ON PRINCIPAL EIGENVALUES FOR PERIODIC PARABOLIC STEKLOV PROBLEMS

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ABSTRACT. Let Ω be a $C^{2+\gamma}$ domain in $\mathbb{R}^N$, $N \geq 2$, $0 < \gamma < 1$. Let $T > 0$ and let L be a uniformly parabolic operator $Lu = \frac{\partial u}{\partial t} - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j}\right) + \sum_j b_j \frac{\partial u}{\partial x_i} + a_0 u$, $a_0 \geq 0.$, whose coefficients, depending on $(x,t) \in \Omega \times \mathbb{R}$, are T periodic in t and satisfy some regularity assumptions. Let A be the $N \times N$ matrix whose i,j entry is a_{ij} and let ν be the unit exterior normal to $\partial\Omega$. Let m be a T periodic function (that may change sign) defined on $\overline{\Omega} \times \mathbb{R}$ whose restriction to $\partial\Omega \times \mathbb{R}$ belongs to $W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times (0,T))$ for some large enough q. In this paper we give necessary and sufficient conditions on m for the existence of principal eigenvalues for the periodic parabolic Steklov problem $Lu = 0$ on $\Omega \times \mathbb{R}$, $\langle A\nabla u, \nu\rangle = \lambda m u$ on $\partial\Omega \times \mathbb{R}$, $u(x,t) = u(x,t+T)$, $u > 0$ on $\Omega \times \mathbb{R}$. Uniqueness and simplicity of the positive principal eigenvalue is proved and a related maximum principle is given.

1. Introduction

Let Ω be a $C^{2+\gamma}$ and bounded domain in $\mathbb{R}^N$, $N \geq 2$, $0 < \gamma < 1$, let $T > 0$, let $\{a_{ij}\}_{1 \leq i,j \leq N}$, $\{b_j\}_{1 \leq, j \leq N}$ be two families of real functions defined on $\Omega \times \mathbb{R}$ satisfying $a_{ij} \in C^{\gamma,\gamma/2}(\overline{\Omega} \times \mathbb{R})$, $b_j \in C^1(\overline{\Omega} \times \mathbb{R})$, $a_{ij} = a_{ji}$, and $\frac{\partial a_{ij}}{\partial x_i} \in C(\overline{\Omega} \times \mathbb{R})$ for $1 \leq i,j \leq N$. Assume also that

$$\sum_{i,j} a_{ij}(x,t)\,\xi_i \xi_j \geq \alpha_0 |\xi|^2$$

for some positive constant α_0 and all $(x,t) \in \overline{\Omega} \times \mathbb{R}$, $\xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N$ and that each $a_{ij}(x,t), b_j(x,t)$ is T periodic in t. Let A be the $N \times N$ matrix whose i,j entry is a_{ij}, let $b = (b_1, ..., b_N)$, let a_0 be a nonnegative and T periodic function belonging to $C^{\gamma,\gamma/2}(\overline{\Omega} \times \mathbb{R})$ and let L be the parabolic operator given by

$$Lu = u_t - div(A\nabla u) + \langle b, \nabla u\rangle + a_0 u \tag{1.1}$$

where $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^N$.

For $q \geq 1$, $\tau > 0$, let $W_q^{2,1}(\Omega \times (0,\tau))$ be the Sobolev space of the functions $u \in L^q(\Omega \times (0,\tau))$, $u = u(x,t)$, $x = (x_1,,,,.x_N)$ such that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_j}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ belong to $L^q(\Omega \times (0,\tau))$ for $1 \leq i,j \leq N$. We are interested in the periodic parabolic Steklov eigenvalue problem

$$(P_\lambda) \quad \begin{cases} Lu = 0 \text{ in } \Omega \times \mathbb{R}, \\ \langle A\nabla u, \nu\rangle = \lambda m u \text{ on } \partial\Omega \times \mathbb{R} \\ u(x,t) \;\; T \text{ periodic in } t \end{cases} \tag{1.2}$$

where ν denotes the unit exterior normal to $\partial\Omega$ and the solution u is taken such that $u_{|\Omega \times (0,T)} \in W_q^{2,1}(\Omega \times (0,T))$ for a fixed and large enough q. The weight function m is assumed T periodic and such that $m_{|\partial\Omega \times (0,T)} \in W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times (0,T))$ (the fractional Sobolev space defined e.g., as in [6], chapter 2, paragraph 3).

Steklov introduced this eigenvalue problem in the elliptic case in connection with the study of the map, nowadays called Dirichlet to Neumann map (see e.g. [7], chapter VI of part B, p 395-404) which is also of interest in the inverse problem of reconstructing the coefficients of L from this map.

We say that $\lambda^* \in \mathbb{R}$ is a *principal eigenvalue* for the weight m if (P_{λ^*}) has a positive (i.e. a nonnegative and non trivial) solution.

