmatrix}a&b\\-b^*&c\end{pmatrix}\begin{pmatrix}\xi\\0\end{pmatrix}\right|\right|$$

This proves the lemma.

□

Next we present length reducing maps $m_\xi : \mathrm{Gr}(\mathcal{H}) \to \mathcal{S}$ from $\mathrm{Gr}(\mathcal{H})$ into the unit sphere $\mathcal{S} = \{\eta \in \mathcal{H} \ \mid \ ||\eta|| = 1\}$. We consider $\mathcal{S}$ as a Riemannian manifold with the metric given by $||W||_\eta = \frac{1}{2}||W||$, for each $\eta \in \mathcal{S}$ and $W \in T\mathcal{S}_\eta$.

For each $\xi \in \mathcal{S}$, consider the evaluation mapping $m_\xi : \mathrm{Gr}(\mathcal{H}) \to \mathcal{S}$ given by:

$$m_\xi(r) = r(\xi).$$

Proposition 3.3 *The mapping $m_\xi : Gr(\mathcal{H}) \to \mathcal{S}$ reduces length.*

Proof: Consider a curve $r(t)$ in $\mathrm{Gr}(\mathcal{H})$ with $\dot{r}(0) = Y \in T(\mathrm{Gr}(\mathcal{H}))_{r(0)}$. The image curve is $m_\xi(r(t)) = r(t)\xi$, then

$$(Tm_\xi)_{r(0)}(Y) = Y\xi.$$

Then

$$\begin{aligned} ||\,(Tm_\xi)_{r(0)}(Y)||_{r(0)\xi} &= ||Y\xi||_{r(0)\xi} \\ &= \frac{1}{2}||Y\xi|| \end{aligned}$$

Now it follows that,

$$||\,(Tm_\xi)_{r(0)}(Y)||_{r(0)\xi} \leq \frac{1}{2}||Y|| = ||Y||_{r(0)}$$

□

For $\xi \in \mathcal{S}$, we define the map F_ξ, $\mathrm{F}_\xi : \mathcal{P} \to \mathcal{S}$ by:

$$\mathrm{F}_\xi(\rho) = \mathrm{F}(\rho)\xi.$$

Observing that F_ξ is the composition: $\mathcal{P} \xrightarrow{\mathrm{F}} \mathrm{Gr}(\mathcal{H}) \xrightarrow{m_\xi} \mathcal{S}$ we get,

Corollary 3.4 *Let $\mathcal{P}$ be a generalized flag. For any Grassmann representation of $\mathcal{A}$ at ρ_0 with respect to r_0, and any $\xi \in \mathcal{H}$, the mapping F_ξ reduces length.*

4 Geometric conditions for minimality

In this section we give 'extrinsic' conditions which guarantee that a short enough arc of a uniparametric group curve $\gamma(t) = L_{e^{tz}}\rho_0$ minimizes length among all curves with the same end-points. The word 'extrinsic' here means that these conditions refer to Grassmann representations.

These conditions are presented in two ways in theorems 4.1 and 4.3.

4.1 Minimality conditions for a given Grassmann representation

Let $\mathcal{P}$ be a generalized flag together with a Grassmann representation at ρ_0 with respect to r_0. Let $X \in (T\mathcal{P})_{\rho_0}$ and a lift Z of X be given.

Definition 4.1 *The pair (X, Z) is in* good position *for the given representation, if there is a unit vector $\xi \in \mathcal{H}$ such that the following conditions are satisfied:*

1. $|Z| = ||X||_{\rho_0}$.
2. *ξ norms Z^2, i.e.* $\widetilde{Z^2}\xi = -\lambda^2\xi, \quad \lambda = |Z|$.
3. $r_0(\xi) = \xi$, $r_0(\widetilde{Z}\xi) = -\widetilde{Z}\xi$

Note that, because of condition (2) above we have $\lambda = |Z| = ||\widetilde{Z}||$.

Theorem 4.1 *Given a Grassmann representation of $\mathcal{P}$ at ρ_0 with respect to r_0, and a pair (X, Z) $(X \neq 0)$ which is in* good position *for this representation, let $\gamma(t) = L_{e^{tZ}}\rho_0$. Then, $\gamma(t)$ minimizes length between the points $\rho_0 = \gamma(0)$ and $\gamma(t)$ if*

$$0 \leq t \leq \frac{\pi}{2|Z|}.$$

In this case,

$$d(\gamma(0)\,,\,\gamma(t)) = \ell_0^t\gamma = t\,|Z|$$

The proof is based on the following lemma:

Lemma 4.2 *With the hypotheses of theorem 4.1, $F_\xi : \mathcal{P} \to \mathcal{S}$ maps the uniparametric group curve $\gamma(t) = L_{e^{tZ}}\rho_0$ into a geodesic $w(t) = F_\xi(\gamma(t))$ in the sphere $\mathcal{S}$. Furthermore, the length of $w(t)$ from 0 to $t > 0$ coincides with the length of $\gamma(t)$, i.e. $\ell_0^t = t|Z|$.*

Proof: (of lemma 4.2) Consider $\mu > 0$ and $\eta \in \mathcal{H}$ with $||\eta|| = 1$ such that $\widetilde{Z}\xi = \mu\eta$. We have,

$$-\lambda^2 = \langle \widetilde{Z^2}\xi \mid \xi \rangle = -\langle \widetilde{Z}\xi \mid \widetilde{Z}\xi \rangle = -\mu^2.$$

