On the causal and anticausal n-dimensional convolution equation related to the Diamond kernel of Marcel Riesz.

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ABSTRACT. In this Note we extend the Theorem 4.1 ([3], p. 37) due to A. Kananthai which says that "Given the linear differential equation of the form

$$(e^{\alpha t}\diamondsuit^k\delta) * u(t) = L^k u(t) = \delta; \qquad (\mathrm{I}, 1)$$

then

$$u(t) = e^{\alpha t}(-1)^k S_{2k} * R_{2k}(t), \qquad (\mathrm{I}, 2)$$

is an elementary solution of (I,1) or, equivalently, of the Diamond kernel of Marcel Riesz of (I,1), where $S_{2k}(t)$ and $R_{2k}(t)$ are defined, respectively by (2,1) and (2,3) of [1] with $\gamma = 2k$.

Our main result is the Theorem V.1, formula (V,3) which expresses that: "Given the linear partial differential equation of the form

$$(e^{\alpha t}\diamondsuit^k\delta) * (P \pm i0)^{\frac{\alpha-n}{2}} = L^k(P \pm i0)^{\frac{\alpha-n}{2}} = \delta. \qquad (\mathrm{I}, 3)$$

Here L is the partial differential operator of Diamond type defined by (IV,2). Then

$$(P \pm i0)^{\frac{2k-n}{2}} = e^{\alpha t}(-1)^k \cdot S_{2k}(P' \pm i0) * R_{2k}(P \pm i0), \qquad (\mathrm{I}, 4)$$

is an elementary solution of (I,3) where $S_{2k}(P' \pm i0)$ and $R_{2k}(P \pm i0)$ are defined by (II,10) and (II,7), respectively.

1. Introduction

From [1], Theorem 3,1, the equation $\diamondsuit^k u(t) = \delta$ has $(-1)^k S_{2k}(t) * R_{2k}(t)$ as an elementary solution and is called the Diamond kernel of Marcel Riesz where $S_2k(t)$ and $R_{2k}(t)$ are defined by the formulae (2,1) and (2,3) respectively of [1], with $\gamma = 2k$,

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where γ is nonnegative and $\diamondsuit^k$ is the Diamond operator iterated defined by (II,6). Otherwise, S.E. Trione ([2]) has proved that $(-1)^k S_{2k}(P' \pm i0) * R_{2k}(P \pm i0)$, which are defined by (II,10) and (II,7), where $k = 1, 2, \ldots$; n integer ≥ 2; is the elementary solution of the operator $\diamondsuit^k (P \pm i0)^{\frac{\alpha-n}{2}}$.

The purpose of this Note is extend the Theorem 4.1, due to A. Kananthai, ([3], p. 37), which says that "Given the linear differential equation of the form

$$(e^{\alpha t} \diamondsuit^k \delta) * u(t) = L^k u(t) = \delta, \tag{I, 1}$$

then

$$u(t) = e^{\alpha t} (-1)^k S_{2k} * R_{2k}(t) \tag{I, 2}$$

is an elemantary solution of (I,1) or, equivalently, of the Diamond kernel of Marcel Riesz of (I,1), where $S_{2k}(t)$ and $R_{2k}(t)$ are defined, respectively by (2,1) and (2,3) of [1] with $\gamma = 2k$.

Our main result is the Theorem V.1, formula (V,3) which expresses that: "Given the linear partial differential equation of the form

$$(e^{\alpha t} \diamondsuit^k \delta) * (P \pm i0)^{\frac{\alpha-n}{2}} = L^k (P \pm i0)^{\frac{\alpha-n}{2}} = \delta. \tag{I, 3}$$

Here L is the partial differential operator of Diamond type defined by (IV,2). Then

$$(P \pm i0)^{\frac{2k-n}{2}} = e^{\alpha t} (-1)^k \cdot S_{2k}(P' \pm i0) * R_{2k}(P \pm i0) \tag{I, 4}$$

is an elementary solution of (I,3) where $S_{2k}(P' \pm i0)$ and $R_{2k}(P \pm i0)$ are defined by (II,10) and (II,7), respectively.

Definitions.

Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional Euclidean space. Consider a non-degenerate quadratic form in n variables of the form

$$P = P(x) = x_1^2 + \cdots + xp^2 - x_{p+1}^2 - \cdots - x_{p+q}^2, \tag{II, 1}$$

where $p + q = n$.

The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{P \pm i\varepsilon |x|^2\}^\lambda, \tag{II, 2}$$

where $\varepsilon > 0$, $|x|^2 = x_1^2 + \cdots + x_n^2$, $\lambda \in \mathbb{C}$.

The distributions $(P \pm i0)^\lambda$ are an important contribution of Gelfand ([4], p. 274). The distributions $(P \pm i0)^\lambda$ are analytic in λ everywhere except at $\lambda = -\frac{n}{2} - k$,

$k = 0, 1, \ldots,$ where they have simple poles ([4], p. 275). The case $\lambda = -\frac{n}{2} - k$, $k = 0, 1, \ldots,$ has been evaluated by S.E. Trione ([11]).

By causal (anticausal) distributions we mean distributions of the form $(P \pm i0)^\lambda$, where

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 - x_n^2 \, . \qquad (\mathrm{II}, 3)$$

The causal distributions are particularly important when $n = 4$ because they appear frequently in the quantum theory of fields.

We defined the n-dimensional ultrahyperbolic operator, iterated k-times (k integer ≥ 1) by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k , \qquad (\mathrm{II}, 4)$$

$p + q = n$.

Also, we define the n-dimensional Laplace operator iterated k-times (k integer ≥ 1) by

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \right)^k . \qquad (\mathrm{II}, 5)$$

The Diamond operator iterated k-times is defined by

$$\diamondsuit^k = \left[\left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \ldots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2 \right]^k , \qquad (\mathrm{II}, 6)$$

$k = 0, 1, \ldots$, and $p + q = n$.

We shall write, by definition, $P' = P'(x) = x_1^2 - x_2^2 - \cdots - x_n^2$.

We designate Γ_+ the interior of the forward cone $\Gamma_+ = \{x \in I\!R^n / x_1 > 0, P > 0\}$ and designates $\overline{\Gamma}_+$ its closure. Similarly, Γ_- designates the domain $\Gamma_- = \{x \in I\!R^n, x_1 < 0, P > 0\}$, and $\overline{\Gamma}_-$ designates its closure.

Now, we define the following functions introduced by S.E. Trione ([5], form. 4, p. 150):

$$R_\alpha = R_\alpha(P \pm i0) = \begin{cases} (P \pm i0)^{\frac{\alpha - n}{2}} & \text{if } x \in \Gamma_+, \\ 0 & \text{if } x \notin \Gamma_+. \end{cases} \qquad (\mathrm{II}, 7)$$

$R_\alpha(P \pm i0)$ is called the generalized ultra-hyperbolic kernel of Marcel Riesz

In the formula (II,7) α is a complex parameter, n the dimension of the space and $K_n(\alpha)$ is the constant due to Y. Nozaki ([6], p.72) defined by

$$K_n(\alpha) = \frac{\Pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)} , \qquad (\mathrm{II}, 8)$$

p is the number of positive terms of

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \ , \qquad \text{(II, 9)}$$

$p + q = n$

$R_\alpha(P \pm i0)$ is an ordinary function if $\operatorname{Re}\alpha \geq n$, and is a distribution of α is $\operatorname{Re}\alpha < n$ and $\operatorname{supp} R_\alpha(P \pm i0) \subset \overline{\Gamma}_+$.

Now, we define the causal (anticausal) distributions $S_\alpha(P' \pm i0)$ as follows:

$$S_\alpha = S_\alpha\left(P' \pm i0\right) = \frac{e^{i\frac{\pi}{2}\alpha} e^{\pm i\frac{\pi}{2}q} \Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \left(P' \pm i0\right)^{\frac{\alpha-n}{2}} \ , \qquad \text{(II, 10)}$$

where $\alpha \in \mathbb{C}$,

$$P' = P'(x) = x_1^2 - x_2^2 - \cdots - x_n^2 \ , \qquad \text{(II, 11)}$$

and q is the number of negative terms of the quadratic form P. The distributional functions $S_\alpha(P' \pm i0)$ are the causal (anticausal) analogues of the elliptic kernel of M. Riesz ([7]), and have analogous properties ([8]).