In this paper we give necessary and sufficient conditions, on a weight m as above, for existence of a positive principal eigenvalue. Uniqueness and simplicity of this positive principal eigenvalue is proved and a related form of the maximum principle is given.

We remark that this weighted eigenvalue problem includes the corresponding elliptic case where the coefficients are time independent.

In section 2, for given T periodic functions f and Φ defined on $\Omega\times\mathbb{R}$ and $\partial\Omega\times\mathbb{R}$ respectively and satisfying $f_{|\Omega\times(0,T)} \in L^q(\Omega\times(0,T))$, $\Phi_{|\partial\Omega\times(0,T)} \in W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))$, we study existence of T periodic solutions $u:\Omega\times\mathbb{R}\to\mathbb{R}$ such that $u_{|\partial\Omega\times(0,T)} \in W_q^{2,1}(\Omega\times(0,T))$ for the problem

$$\begin{cases} Lu = f\,on\ \Omega\times\mathbb{R} \\ b_0 u + \langle A\nabla u,\nu\rangle = \Phi \text{ on } \partial\Omega\times\mathbb{R} \\ u(x,t)\ T\ periodic\ in\ t. \end{cases}$$

We prove that, under suitable hypothesis on a_0 and b_0, this problem has a unique solution. and we state the boundedness (with respect to the natural topologies involved) of the corresponding solution operator $u = S_{b_0}(f,\Phi)$ (see Theorem 2.5). We prove also (see Theorem 2.6) the compactness and the strong positivity of the operator $\Phi \to S_{b_0}(0,\Phi)$.

In section 3 we study the following one parameter family of principal eigenvalue problems: given $\lambda\in\mathbb{R}$ we prove that there exists a unique principal eigenvalue $\mu=\mu_m(\lambda)$ for the problem

$$\begin{cases} Lu = 0 \text{ on } \Omega\times\mathbb{R} \\ \langle A\nabla u,\nu\rangle - \lambda m u = \mu u \text{ on } \partial\Omega\times\mathbb{R} \\ u(x,t)\ T \text{ periodic in } t. \\ u>0 \text{ on } \Omega\times\mathbb{R}, \end{cases}$$

we show that $\mu_m(\lambda)$ is concave and real analytic in λ and its behavior near zero and at infinity is studied.

In section 4, using properties of the function μ_m, we prove that, for the case $a_0>0$, the condition $P(m) := \int_0^T \max_{x\in\Omega} m(x,t)\,dt > 0$ is a necessary and sufficient condition for the existence of a positive principal eigenvalue for the weighted problem (1.2) and that, for the case $a_0=0$, there exists a positive principal eigenvalue for (1.2) if and only if $P(m)>0$ and $\int_{\Omega\times(0,T)}\Psi m<0$ where Ψ is a positive (unique up a multiplicative constant and belonging to $C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega}\times\mathbb{R}\right)$) for the T periodic problem

$$\begin{cases} \frac{\partial\Psi}{\partial t} + div\,(A\nabla\Psi) + \langle b,\nabla\Psi\rangle + div\,(b)\,\Psi = 0 \text{ on } \Omega\times\mathbb{R} \\ \langle A\nabla\Psi,\nu\rangle + \langle b,\nu\rangle\,\Psi = 0 \text{ on } \partial\Omega\times\mathbb{R} \end{cases}$$

2. Preliminaries

We recall the following well known facts concerning Sobolev spaces (see e.g. [6], Lemma 3.3, p 80 Lemma 3.4, p. 82)

(i): if $u \in W_q^{2,1}(\Omega \times (0,\tau))$, $q > 1$, $\tau > 0$ then

$$u_{|\partial\Omega\times(0,\tau)} \in W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times (0,\tau))$$

and the restriction map (understood in the trace sense) is continuous.

(ii): For $\tau > 0$ and q large enough, it holds that

$$W_q^{2,1}(\Omega \times (0,\tau)) \subset C^{1+\gamma,\frac{1+\gamma}{2}}\left(\overline{\Omega} \times [0,\tau]\right) \tag{2.1}$$

with continuous inclusion.

(iii): For $\tau > 0$ and q large enough, it holds that

$$W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times (0,\tau)) \subset C^{1+\gamma,\frac{1+\gamma}{2}}(\partial\Omega \times [0,\tau]) \tag{2.2}$$

with continuous inclusion.