Hence $\lambda = \mu$ and $\widetilde{Z}$ leaves invariant the two-dimensional subspace $\mathcal{E}$ generated by ξ and η, as well as its orthogonal complement $\mathcal{E}^\perp$. The set $B = \{\xi, \eta\}$ forms an orthonormal base for $\mathcal{E}$. The matrix M of $\widetilde{Z}$ with respect to the base B is:

$$M = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$$

The reflection r_0 leaves stable both $\mathcal{E}$ and $\mathcal{E}^\perp$, and its restriction to $\mathcal{E}$ has matrix R_0 with respect to the base B:

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notice that the curve $w(t)$ sits inside the subspace $\mathcal{E}$ and

$$e^{tM} = \cos(\lambda t)I_0 + \sin(\lambda t)J_0,$$

where

$$I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and therefore $w(t) = \cos(2\lambda t)\xi + \sin(2\lambda t)\eta$. So $w(t)$ is a geodesic in the sphere $\mathcal{S}$.

Finally, we compute the length of $w(s)$ from $s = 0$ to $s = t$,

$$\begin{aligned} \ell_0^t &= \int_0^t ||w'(s)||_{w(s)}\, ds = \int_0^t \frac{1}{2}||w'(s)||\, ds \\ &= \int_0^t \frac{1}{2}|| - 2\lambda \sin(2\lambda s)\xi + 2\lambda\cos(2\lambda s)\eta||\, ds \\ &= \int_0^t \frac{1}{2}(2\lambda)\, ds = t\lambda \end{aligned}$$

□

Now we prove **theorem 4.1**. Consider a curve $\delta(s)$ in $\mathcal{P}$ $(0 \leq s \leq t)$ which connects $\rho_0 = \gamma(0)$ and $\gamma(t)$, for $0 \leq t \leq \frac{\pi}{2|Z|}$. Consider $v(s) = \mathrm{F}_\xi(\delta(s))$ and $w(s) = \mathrm{F}_\xi(\gamma(s))$. These curves join $\xi = w(0)$ to $w(t)$. We have the following inequalities:

1. $\ell_0^t w \leq \ell_0^t v$ if $0 \leq t \leq \frac{\pi}{2|Z|}$, because w is a geodesic in the Riemannian manifold $\mathcal{S}$.

2. $\ell_0^t \delta \geq \ell_0^t v$, because F_ξ reduces length according to corollary 3.4.

Now we observe that, $\ell_0^t \gamma = \ell_0^t w$ by lemma 4.2. Then, combining these observations we get,

$$\ell_0^t\gamma = \ell_0^t w \leq \ell_0^t v \leq \ell_0^t \delta.$$

□

4.2 An alternative sufficient geometric condition for minimality

Here we present a result, similar to theorem 4.1 in the previous section, about uniparametric group curves which are minimal.

Definition 4.2 *Let $X \in (T\mathcal{P})_{\rho_0}$. We say that a representation of the C^*-algebra $\mathcal{A}$ is* adapted *to X if the following conditions are satisfied:*

1. *There exists a lift Z of X such that $|Z| = ||X||$.*

2. *There is a unit vector $\xi \in \mathcal{H}$ which is a norming eigenvector for $\widetilde{Z^2}$, i.e.*

$$\widetilde{Z^2}\xi = -\lambda^2\xi, \quad \text{with } \lambda = |Z|$$

3. *For each* $\tilde{b} \in \widetilde{\mathcal{U}_{\rho_0}}$ *the vector* $\tilde{b}\xi \in \mathcal{H}$ *is orthogonal to* $\widetilde{Z}\xi$.

We will say that such a Z *is* adapted *to* X.

We note that in the previous definition the representation may not be faithful.

Theorem 4.3 *Given a representation of* $\mathcal{A}$ *adapted to* $X \in (T\mathcal{P})_{\rho_0}$, *the uniparametric group curve* $\gamma(t) = L_{e^{tZ}}\rho_0$ *minimizes length up to* $t = \frac{\pi}{2||X||}$ *for any* Z *adapted to* X.

Proof: Consider a representation of $\mathcal{A}$ adapted to $X \in (T\mathcal{P})_{\rho_0}$ (cf. definition 4.2). Define the reflection r_0 in $\mathcal{H}$ as follows,

$$r_0(\zeta) = \zeta \quad \text{if} \quad \zeta \in S_{\rho_0}$$
$$r_0(\zeta) = -\zeta \quad \text{if} \quad \zeta \in S_{\rho_0}^{\perp},$$

where S_{ρ_0} is the closure of the vector space generated by the set $\Omega = \{\zeta \in \mathcal{H} \mid \zeta = \tilde{b}\xi; \tilde{b} \in \widetilde{\mathcal{U}_{\rho_0}}\}$. Observe that the commutant of r_0 contains $\widetilde{\mathcal{U}_{\rho_0}}$. In fact it is clear that the set Ω is invariant under $\widetilde{\mathcal{U}_{\rho_0}}$ and therefore so are S_{ρ_0} and $S_{\rho_0}^{\perp}$.