Section I.

Lemma III.1. *$S_\gamma(P' \pm i0)$ and $R_\gamma(P \pm i0)$ are homogeneous distributions of order $\alpha - n$. Moreover, they are tempered distributions.*

Proof. The Lemma 2.1, p. 34 of [3], in the generalized case of our thesis, is valid taking into account the following comment:

We know that (cf. form. (II.2))

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{P(x) \pm i\varepsilon |x|^2\}^\lambda, \qquad \text{(III, 1)}$$

so

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} (1 + i\varepsilon) \left[x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2\right]^\lambda \ . \qquad \text{(III, 2)}$$

Therefore, by taking the limit when ε tends to zero, the formula of the thesis is valid because the expression which appears between the clap is true by A. Kananthai ([3], Lemma 2.1, 34-35)

Lemma III.2. *$S_\gamma(P' \pm i0)$ and $R_\gamma(P \pm i0)$ are tempered distributions.*

Proof. By following A. Kananthai ([3]) we know that W.F. Donoghue ([9], 156-159) establishes that every homogeneous distribution is a tempered distribution. ∎

Lemma III.3. *The convolution* $S_\gamma(P' \pm i0) * R_\gamma(P \pm i0)$ *exists and is a tempered distribution.*

Proof. Taking into account the formula (II,3;11'), p. 41 of [8] and [10], the convolution $S_\gamma(P' \pm i0) * R_\gamma(P \pm i0)$ exists.

Otherwise, by Lemma III.2, $S_\gamma(P' \pm i0)$ and $R_\gamma(P \pm i0)$ are tempered distributions. Therefore, the convolution $S_\gamma(P' \pm i0) * R_\gamma(P \pm i0)$ is a tempered distribution.

Then, the Lemma III.3 is proved. ■

Section II.

Lemma IV.1.

$$e^{\alpha t} \diamondsuit^k \delta = L^k \delta \ , \tag{IV, 1}$$

where $t = (t_1, t_2, \ldots, t_n)$ *is a variable,* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *is a constant and both are the points in the* n*-dimensional Euclidean space* $I\!R^n$, $k = 0, 1, \ldots.$

In the thesis (IV,1), $\diamondsuit^k$ *is defined by (II,6) and* L *is the partial differential operator of Diamond type and is defined by*

$$\begin{aligned} L \overset{\text{def}}{=} & \diamondsuit + \sum_{r=1}^{n} \alpha_r^2 \Box - 2 \sum_{r=1}^{n} \sum_{i=1}^{n} \left(\alpha_r \frac{\partial^3}{\partial t_i^2 \partial t_r} + \alpha_i \frac{\partial^3}{\partial t_i \partial t_r^2} \right) \\ & + 2 \sum_{r=1}^{n} \sum_{j=p+1}^{p+q} \left(\alpha_r \frac{\partial^3}{\partial t_j^2 \partial t_r} + \alpha_j \frac{\partial^3}{\partial t_j \partial t_r^2} \right) \\ & + 4 \left(\sum_{r=1}^{n} \sum_{i=1}^{p} \alpha_r \alpha_j \frac{\partial^2}{\partial t_j \partial t_r} - \sum_{r=1}^{n} \sum_{j=p+1}^{p+q} \alpha_r \alpha_j \frac{\partial^2}{\partial t_j \partial t_r} \right) \\ & - 2 \sum_{r=1}^{n} \alpha_r^2 \left(\sum_{i=1}^{p} \alpha_i \frac{\partial}{\partial t_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial}{\partial t_j} \right) \\ & + \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right) \Delta - 2 \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right) \cdot \sum_{r=1}^{n} \alpha_r \frac{\partial}{\partial t_\gamma} \\ & + \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right) \sum_{r=1}^{n} \alpha_r^2 \ , \end{aligned} \tag{IV, 2}$$

where

$$\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial t_1^2} - \sum_{j=p}^{p+q} \frac{\partial^2}{\partial t_j^2} \tag{IV, 3}$$

is the n*-dimensional ultrahyperbolic operator, iterated* k*-times (*k *integer* ≥ 1*) defined by (II,4),* $p+q=n$*,* n *is the dimensional of the space and*

$$\Delta^k = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial t_i^2}\right)^k \quad , \tag{IV,4}$$

is the n*-dimensional Laplace operator iterated* k*-times (*k *integer* ≥ 1*) defined by (II,5).*