We fix, from now on, $\tau > T$ and a large enough q such that (ii) and (iii) hold.■

We recall also the following

Lemma 2.1. *Let $b_0 \in W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,\tau))$, $b_0 \geq 0$. Suppose also that $f \in L^q(\Omega\times(0,\tau))$, $\varphi \in W_q^{2-\frac{2}{q}}(\Omega)$ and $\Phi \in W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times (0,\tau))$ and that the compatibility condition*

$$b_0(.,0)\,\varphi + \langle A\nabla\varphi, \nu\rangle = \Phi(.,0) \text{ on } \partial\Omega \tag{2.3}$$

is fulfilled, then the problem

$$\begin{cases} Lu = f \text{ on } \Omega \times (0,\tau) \\ b_0 u + \langle A\nabla u, \nu\rangle = \Phi \text{ on } \partial\Omega \times (0,\tau) \\ u(.,0) = \varphi \text{ on } \Omega \end{cases} \tag{2.4}$$

has a unique solution $u \in W_q^{2,1}(\Omega \times (0,\tau))$. Moreover, there exists a positive constant c independent of f, φ and Φ such that

$$\|u\|_{W_q^{2,1}(\Omega\times(0,T))} \leq c\left(\|f\|_{L^q(\Omega\times(0,T))} + \|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))} + \|\varphi\|_{W_q^{2-\frac{2}{q}}(\Omega)}\right). \tag{2.5}$$

For a proof of Lemma 2.1, see [6], (Theorem 9.1, p. 341, concerning to the Dirichlet problem and its extension, to our boundary conditions, indicated there, at the end of chapter 4, paragraph 9, p. 351).

For regular data, the following result holds (see e.g. [6], Theorem 5.3, p. 320):

Lemma 2.2. *Suppose that $b_0 \in C^{1+\gamma\frac{1+\gamma}{2}}(\partial\Omega \times [0,\tau])$, $b_0 \geq 0$. Suppose that $f \in C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega} \times [0,\tau])$, $\varphi \in C^{2+\gamma}(\overline{\Omega})$, $\Phi \in C^{1+\gamma\frac{1+\gamma}{2}}(\partial\Omega \times [0,\tau])$ and that the compatibility con-*

Moreover, there exists a positive constant c independent of f, φ and Φ such that

$$\begin{aligned} &\|u\|_{C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega}\times[0,\tau]\right)} \\ &\leq c\left(\|f\|_{C^{\gamma,\frac{\gamma}{2}}\left(\overline{\Omega}\times[0,\tau]\right)}+\|\Phi\|_{C^{1+\gamma\frac{1+\gamma}{2}}(\partial\Omega\times[0,\tau])}+\|\varphi\|_{C^{2+\gamma}\left(\overline{\Omega}\right)}\right) \end{aligned} \tag{2.6}$$

Remark 2.3. If in addition to the hypothesis of Lemma 2.1 we have also that $f \in C^{\gamma,\frac{\gamma}{2}}\left(\overline{\Omega}\times[0,\tau]\right)$, then the solution u of (2.4) satisfies

$$u \in C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega}\times[\delta,\tau]\right) \tag{2.7}$$

for all $\delta > 0$. Moreover, for such a δ, there exists a positive constant c_δ independent of f and Φ such that

$$\begin{aligned} &\|u\|_{C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega}\times[\delta,\tau]\right)} \\ &\leq c_\delta\left(\|f\|_{C^{\gamma,\frac{\gamma}{2}}\left(\overline{\Omega}\times[0,\tau]\right)}+\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,\tau))}+\|\varphi\|_{W_q^{2-\frac{2}{q}}(\Omega)}\right). \end{aligned} \tag{2.8}$$

Indeed, let $h \in C^\infty(\mathbb{R})$ be such that $0 \leq h \leq 1$, $h(t) = 0$ for $t < \frac{\delta}{4}$, $h(t) = 1$ for $t \geq \frac{3\delta}{4}$, let $\widetilde{u}(x,t) = u(x,t)\,h(t)$, let $\widetilde{f}(x,t) = u(x,t)\,h'(t) + f(x,t)\,h(t)$ and let $\widetilde{\Phi}(x,t) = \Phi(x,t)\,h(t)$. Then

$$\begin{cases} L\widetilde{u} = \widetilde{f} \text{ on } \Omega\times(0,\tau) \\ b_0\widetilde{u} + \langle A\nabla\widetilde{u},\nu\rangle = \widetilde{\Phi} \text{ on } \partial\Omega\times(0,\tau) \\ \widetilde{u}(.,0) = 0 \text{ on } \Omega. \end{cases} \tag{2.9}$$

By Lemma 2.1 this problem has a unique solution in $W_q^{2,1}(\Omega\times(0,\tau))$. Since $f \in C^{\gamma,\frac{\gamma}{2}}\left(\overline{\Omega}\times[0,\tau]\right)$ and $\widetilde{\Phi} \in C^{1+\gamma,\frac{1+\gamma}{2}}(\partial\Omega\times[0,\tau])$, Lemma 2.2 says that it has also a unique solution $\widetilde{u} \in C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega}\times[0,\tau]\right)$ and so, since $h \equiv 1$ on $[\delta,\tau]$, we obtain (2.7).