Next we observe that the pair (X, Z) is in good position with respect to r_0 in the representation (cf. definition 4.1). In fact it enough to see that $\widetilde{Z}\xi \in S_{\rho_0}^{\perp}$. But for $\tilde{b} \in \widetilde{\mathcal{U}_{\rho_0}}$ we have $\langle \tilde{b}\xi \mid \widetilde{Z}\xi \rangle = 0$, which shows that $\widetilde{Z}\xi$ is orthogonal to Ω and therefore to S_{ρ_0}. Now the proof follows from theorem 4.1.

□

5 The Minimality Theorem

In the previous sections, we assumed a Grassmann representation of $\mathcal{A}$, and the minimality theorems there depend on the equivariant map $\mathrm{F} : \mathcal{P} \to \mathrm{Gr}(\mathcal{H})$ given by the representation. Now we come to one of the main results in this paper, namely theorem I, which gives an *intrinsic* condition for minimality. The proof, however, is based on the construction of a representation and an isotropic reflection in order to apply the theorems of section 4.

Theorem I *Let* $\mathcal{P}$ *be a generalized flag. Consider* $\rho \in \mathcal{P}$ *and* $X \in (T\mathcal{P})_{\rho}$. *Suppose that there exists* $Z \in \mathcal{A}^{ant}$ *which is a 'minimal' lift of* X *i.e.* $|Z| = ||X||_{\rho}$. *Then the* uniparametric group curve $\gamma(t)$ *defined by* $\gamma(t) = L_{e^{tZ}}\rho_0$ *has minimal length in the class of all curves in* $\mathcal{P}$ *joining* $\gamma(0)$ *to* $\gamma(t)$ *for each* t *with* $|t| \leq \frac{\pi}{2|Z|}$.

That is, uniparametric group curves $\gamma(t) = L_{e^{tZ}}\rho_0$ are minimal if Z is a lift of $\gamma'(0) = X$ which realizes the (quotient) norm of X.

Remark: In the case of $W^* - algebras$ there always exists $Z \in \mathcal{A}$ which is a 'minimal' lift of X (see theorem II).

The proof of theorem I will be presented at the end of this section after some preliminary results.

To find a representation of $\mathcal{A}$ adapted to a tangent vector $X \in (T\mathcal{P})_{\rho_0}$, we need *states* of the algebra $\mathcal{A}$ *'adapted'* to X.

Consider the state $\varphi : \mathcal{A} \to \mathbb{C}$ of the algebra $\mathcal{A}$ given by the vector ξ in definition 4.2,

$$\varphi(a) = \langle \widetilde{a}\xi \mid \xi \rangle \ , \forall\, a \in \mathcal{A}.$$

With this state φ, we define the inner-product in $\mathcal{A}$,

$$\langle a' \mid a \rangle = \varphi(a^* a') = \langle \widetilde{a'}\xi \mid \widetilde{a}\xi \rangle.$$

Remark 1 *The second condition (2) in definition 4.2, namely: There is a unit vector $\xi \in \mathcal{H}$ which is a norming eigenvector for $\widetilde{Z^2}$, i.e.*

$$\widetilde{Z^2}\xi = -\lambda^2 \xi, \ \ \textit{with } \lambda = |Z|,$$

is equivalent to the condition:

$$Z^2 + \lambda^2 \in \ker \varphi, \ \ \lambda = |Z|.$$

in fact, the symmetric element $Z^2 + \lambda^2 \in \mathcal{A}$ is positive because $|Z|^2 = \lambda^2$. Then $(\widetilde{Z}^2 + \lambda^2)\xi = 0$ is equivalent to

$$\langle (\widetilde{Z}^2 + \lambda^2)\xi \mid \xi \rangle = 0.$$

Remark 2 *The third condition (3) in definition 4.2, namely: For each $\widetilde{b} \in \widetilde{\mathcal{U}_{\rho_0}}$ the vector $\widetilde{b}\xi \in \mathcal{H}$ is orthogonal to $\widetilde{Z}\xi$, i.e.*

$$\langle \widetilde{b}\xi \mid \widetilde{Z}\xi \rangle = 0, \ \ \forall\, \widetilde{b} \in \widetilde{\mathcal{U}_{\rho_0}},$$

is equivalent to the condition: For each $b \in \mathcal{U}_{\rho_0}$, $Zb \in \ker \varphi$.

Indeed to observe that,

$$\varphi(Zb) = \langle \widetilde{Zb}\xi \mid \xi \rangle = -\langle \widetilde{b}\xi \mid \widetilde{Z}\xi \rangle.$$

Remarks 1 and 2 lead us to the following,

Definition 5.1 *Let $X \in (T\mathcal{P})_{\rho_0}$. We say that a state φ is* adapted *to the tangent vector X if it admits a lift Z such that:*

1. *$|Z| = ||X||_{\rho_0}$.*
2. *For each $b \in \mathcal{U}_{\rho_0}$, $Zb \in \ker \varphi$.*
3. *$Z^2 + \lambda^2 \in \ker \varphi$, $\lambda = |Z|$.*

Clearly to have a representation of $\mathcal{A}$ adapted to a vector X is equivalent to the existence of a state φ adapted to the vector X.

Next we prove a useful lemma for which we need some notation. We fix $Z \in \mathcal{A}^{\text{ant}}$. Let $M = \{bZ + Zb' \in \mathcal{A} \mid b, b' \in \mathcal{B}_{\rho_0}\}$. Notice that the symmetric part of M is

$$M^{\text{sim}} \ = \ \{bZ - Zb^* \mid b \in \mathcal{B}_{\rho_0}\} = \{bZ + Zb \mid b \in \mathcal{B}_{\rho_0}\}.$$

Observe that condition (2) in the definition above is equivalent to require that $M^{\text{sim}} \subset \ker\varphi$, because, as it is well known, $\mathcal{U}_{\rho_0}$ linearly generates $\mathcal{B}_{\rho_0}$, and because $\ker\varphi$ is $*$-closed.