The formula

$$\diamondsuit = \square\,\Delta \tag{IV,5}$$

is valid, by immediately calculation, and $e^{\alpha t}\diamondsuit^k\delta$ *is a tempered distribution of order* $4k$*.*

<u>*Proof.*</u> By taking into account the definitory formula (1,1), p. 28 of [1], and remembering (II,2) we can evaluate $L(P \pm i0)$. Noting that L is defined by (IV,2) and

$$(P \pm i0)^\lambda = \lim_{\varepsilon\to 0}(1+i\varepsilon)\left[x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2\right] \tag{IV,6}$$

we arrive at our thesis (following A. Kananthai [3]).

Section III.

We note, by taking into account the Lemmaa IV.1, that the following formula is valid:

$$(e^{\alpha t}\diamondsuit^k\delta) * (P \pm i0)^{\frac{\alpha-n}{2}} = L^k\delta * (P \pm i0)^{\frac{\alpha-n}{2}} = L^k(P \pm i0)^{\frac{\alpha-n}{2}} \quad , \tag{V,1}$$

where L is defined by (IV,2).

Now we can state our main theorem

Theorem V.I. *Given the linear partial differential equation of the form*

$$(e^{\alpha t}\diamondsuit^k\delta) * (P \pm i0)^{\frac{\alpha-n}{2}} = L^k(P \pm i0)^{\frac{\alpha-n}{2}} = \delta \ . \tag{V,2}$$

Then,

$$(P \pm i0)^{\frac{\alpha-n}{2}} = e^{\alpha t}(-1)^k S_{2k}(P' \pm i0) * R_{2k}(P \pm i0) \ , \tag{V,3}$$

is an elementary solution of (V,2) or, the same, of the Diamond kernel of Marcel Riesz solution of (V,2), where $S_{2k}(P' \pm i0)$ *and* $R_{2k}(P \pm i0)$ *are defined by the formulas (II,10) and (II,7), respectively, with* $\gamma = 2k$*.*

Proof. By S.E. Trione ([2]), $(-1)^k S_{2k}(P' \pm i0)$ is an elementary solution of the homogeneous ultrahyperbolic operator iterated k-times

$$L^k = \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2} \right)^k$$

and also by Trione [5], $R_{2k}(P \pm i0)$ is an elementary solution of the n-dimensional ultrahyperbolic operator $\Box^k$ iterated k-times, here

$$\Box^k \overset{\text{def}}{=} \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k , \qquad (V,4)$$

$p + q = n$, $k =$ integer ≥ 1.

We know, by an elementary calculation that

$$\Diamond = \Box \Delta . \qquad (V,5)$$

Therefore, by iteration, we obtain immediately that

$$\Diamond^k = \Box^k \Delta^k . \qquad (V,6)$$

Now, we consider

$$e^{\alpha t} \left(\Box^k \Delta^k \delta \right) * R_{2k}(P \pm i0) = \delta . \qquad (V,7)$$

By Lemma III.3, with $\gamma = 2k$, $(-1)^k S_{2k}(P' \pm i0) * R_{2k}(P \pm i0)$ exists and is a tempered distribution.

By convolving both sides of the equation (V,7) by $e^{\alpha t}[(-1)^k S_{2k}(P' \pm i0) * R_{2k}(P \pm i0)]$, we obtain

$$(e^{\alpha t} \delta) * (P \pm i0)^{\frac{\alpha - n}{2}} = e^{\alpha t} (-1)^k S_{2k}(P' \pm i0) * R_{2k}(P \pm i0) . \qquad (V,8)$$

It follows that

$$(P \pm i0)^{\frac{\alpha - n}{2}} = e^{\alpha t} (-1)^k S_{2k}(P' \pm i0) * R_{2k}(P \pm i0) , \qquad (V,9)$$

formula (V,9) is identical to (V,3) which is the thesis to our main Theorem. So this finishes the proof of Theorem (V,I). ∎

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