Also,

$$\left\|\widetilde{\Phi}\right\|_{C^{1+\gamma,\frac{1+\gamma}{2}}(\partial\Omega\times[0,T])} \leq c'_\delta\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,\tau))}$$

for some constant c'_δ independent of Φ and so, using (2.5) and the definition of $\widetilde{f}$ we get that

$$\begin{aligned} &\left\|\widetilde{f}\right\|_{C^{\gamma,\frac{\gamma}{2}}\left(\overline{\Omega}\times[0,\tau]\right)} \\ &\leq c''_\delta\left(\|f\|_{C^{\gamma,\frac{\gamma}{2}}\left(\overline{\Omega}\times[0,\tau]\right)}+\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,\tau))}+\|\varphi\|_{W_q^{2-\frac{2}{q}}(\Omega)}\right) \end{aligned}$$

for some positive constant c''_δ independent of f, Φ. Then (2.6), applied to problem (2.9), gives (2.8).■

Let $b_0 \in W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,\tau))$, $b_0 \geq 0$. For $s > 1+\frac{1}{q}$, let $W^s_{q,B_0}(\Omega)$ be the Banach space of the functions $\varphi \in W^s_q(\Omega)$ satisfying $b_0(.,0)\,\varphi + \langle A(.,0)\,\nabla\varphi,\nu\rangle = 0$ on $\partial\Omega$.

$W^{2-\frac{2}{q}}_{q,B_0}(\Omega)$ and $W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,\tau))$ provided with their natural orders are ordered

As usual, for $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ (respectively $f : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $f : \Omega \rightarrow \mathbb{R}$) we will write $f > 0$ to mean $f(x,t) \geq 0$ and f non identically zero.

Let $U : W_{q,B_0}^{2-\frac{2}{q}}(\Omega) \rightarrow W_{q,B_0}^{2-\frac{2}{q}}(\Omega)$ be defined by $U\varphi = u(.,T)$ where $u \in W_q^{2,1}(\Omega \times (0,\tau))$ is the solution (given by Lemma 2.1) of

$$\begin{cases} Lu = 0 \text{ on } \Omega \times (0,\tau) \\ b_0 u + \langle A\nabla u, \nu \rangle = 0 \text{ on } \partial\Omega \times (0,\tau) \\ u(.,0) = \varphi \text{ on } \Omega \end{cases} \tag{2.10}$$

We have

Lemma 2.4. *Suppose that $b_0 \in W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times (0,\tau))$, $b_0 \geq 0$. Then U is a compact operator. Moreover, if either $a_0 > 0$ or $b_0 > 0$ in their respective domains, then U is a strongly positive operator with positive spectral radius $\rho < 1$.*

Proof. From Lemma 2.1, the solution u of (2.10) satisfies

$$\|u\|_{W_q^{2,1}(\Omega \times (0,T))} \leq c\, \|\varphi\|_{W_q^{2-\frac{2}{q}}(\Omega)}. \tag{2.11}$$

Let $h, \widetilde{u}, \widetilde{f}$ and $\widetilde{\Phi}$ be as in remark 2.3, taking there $f = 0$, $\Phi = 0$. From (2.11) and (2.1) we have $\left\|\widetilde{f}\right\|_{C^{\gamma,\frac{\gamma}{2}}\left(\overline{\Omega} \times [0,T]\right)} \leq c\, \|\varphi\|_{W_q^{2-\frac{2}{q}}(\Omega)}$ and so (2.8) applied to (2.10) implies

$$\|u(.,T)\|_{C^{2+\gamma}(\overline{\Omega})} \leq c\, \|\varphi\|_{W_q^{2-\frac{2}{q}}(\Omega)} \tag{2.12}$$

for some positive constant c independent of φ. Now, (2.12) implies the compactness assertion of the lemma.

Suppose now that for some $\varphi > 0$ in $W_{q,B_0}^{2-\frac{2}{q}}(\Omega)$, the minimum of $U\varphi = u(.,T)$ is non positive. then the minimum of u on $\overline{\Omega} \times (0,T)$ is non positive and it is achieved at some $(x_0,t_0) \in \overline{\Omega} \times (0,T]$. If $x_0 \in \Omega$, the parabolic maximum principle (as stated e.g., in [4], Proposition 13.3, p. 33) implies that u is a constant on $\overline{\Omega} \times [\delta, T]$ for all $\delta > 0$, so φ is a non positive constant, contradiction. If $x_0 \in \partial\Omega$, the same principle states that $\langle A\nabla u, \nu \rangle < 0$ at (x_0,t_0) contradicting $b_0(x_0,t_0)\, u(x_0,t_0) + \langle A\nabla u, \nu \rangle (x_0,t_0) = 0$. So U is a strongly positive operator on $W_q^{2-\frac{2}{q}}(\Omega)$. Now, Krein Rutman Theorem (as stated e.g. in [1], Theorem 3.1) gives that its spectral radius ρ is a positive eigenvalue with positive eigenfunctions. Let $\varphi_\rho \in W_q^{2-\frac{2}{q}}(\Omega)$ such eigenfunction. To see that $\rho < 1$, we proceed by contradiction. Suppose $\rho \geq 1$. Then $U(\varphi_\rho) = \rho\varphi_\rho \geq \varphi_\rho$, i.e., the solution of (2.10) (taking there $\varphi = \varphi_\rho$) would satisfy $u(.,T) \geq \varphi_\rho$, but the maximum principle states that u is a constant or $\max_{\overline{\Omega} \times [\delta,T]} u(x,t)$ is attained at some $(x_0,t_0) \in \partial\Omega \times [0,T]$ and so $a_0 = 0$ or $b_0(x_0,t_0) < 0$ respectively.■