We denote by S the real subspace of $\mathcal{A}^{\text{sim}}$ generated by the subset M^{sim} and the element $Z^2 + \lambda^2$ in $\mathcal{A}$ where $\lambda = |Z|$.

Lemma 5.1 *The following are equivalent,*

1. *There exists a real linear form on $\mathcal{A}^{sim}$ which is a state.*
2. *$S \cap C = \emptyset$, where C is the open cone of positive invertible elements of $\mathcal{A}$.*

Proof: (2)→(1) By the Hann-Banach theorem, if the real vector space S does not intersect C, then there is a linear functional which is positive on C and vanishes on S. This linear functional, when normalized, is the desired state.
(1)→(2) On the other hand any state is strictly positive on invertible positive elements of $\mathcal{A}^{\text{sim}}$, hence the cone C does not intersect S.

□

The equivalence of statements (3) and (4) in proposition 5.2 is the basis of the proof of the minimality theorem I.

Proposition 5.2 *Let $\mathcal{P}$ be a generalized flag over the unitary group of the C^*-algebra $\mathcal{A}$. Let X be a tangent vector of $\mathcal{P}$and let $Z \in \mathcal{A}^{ant}$ be a lift of X. Then with the notation above, the following statements are equivalent:*

1. *There is a representation of $\mathcal{A}$, adapted to X.*
2. *There is a state φ adapted to the vector X.*
3. *S contains no invertible positive elements of $\mathcal{A}$.*
4. *$|Z^2| \leq |Z^2 + m|$, $\forall\, m \in M^{sim}$.*

Proof:(1) ⇔ (2): follows from remarks 1 and 2.
(2) ⇔ (3): follows from lemma 5.1.
(3) ⇒ (4): Suppose that condition 3 is satisfied and that $|Z^2| > |Z^2 + m_0|$ for some $m_0 \in M^{\text{sim}}$. Then, thinking of the elements of $\mathcal{A}$ as operators, the operator λ^2 is larger that $-Z^2 + m_0$, i.e. $\lambda^2 + Z^2 - m_0 > 0$ which is a positive invertible element in S and this contradicts the hypothesis.
(4) ⇒ (3): Suppose also that there is a positive invertible element of the form $s(\lambda^2 + Z^2) + m$ with $s \in \mathbb{R}$ and $m \in M^{\text{sim}}$, then we would have

$$s(\lambda^2 + Z^2) + m \geq \sigma > 0, \text{ for some } \sigma \in \mathbb{R}.$$

Clearly we can consider $s > 0$, and then,

$$\lambda^2 + Z^2 + m \geq \tau > 0, \text{ for some } \tau \in \mathbb{R}.$$

Then $\lambda^2 > -Z^2 - m$ as operators, and then $|Z^2| > |Z^2 + m|$, which contradicts condition 4.

□

The last tool in the proof of theorem I is the following 'convexity' result.

Lemma 5.3 *In the setting above, suppose that* $|(Z+b)^2| \geq |Z^2|$ *for all* $b \in \mathcal{B}_{\rho_0}$. *Then* $|Z^2| \leq |Z^2 + bZ + Zb|$, $\forall\, b \in \mathcal{B}_{\rho_0}$.

This lemma has a simple geometrical interpretation which is illustrated in Figure 1 based on the fact that the expression $bZ + Zb$ is the derivative at $t = 0$ of the expression $(Z + t\,b)^2$. **Proof:** Consider for $t > 0$ the $\mathcal{A}$ valued

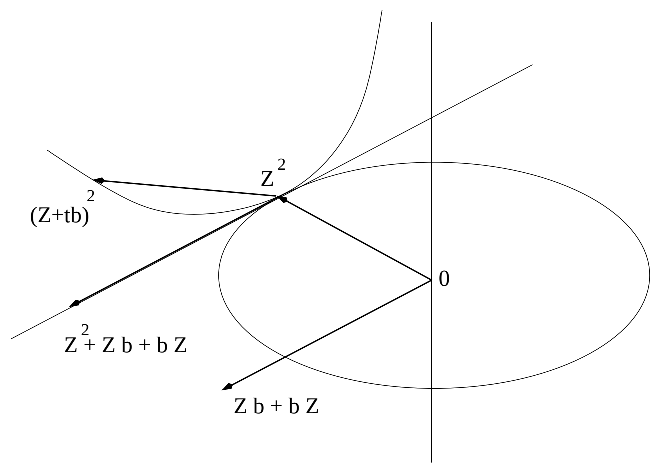

Figure 1: The point $Z^2 + bZ + Zb$ lies outside the ball of radius $|Z^2|$.

function $h(t) = Z^2 + ((Z + tb)^2 - Z^2)/t$. First we show that $|h(t)| \geq |Z^2|$. Suppose on the contrary that $|h(t)| < |Z^2|$, then the convex combination $t\,h(t) + (1-t)Z^2$ has norm $|t\,h(t) + (1-t)Z^2| < |Z^2|$ for all $0 < t < 1$. Notice that

$$\begin{aligned} t\,h(t) + (1-t)Z^2 &= \\ &= t\,Z^2 + (Z + tb)^2 - Z^2 + (1-t)Z^2 \\ &= (Z + tb)^2. \end{aligned}$$

Then we get $|(Z + tb)^2| < |Z^2|$ which contradicts the hypothesis. Observe that $\lim_{t \to 0} h(t) = Z^2 + bZ + Zb$. Consider the inequality $|h(t)| \geq |Z^2|$ and by taking the limit when $t \to 0$ we get $|Z^2 + bZ + Zb| \geq |Z^2|$ as desired.

□

Finally we present the proof of theorem I.

Proof: (of theorem I). By theorem 4.3, it is enough to show that there is some representation of $\mathcal{A}$ adapted to X. By hypothesis we have $|Z+b| \geq |Z|$, for all $b \in \mathcal{B}_{\rho_0}$. Both Z and b are antisymmetric in $\mathcal{A}$, and the condition $|Z + b| \geq |Z|$ for all $b \in \mathcal{B}_{\rho_0}$ is equivalent to $|(Z + b)^2| \geq |Z^2|$ for all $b \in \mathcal{B}_{\rho_0}$. From lemma 5.3 we get $|Z^2| \leq |Z^2 + bZ + Zb|$, $\forall\, b \in \mathcal{B}_{\rho_0}$. But then from proposition 5.2 we get that there is some representation of $\mathcal{A}$ adapted to X as desired.

□

Remark: In theorem I, the analytical hypothesis $|(Z+b)| \geq |Z|$, $\forall\, b \in \mathcal{B}_{\rho_0}$ implies the geometrical condition that the pair (X, Z) is in good position in some representation of the C^*-algebra $\mathcal{A}$.

6 Existence of minimal curves with given initial velocity

In this section we consider the question of the existence of minimal curves with a given initial velocity vector X. From theorem I, we need to know that there is lift Z which realizes norm of X, i.e. $|Z| = ||X||_{\rho_0}$, or equivalently $|Z+b| \geq |Z|$ for all $b \in \mathcal{B}_{\rho_0}$. The theorem below says that if $\mathcal{A}$ is a W^*-algebra such a lift exists.

Theorem II *Let $\mathcal{A}$ be a W^*-algebra, and let $\mathcal{P}$ be a generalized flag over $\mathcal{U}$, the unitary group of $\mathcal{A}$. Let $\rho_0 \in \mathcal{P}$ and consider any tangent vector $X \in (T\mathcal{P})_{\rho_0}$. Then:*

- *There is a lift Z of X which satisfies $|Z| = ||X||_{\rho_0}$.*
- *The uniparametric group curve $\gamma(t) = L_{e^{tZ}}\rho_0$ has minimal length in the class of all curves in $\mathcal{P}$ joining $\gamma(0)$ to $\gamma(t)$ for each t with $|t| \leq \frac{\pi}{2|Z|}$.*

The proof of the theorem II is presented at the end of this section. We shall need a theorem about minimality in the norm. Suppose that $\mathcal{A}$ is a W^*-algebra and that $\mathcal{B}$ is a weakly closed W^*-subalgebra of $\mathcal{A}$. Observe that the symmetric and antisymmetric parts of $\mathcal{A}$ and $\mathcal{B}$ are also weakly closed (see [18, p.14]).

Theorem 6.1 *Suppose that $\mathcal{A}$ is a W^*-algebra and that $\mathcal{B}$ is a weakly closed W^*-subalgebra of $\mathcal{A}$. In the quotient space $\mathcal{A}^{sim}/\mathcal{B}^{sim}$, the (quotient) norm of each class is realized by some element in that class.*

Proof: Let $Z \in \mathcal{A}^{\text{sim}}$ and, for every natural number n, let $b_n \in \mathcal{B}^{\text{sim}}$ such that,

1. $|Z + b_n|$ is a decreasing sequence of numbers.
2. If $z = \inf\{|Z+b| \mid b \in \mathcal{B}^{\text{sim}}\}$ is the norm of the class of Z in the quotient $\mathcal{A}^{\text{sim}}/\mathcal{B}^{\text{sim}}$, then

$$z \leq |Z + b_n| \leq z + \frac{1}{n}. \tag{I}$$

It is clear that the set of b_n's is bounded for $|b_n| \leq |Z+b_n| + |Z| \leq z+1+|Z|$, hence this set is weakly compact. Then there exists $b \in \mathcal{B}^{\text{sim}}$ which is in the weak closure of any tail $\{b_k \mid k \geq n\}$. It is clear that $Z + b \in \mathcal{B}^{\text{sim}}$ is in the weak closure of any tail $D_n = \{Z + b_k \mid k \geq n\}$. The theorem will be proved once we show that $|Z+b| = z$. Suppose on the contrary that $|Z+b| > z$, so there exists a natural number n_0 such that $|Z+b| > z + 1/n_0$. Denote by $\mathcal{A}_*$ a predual of the W^*-algebra $\mathcal{A}$. Then,

$$|Z+b| = \sup\{|\langle \eta, Z+b\rangle| \ | \ \eta \in \mathcal{A}_*, \ |\eta| = 1\},$$

where $\langle \eta, Z+b\rangle$ indicates de value of $Z+b$ at η. We can choose then $\xi \in \mathcal{A}_*$ of unit length, such that

$$|Z+b| \geq |\langle \xi, Z+b\rangle| > z + \frac{1}{n_0}.$$

Now $Z+b$ is in the weak closure of the tails D_n for any natural n. Then for any $\varepsilon > 0$ there exist arbitrarily large numbers n such that

$$||\langle \xi, Z+b\rangle| - |\langle \xi, Z+b_n\rangle|| < \varepsilon.$$

Taking ε small enough, we can find some $n > n_0$ such that $|\langle \xi, Z+b_n\rangle|$ is larger that $z+1/n_0$. But $|Z+b_n| \geq |\langle \xi, Z+b_n\rangle|$, and we get that $|Z+b_n| > z+1/n_0 > z+1/n$ which contradicts inequality (I) above.