Let $W_{q,T}^{2,1}(\Omega \times \mathbb{R})$ (respectively $W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times \mathbb{R})$) be the Banach space of the T periodic functions $v : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$v_{|\Omega \times (0,T)} \in W_q^{2,1}(\Omega \times (0,T))$$

(respectively $v_{|\partial\Omega \times (0,T)} \in W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times (0,T))$), equipped with the norm $\|v\|_{W_q^{2,1}(\Omega \times (0,T))}$

Theorem 2.5. *Let $b_0, \Phi \in W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times \mathbb{R})$, $b_0 \geq 0$. If $a_0 > 0$ or $b_0 > 0$ and if $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is T periodic and satisfies $f_{|\Omega\times(0,T)} \in L^q(\Omega \times (0,T))$ then the problem*

$$\left\{ \begin{array}{c} Lu = f \, on \ \Omega \times \mathbb{R} \\ b_0 u + \langle A\nabla u, \nu\rangle = \Phi \text{ on } \partial\Omega \times \mathbb{R} \\ u(x,t) \ T \ periodic \ in \ t \end{array} \right. \tag{2.13}$$

has a unique solution $u \in W_{q,T}^{2,1}(\Omega \times \mathbb{R})$. Moreover, there exists a positive constant c independent of f and Φ such that

$$\|u\|_{W_q^{2,1}(\Omega\times(0,T))} \leq c\left(\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))} + \|f\|_{L^q(\Omega\times(0,T))} \right). \tag{2.14}$$

If in addition to the above hypothesis we have $f \in C^{\gamma,\frac{\gamma}{2}}\left(\overline{\Omega}\times\mathbb{R}\right)$ then $u \in C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega}\times\mathbb{R}\right)$ and

$$\|u\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times\mathbb{R})} \tag{2.15}$$
$$\leq c\left(\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))} + \|f\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times\mathbb{R})} \right)$$

for some constant c independent of f and Φ

Proof. Let us start constructing a function $\varphi_1 \in W^{2-\frac{2}{q}}(\Omega)$ satisfying

$$b_0(,.0)\varphi_1 + \langle A(.,0)\nabla\varphi_1, \nu\rangle = \Phi(.,0) \tag{2.16}$$

and such that

$$\|\varphi_1\|_{W^{2-\frac{2}{q}}(\Omega)} \leq c\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))} \tag{2.17}$$

for some constant c independent of Φ. To do so, consider $F(\overline{x}, s) = \overline{x} - sA(\overline{x}, 0)\nu(\overline{x})$ on $\partial\Omega \times (-\varepsilon, \varepsilon)$. Since $\overline{x}(x) = x$ for $x \in \partial\Omega$ and $A(\overline{x}, 0)\nu(\overline{x})$ is non tangential to $\partial\Omega$ at $\overline{x} \in \partial\Omega$, F defines a diffeomorphism onto a neighborhood V of $\partial\Omega$ in $\mathbb{R}^N$ for some $\varepsilon > 0$. So we have $F^{-1}(x) = (\overline{x}(x), s(x))$ for $x \in V$.

Then we solve the (non characteristic) Cauchy problem

$$\left\{ \begin{array}{c} b_0(\overline{x}(x),0)w + \langle A(\overline{x}(x),0)\nabla w, \nu(\overline{x}(x))\rangle = \Phi(\overline{x}(x),0), \ x \in V \\ w = 0 \text{ on } \partial\Omega \end{array} \right.$$

the solution is, for $x = \overline{x} - sA(\overline{x}, 0)\nu(\overline{x})$,

$$w(x) = \Phi(\overline{x}, 0)\int_0^s e^{b_0(\overline{x},0)(\eta - s)}d\eta, \qquad x \in V$$

Thanks to a cut off function h associated to V, we can extend w to Ω by $\varphi_1 = hw$ which satisfies (2.16) and (2.17).

Let $u_1 \in W^{2,1}(\Omega \times (0,\tau))$ be the solution, given by lemma 2.1, of the problem

$$\left\{ \begin{array}{c} Lu_1 = f \text{ on } \Omega \times (0,\tau) \\ b_0 u_1 + \langle A\nabla u_1, \nu\rangle = \Phi \text{ on } \partial\Omega \times (0,\tau) \end{array} \right. \tag{2.18}$$

Thus, taking into account (2.17) and the estimate given by Lemma 2.1 we obtain

$$\|u_1\|_{W_q^{2,1}(\Omega\times(0,T))} \leq c\left(\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))}+\|f\|_{L^q(\Omega\times(0,T))}\right) \tag{2.19}$$

Since b_0 is T periodic we have $u_1(.,T)-u_1(.,0)\in W_{q,B_0}^{2-\frac{2}{q}}(\Omega)$. Let $\varphi_2\in W_{q,B_0}^{2-\frac{2}{q}}(\Omega)$ be defined by

$$\varphi_2=(I-U)^{-1}(u_1(.,T)-u_1(.,0)) \tag{2.20}$$

from (2.19) we get

$$\|\varphi_2\|_{W_q^{2-\frac{2}{q}}(\Omega)}\leq c\left(\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))}+\|f\|_{L^q(\Omega\times(0,T))}\right) \tag{2.21}$$

with the constant c independent of Φ and f. Let $u_2\in W_q^{2,1}(\Omega\times(0,T))$ be the solution of the problem

$$\begin{cases} Lu_2=0 \text{ on } \Omega\times(0,\tau)\\ b_0u_2+\langle A\nabla u_2,\nu\rangle=0 \text{ on } \partial\Omega\times(0,\tau)\\ u(.,0)=\varphi_2 \text{ on } \Omega\end{cases} \tag{2.22}$$

taking into account (2.21), lemma 2.1 gives

$$\|u_2\|_{W_q^{2,1}(\Omega\times(0,T))} \leq c\left(\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))}+\|f\|_{L^q(\Omega\times(0,T))}\right). \tag{2.23}$$

Thus $u:=u_1+u_2$ solves (2.13) on $\Omega\times(0,\tau)$. From (2.20), u satisfies $u(.,0)=u(.,T)$. Also, (2.19) and (2.23) give (2.15). Moreover, it is easy to see that $u(x,t)-u(x,t+T)$ is identically zero for $0\leq t\leq\tau-T$. So, the T periodic extension of u (still denoted by u) solves (2.13) on $\Omega\times\mathbb{R}$. The uniqueness assertion of the lemma follows easily from the maximum principle.

Observe also that if $f\in C^{\gamma,\frac{\gamma}{2}}(\Omega\times\mathbb{R})$, then, taking into account Remark 2.3, the periodicity of u implies that $u\in C^{2+\gamma,1+\frac{\gamma}{2}}(\Omega\times\mathbb{R})$. From (2.14) we have

$$\|u(.,0)\|_{W_q^{2-\frac{1}{q}}(\Omega)}\leq c\left(\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))}+\|f\|_{C^{\gamma,\frac{\gamma}{2}}(\Omega\times\mathbb{R})}\right).$$

and so, remark 2.3 applied to (2.13) gives

$$\|u(.,T)\|_{C^{2+\gamma}(\overline{\Omega})}\leq c\left(\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))}+\|f\|_{C^{\gamma,\frac{\gamma}{2}}(\Omega\times\mathbb{R})}\right).$$

So, by the periodicity of u, the same estimate holds for $u(.,0)$. Then (2.15) follows from the estimate given in Lemma 2.2.■

Theorem 2.6. *Let a_0, b_0 and Φ be as in Theorem 2.5 and let* *be the operator defined by* $S_{b_0}\Phi = u_{|\partial\Omega\times\mathbb{R}}$, *where* u *is the* T *periodic solution of*

$$\left\{\begin{array}{c} Lu = 0 \text{ on } \Omega\times\mathbb{R} \\ b_0 u + \langle A\nabla u, \nu\rangle = \Phi \text{ on } \partial\Omega\times\mathbb{R} \\ u\ T \text{ periodic in } t \end{array}\right. \tag{2.25}$$

given by Theorem 2.5. Then S_{b_0} *is a compact strongly positive operator.*

Proof. Theorem 2.5 gives

$$\|u\|_{C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega}\times[0,T]\right)} \leq c\,\|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))} \tag{2.26}$$

From this estimate it follows the compactness of S_{b_0} and, taking into account the regularity of the solution of (2.25), the assertion about the strong positivity of S_{b_0} follows easily from the stated hypothesis on a_0 and b_0 and the maximum principle.■

Corollary 2.7. *Let* a_0, b_0, S_{b_0} *be as in Theorem 2.6 and let* ρ *be the spectral radius of* S_{b_0}. *Then* ρ *is positive and it is an algebraically simple eigenvalue of* S_{b_0} *with positive associated eigenfunctions. Moreover, no other eigenvalue of* S_{b_0} *has positive associated eigenfunctions* .

Proof. Follows from Theorem 2.6 and the Krein Rutman Theorem.■

3. A one parameter eigenvalue problem

Let $m \in W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times\mathbb{R})$, fixed from now on. In order to study principal eigenvalues for the weighted problem (1.2) we can assume, without lost of generality that $\|m\|_\infty \leq \frac{1}{2}$.