□

Therefore,

Corollary 6.2 *Suppose that $\mathcal{A}$ is a W^*-algebra and that $\mathcal{B}$ is a weakly closed W^*-subalgebra of $\mathcal{A}$. In the quotient space $\mathcal{A}^{ant}/\mathcal{B}^{ant}$, the (quotient) norm of each class is reached by some element in that class.*

Proof of theorem II. From corollary 6.2 we get that there is a lift Z of X which satisfies $|Z| = ||X||_{\rho_0}$. The minimality of the given uniparametric group curve follows from theorem I.

□

We conclude this section with the following observation about the diameter of any generalized flag over a von Neumann algebra as a consequence of theorems I and II.

Proposition 6.3 *Let $\mathcal{P}$ be a generalized flag over a von Neumann algebra $\mathcal{A}$. Then the diameter $d(\mathcal{P})$ of $\mathcal{P}$ satisfies $\pi/2 \leq d(\mathcal{P}) \leq \pi$.*

Proof: Let X be a non zero tangent vector to $\mathcal{P}$ at any point ρ. Let Z be a minimal lift of X. Then the distance between the points ρ and $\gamma(\frac{\pi}{2|Z|})$ is $\pi/2$, where $\gamma(t) = L_{e^{tZ}}\rho$ (by theorem I). Therefore $d(\mathcal{P}) \geq \pi/2$. On the other hand, let $\rho, \sigma \in \mathcal{P}$. Let $g \in \mathcal{U}$ be such that $L_g\rho = \sigma$. Since we are assuming that $\mathcal{A}$ is a von Neumann algebra, there exists a symmetric element $S \in \mathcal{A}$ such that $e^{iS} = g$ and $|S| \leq \pi$ (see [13]). Thus the curve $L_{e^{itS}}\rho$ has length less than or eaual to π and joins ρ to σ.

□

The examples (see section 7.4) suggests that the diameter of $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ is an interesting invariant of the pair $(\mathcal{A}, \mathcal{B})$, which we plan to consider in a forthcoming paper (see also concluding remarks).

7 Examples

7.1 The simplest example

The simplest possible example is given by the complex projective line $\mathbb{C}P^1 = S^2$ as a homogeneous space of $U(2)$, and the isotropy is given by matrices of the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix} .$$

Tangent vectors canonically lift to vectors in the Lie algebra $u(2)$ of the form

$$\begin{pmatrix} 0 & -\bar{\beta} \\ \beta & 0 \end{pmatrix} .$$

The isotropy acts transitively on the directions in $T_x\mathbb{C}P^1$, and therefore the only invariant metrics are multiples of the canonical round metric given by $g(\beta, \beta) = |\beta|^2$. In this case, the compositions of length reducing maps collapse, and one is left with the identity on $\mathbb{C}P^1$: we choose the identity of $U(2)$ as Grassmann representation, and than for any tangent vector

$$\xi = (T\Pi)_1 \begin{pmatrix} 0 & -\bar{\beta} \\ \beta & 0 \end{pmatrix} .$$

The reflection

$$r_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is an isotropic reflection adapted to ξ. The isotropy of r_0 coincides with the isotropy of $\mathbb{C}P^1$, and the composition

$$\mathcal{P} \xrightarrow{\mathrm{F}} \mathrm{Gr}(\mathcal{H}) \xrightarrow{m_\xi} \mathcal{S}$$

is the identity.

An important feature of this example (and in fact of all Grassmann manifolds, see [16]) is that the quotient norm coincides with the connection norm; that is, the canonical lift above actually realizes the minimum norm of all possible lifts.

7.2 A not so simple example

Let us consider the space $\mathcal{P} = U(3)/U(1) \times U(1) \times U(1)$ of 3-flags in $\mathbb{C}^3$. The isotropy is composed of diagonal unitary matrices, and a tangent vector $X \in T_\rho\mathcal{P}$ has a canonical lift to $u(3)$ of the form

$$\begin{pmatrix} 0 & -\bar{a} & -\bar{b} \\ a & 0 & -\bar{c} \\ b & c & 0 \end{pmatrix} .$$

In contrast to the previous example, the canonical lift does *not* in general realize the quotient norm, and the problem of finding such minimal lifts is in general quite hard. There are special configurations, however, in which the canonical lift actually realizes the norm, e.g. vectors of the form

$$Z = \begin{pmatrix} 0 & -\bar{a} & -\bar{b} \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} .$$

For such Z and any flag $\rho \in \mathcal{P}$, the curve $L_{e^{tZ}}\rho$ is minimal up to length $\pi/2$. For this kind of vectors, we can choose as representation the identity of $End(\mathbb{C}^3)$, and as isotropy reflection the matrix

$$r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

With minor modifications, in this example we can substitute $\mathbb{C}^3$ with a general Hilbert space $\mathcal{H}$, and any (finite) number of flags. More precisely, if $\mathcal{H}$ is decomposed as $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, any operator $A \in End(\mathcal{H})$ can be written in matrix form A_{ij} with respect to the decomposition, $A_{ij} : \mathcal{H}_j \to \mathcal{H}_i$. If D is the space of diagonals matrices, the homogeneous space $U(\mathcal{H})/D$ is the space of n-flags in $\mathcal{H}$, and the same discussion applies.