For ε positive and small enough (i.e. such that $1-\varepsilon\left(1-\|m\|_\infty\right) > 0$) and $\lambda > -\varepsilon$, let

$$S_{\lambda,m} : W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times\mathbb{R}) \to W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times\mathbb{R})$$

be the operator defined by $S_{\lambda,m}\Phi = u_{|\partial\Omega\times\mathbb{R}}$ where $u \in W_{q,T}^{2,1}(\Omega\times\mathbb{R})$ is the solution of the problem

$$\left\{\begin{array}{c} Lu = 0 \text{ on } \Omega\times\mathbb{R} \\ u + \lambda(1-m)\,u + \langle A\nabla u, \nu\rangle = \Phi \text{ on } \partial\Omega\times\mathbb{R} \\ u(x,t)\ \ T \text{ periodic in } t \end{array}\right. \tag{3.1}$$

and let $\mu_m(\lambda)$ be defined by

$$\frac{1}{1+\lambda+\mu_m(\lambda)} = \rho_{\lambda,m} \tag{3.2}$$

where $\rho_{\lambda,m}$ is the spectral radius of $S_{\lambda,m}$.

Remark 3.1. From corollary 2.7 it follows that $\mu_m(\lambda)$ can be characterized as the unique real number μ such that the problem

$$\left\{\begin{array}{c} Lu_\lambda = 0 \text{ on } \Omega\times\mathbb{R} \\ \langle A\nabla u_\lambda, \nu\rangle = \lambda m u_\lambda + \mu u_\lambda \text{ on } \partial\Omega\times\mathbb{R} \\ u_\lambda(x,t)\ \ T \text{ periodic in } t \end{array}\right. \tag{3.3}$$

has a positive solution $u_\lambda \in W_{q,T}^{2,1}(\Omega\times\mathbb{R})$. Since $\lambda m = (-\lambda)(-m)$, the above characterization of $\mu_m(\lambda)$ implies that $\mu_m(-\lambda) = \mu_{-m}(\lambda)$ for $\lambda\in(-\varepsilon,\varepsilon)$. We extend μ_m to the whole

Observe also that, for fixed $\lambda \in \mathbb{R}$, the solution space in $W^{2,1}_{q,T}(\Omega \times \mathbb{R})$ of the problem

$$\left\{ \begin{array}{c} Lu = 0 \text{ on } \Omega \times \mathbb{R} \\ \langle A \nabla u, \nu \rangle = \lambda m u + \mu_m(\lambda) u \text{ on } \partial\Omega \times \mathbb{R} \\ u(x,t) \ \ T \text{ periodic in } t \end{array} \right. \tag{3.4}$$

is one dimensional and it is contained in $C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega} \times \mathbb{R}\right)$. Moreover, by Corollary 2.7, positive solutions have a positive minimum on $\overline{\Omega} \times \mathbb{R}$.■

From the above characterization of $\mu_m(\lambda)$, our problem P_λ on principal eigenvalues is equivalent to find the zeroes of the function μ_m.

Lemma 3.2. *Suppose* $v \in C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega} \times \mathbb{R}\right)$ *such that*

$$\left\{ \begin{array}{c} Lv \geq 0 \text{ on } \Omega \times \mathbb{R} \\ \langle A \nabla v, \nu \rangle \geq \lambda m v + \overline{\mu} v \text{ on } \partial\Omega \times \mathbb{R} \\ v > 0 \text{ on } \Omega \times \mathbb{R} \\ v(x,t) \ \ T \text{ periodic in } t \end{array} \right. \tag{3.5}$$

for some $\lambda, \overline{\mu} \in \mathbb{R}$. *Then* $\mu_m(\lambda) \geq \overline{\mu}$. *If in addition either* $Lv > 0$ *or* $\langle A \nabla v, \nu \rangle > \lambda m v + \overline{\mu} v$ *then* $\mu_m(\lambda) > \overline{\mu}$.

Proof. We proceed by contradiction. Suppose that $\mu_m(\lambda) < \overline{\mu}$. From (3.5) we have, for r large enough

$$\left\{ \begin{array}{c} Lv \geq 0 \text{ on } \Omega \times \mathbb{R} \\ (r + \lambda(1-m)) v + \langle A \nabla v, \nu \rangle > (r + \lambda + \mu_m(\lambda)) v > 0 \text{ on } \partial\Omega \times \mathbb{R} \\ v(x,t) \ \ T \text{ periodic in } t \end{array} \right.$$

Then the maximum principle implies that v is bounded from below for some positive constant. Let u_λ be a positive solution of (3.4). It follows that there exists a positive constant c such that $u_\lambda \leq cv$ on $\overline{\Omega} \times \mathbb{R}$. Take c minimal with respect to this property and let $w = cv - u_\lambda$. Then $Lw \geq 0$, $(r + \lambda(1-m)) w + \langle A \nabla w, \nu \rangle > 0$ on $\partial\Omega \times \mathbb{R}$. Now, the maximum principle implies that $\min_{\overline{\Omega} \times [0,T]} w > 0$ and this leads to a contradiction with the choice of c. Finally, note that the above argument gives also the last assertion of the lemma.■