The kind of vectors for which we can actually compute the minimum on the fiber, as in Z above, can be given for any generalized flag:

7.3 Grassmann-like examples

Here we look at the case where we have a projection $p \in \mathcal{A}$ which is in the commutant of $\mathcal{B}_{\rho_0}$.

In this situation we can describe special tangent vectors as follows:

Theorem 7.1 *Suppose there is a self-adjoint projector p in $\mathcal{A}$ which is in the commutant of the C^*-subalgebra $\mathcal{B}_{\rho_0}$ in $\mathcal{A}$. Suppose that a tangent vector $X \in (T\mathcal{P})_{\rho_0}$ admits a lift Z such that $|Z| = ||X||_{\rho_0}$, and that $Z \in \mathcal{A}$ has degree one with respect to p, i.e. $pZ = Z(1-p)$. Then X is the initial velocity of a uniparametric group curve which minimizes length in $\mathcal{P}$ up to $t = \frac{\pi}{2||X||_{\rho_0}}$.*

Proof: According to theorem 4.3, it is enough to show that $\mathcal{A}$ admits a representation which is adapted to X. From the general theory of representations of C^*-algebras (see [15] or [13]), we can choose a representation of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ where there is a vector η which is norming for $\widetilde{Z^2}$. The orthogonal projection $\tilde{p}$, produces in $\mathcal{H}$ an orthogonal decomposition in which we can write $\widetilde{Z}$ in the form,

$$\widetilde{Z} = \begin{pmatrix} 0 & -z^* \\ z & 0 \end{pmatrix}, \text{ hence}$$

$$\widetilde{Z^2} = -\begin{pmatrix} z^*z & 0 \\ 0 & zz^* \end{pmatrix}.$$

Notice that in such decomposition, η is given as $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ and satisfies that,

$$z^* z(\eta_1) = -\lambda^2 \eta_1, \quad \lambda = |Z| (= ||\widetilde{Z}||).$$

We consider first the case $\eta_1 \neq 0$. The idea is that we can set $\xi = \begin{pmatrix} \eta_1' \\ 0 \end{pmatrix}$, the normalization of $\begin{pmatrix} \eta_1 \\ 0 \end{pmatrix}$ in $\mathcal{H}$. With this ξ we can check that the representation previously chosen is adapted to X. In fact, any $b \in \mathcal{B}_{\rho_0}$ commutes with p, and $\tilde{b}$ is "presented" as a diagonal matrix of operators,

$$\tilde{b} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Hence the orthogonality condition is satisfied because,

$$\widetilde{Z}\xi = \begin{pmatrix} 0 & -z^* \\ z & 0 \end{pmatrix} \begin{pmatrix} \eta_1' \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ z\eta_1' \end{pmatrix},$$

and

$$\tilde{b}\xi = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} \eta_1' \\ 0 \end{pmatrix} = \begin{pmatrix} b_1\eta_1' \\ 0 \end{pmatrix}.$$

In the case that $\eta_1 = 0$ we proceed similarly, considering the component $\eta_2 (\neq 0)$ instead of η_1.

□

The simplest concrete case in which this situation arises non-trivially is as in section 7.2.

A more elaborate example occurs in the context of the orbit of a spectral measure. Let μ be a spectral measure, and let $\mathcal{M}_\mu$ be the orbit of μ under the action of the unitary group (cf. section 2). The spectral measure μ is a map from some measure space Ω into the C^*-algebra $\mathcal{A}$. Let us fix an element $Y_0 \in \Omega$, and consider an antisymmetric element $a \in \mathcal{A}^{ant}$. Let $Z = [a, \mu(Y_0)]$. It is easy to show that Z satisfies the conditions of theorem 7.1, where the projector p is $\mu(Y_0)$. Thus the curve $L_{e^{tZ}}\mu$ is a geodesic in $\mathcal{M}_\mu$ which minimizes up to length $\pi/2$.

This last example actually encompasses many others (including the one in section 7.2): just recall that the flag manifolds are also orbits of spectral measures of finite measure spaces.

7.4 An example concerning the diameter

It is known ([16]) that the diameter of Grassmann manifolds is $\pi/2$ (and therefore these are sort of "Blaschke manifolds", see [8]). Also proposition 6.3 tells us that the diameter of a generalized flag lies in the interval $[\pi/2, \pi]$. The diamater of the unitary group, as a trivial generalized flag, is π, but it remains to study the diameter of non-trivial generalized flags. Here we show non-trivial examples of diameters close to π:

Consider the C^*-algebra $\mathcal{A} = \mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$, whose unitary group is the n-torus $\mathbb{T}^n$. Let the subalgebra $\mathcal{B}$ be the scalars, and we look at the homogeneous space $\mathcal{P}_n = U(\mathcal{A})/U(\mathcal{B}) = \mathbb{T}^n/\Delta$ where Δ is the diagonal of $\mathbb{T}^n$.

Consider the point $\rho \in \mathcal{P}_n$ given by the projection of

$$(e^{-i\pi}, e^{-i\frac{n-1}{n}\pi}, \dots e^{-i\pi/n}, e^{i\pi/n}, \dots, e^{i\frac{n-1}{n}\pi}, e^{-i\pi}) = u \in U(\mathcal{A}),$$

i.e. $u = \mathrm{diag}(e^{ik\pi/n})$, $k = -n, \dots, n$ omitting $k = 0$. The fiber over ρ is given by points of the form $u_r = e^{ir}u = \mathrm{diag}(e^{i(r+k\pi/n)})$, $r \in \mathbb{R}$. A moments' reflection shows that if $u_r = e^Z$, $Z = (iz_{-n}, \dots iz_n)$, then at least one of the z_i must be close to π if n is big enough. This shows that $\mathrm{diam}(\mathcal{P}_n) \to \pi$ as $n \to \infty$. It is easy to make an example along these lines on $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ where $\mathcal{A} = \Pi_{n\in\mathbb{Z}}\mathbb{C}$, $\mathcal{B}$ the scalars, where the diameter is exactly π.

8 Concluding Remarks

8.1 The quotient norm

The results of this paper show the relevance of the problem of effectively computing quotient norms and the vectors for which the quotient norm may be realized. The reader is invited to compute the quotient norm of a generic vector in the simplest non-trivial example to get an idea of the algebraic difficulty: let $\mathcal{A}$ be the space of 3×3 (say Hermitian) matrices, and $\mathcal{B} \subset \mathcal{A}$ be the subspace of diagonal matrices (cf. section 7).

On the other extreme of generality, let S be the reader's favourite symmetric operator and let μ_S be the spectral measure associated to S. Compute the quotient Finsler norm on the orbit of μ_S in terms of natural invariants of S.

8.2 Other metrics

We want to remark that theorems I and II, which are about quotient metrics, have easy partial generalizations to other invariant metrics. Let us consider invariant (Banach) Finsler norms N on $\mathcal{P}$ such that, on each tangent space, are topologically equivalent to the quotient metric. By the Banach open mapping theorem, this is equivalent to $N_\rho(X) \geq c||X||_\rho$, and multiplying by a positive constant if necessary (which does not influence minimal curves), we can assume that $c = 1$ and $N_\rho(X) \geq ||X||_\rho$. We shall call such metrics *q-equivalent metrics.* An important example of such metrics are connection metrics, defined by reductive structures (see section 2).

A vector $X \in (T\mathcal{P})_\rho$ will be called *minimal* if $N_\rho(X) = ||X||_\rho$.

Proposition 8.1 *Theorems I and II are valid for minimal vectors in q-equivalent metrics.*

Proof: Let $X \in (T\mathcal{P})_\rho$ be a minimal vector. Let $\gamma(t) = L_{e^{tZ}}\rho$, where Z is a lift of X satisfying $|Z| = N(X)$ $(= ||X||_\rho)$. Given $t_1 \in [0, \pi/2]$, let $\sigma(t)$ be a curve joining $\gamma(0) = \rho$ with $\gamma(t_1)$. Then, denoting by ℓ_N the length of a curve in the Finsler structure given by N, we have

$$\ell_N(\sigma) \geq \ell(\sigma) \geq \ell(\gamma) = \ell_N(\gamma)\,,$$

which shows that γ is minimal in the metric N.

□

These results suggest the following interesting problem: given a norm N on a quotient $Q = B/B_0$ of Banach spaces which majorizes the quotient norm, characterize the set of vectors $X \in Q$ such that $N(X)$ coincides with the quotient norm of X in Q.

An important special case of q-equivalent metrics occurs when N is the connection norm associated to a conditional expectation.

The problem of studying the cone of minimal lifts of vectors $X \in (T\mathcal{P})_\rho$ for a fixed ρ is clearly central to our work and seems to be difficult. Partial results in this direction can be obtained. For example it can be shown that if $Z \in \mathcal{A}$ is a minimal lift of X, then both $i|Z|$ and $-i|Z|$ belong to the spectrum of Z.

8.3 Operator theory and metric geometry

In the 'hyperbolic' case, the study of the metric geometry of C^*-algebra related homogeneous spaces has been fruitful, cf. section 1.5 of the introduction. What can be said in the 'elliptic' case?

In the hyperbolic case, the combinatorial topology is sort of trivial due to Cartan-Hadamard type theorems: the exponential maps are diffeomorphisms ([5],[7],[17]). In the elliptic case, the spaces $\mathcal{P}$ have non-trivial 'cut-loci' at distances beyond $\pi/2$. What is the significance of these sets?

8.4 Diameter of generalized flags

As we have said, the problems related to what information about the pair $(\mathcal{A}, \mathcal{B})$ is contained in the value of the diamter of $\mathcal{P} = U(\mathcal{A})/U(\mathcal{B})$ is interseting. In particular,

- What is the set $\mathcal{D} \subset [\pi/2, \pi]$ of diameters of generalized flags $\mathcal{P}$?
- Characterize the set of generalized flags whose diameter is minimal $(\pi/2)$, and maximal (π).

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