Lemma 3.3. μ_m *is a concave function.*

Proof. Let $\lambda_0, \lambda_1 \in \mathbb{R}$ and let $u_{\lambda_0}, u_{\lambda_1}$ be positive solutions of (3.4) for $\lambda = \lambda_0, \lambda_1$ respectively. For $\theta \in (0,1)$, let $u_\theta = u^\theta_{\lambda_0} u^{1-\theta}_{\lambda_1}$, so $u_\theta \in C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega} \times \mathbb{R}\right)$, u_θ is T periodic and $u_\theta(x,t) > 0$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$. For $w \in \mathbb{R}^N$ and $(x,t) \in \Omega \times \mathbb{R}$, let $\|w\|^2_{A(x,t)} = \langle A(x,t) w, w \rangle$. We recall that for regular $u, v \in C^{2,1}(\Omega \times \mathbb{R}) \to \mathbb{R}$ such that $u(x,t) > 0$ and $v(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$ and for $\beta \in \mathbb{R}$ it holds that $L(v^\beta) = \beta v^{\beta-1} Lv$

definition of $\|.\|_{A(x,t)}$, a direct computation shows that

$$\left(Lu_\theta\right)(x,t)$$
$$=\theta\left(1-\theta\right)\left\|\left[\left(\frac{u_{\lambda_1}}{u_{\lambda_0}}\right)^{\frac{1-\theta}{2}}\frac{\nabla u_{\lambda_0}}{u_{\lambda_0}^{1/2}}-\left(\frac{u_{\lambda_0}}{u_{\lambda_1}}\right)^{\frac{\theta}{2}}\frac{\nabla u_{\lambda_1}}{u_{\lambda_1}^{1/2}}\right](x,t)\right\|_{A(x,t)}^2$$

for $(x,t)\in\Omega\times\mathbb{R}$ and so $Lu_\theta\geq 0$ on $\Omega\times\mathbb{R}$. Another computation shows that

$$\langle A\nabla u_\theta,\nu\rangle=\left(\theta\lambda_0+\left(1-\theta\right)\lambda_1\right)mu_\theta+\left(\theta\mu_m\left(\lambda_0\right)+\left(1-\theta\right)\mu_m\left(\lambda_1\right)\right)u_\theta$$

on $\partial\Omega\times\mathbb{R}$. Then the lemma follows from lemma 3.2.■

Remark 3.4. Lemma 3.3 implies that μ_m is continuous. Moreover, taking into account Corollary 2.7, we can apply ([3] lemma 1.3) (proceeding e.g. as in [5], Remark 3.9 and Lemma 3.10) to obtain that $\mu_m(\lambda)$ is real analytic in λ for $\lambda>-\varepsilon$ for some small enough positive ε, and since $\mu_m(-\lambda)=\mu_{-m}(\lambda)$ get that μ_m is real analytic on the whole real line. Moreover, a positive solution u_λ for (3.4) can be chosen such that $\lambda\rightarrow u_{\lambda|\partial\Omega\times(0,T)}$ is a real analytic map from $\mathbb{R}$ into $W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}\left(\partial\Omega\times\mathbb{R}\right)$.

Observe also that if $a_0=0$ then $\mu_m(0)=0$ and that, in this case, the eigenfunctions associated for (3.4) are the constant functions. Finally, for the case $a_0>0$, applying Lemma 3.2 with $v=1$, $\lambda=0$ and $\overline{\mu}=0$ we obtain that $\mu_m(0)>0$ if .■

Lemma 3.5. *Let $m_1,m_2\in W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}\left(\partial\Omega\times\mathbb{R}\right)$. Suppose that $m_1<m_2$. Then $\mu_{m_1}(\lambda)>\mu_{m_2}(\lambda)$ for all $\lambda>0$.*

Proof. Since for $c\in\mathbb{R}-\{0\}$ $\mu_{cm_j}(\lambda)=\mu_{m_j}\left(\frac{\lambda}{c}\right)$, $j=1,2$, we can assume, without lost of generality, that $\|m_j\|_\infty<\frac{1}{2}$, $j=1,2$. For $\lambda>0$, let S_{λ,m_j} be defined as at the beginning of the section. Let $\Phi\in W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}\left(\partial\Omega\times\mathbb{R}\right)$ such that $\Phi>0$, let $u_j=S_{\lambda,m_j}\Phi$, $j=1,2$ and let $v=u_1-u_2$. A computation shows that v satisfies $Lv=0$ on $\Omega\times\mathbb{R}$ and $\langle A\nabla v,\nu\rangle+\lambda\left(1-m_1\right)v=\lambda\left(m_1-m_2\right)v$ on $\partial\Omega\times\mathbb{R}$, thus Theorem 2.6 implies $v<0$. Then $S_{\lambda,m_1}<S_{\lambda,m_2}$, this gives $\rho_{\lambda,m_1}<\rho_{\lambda,m_2}$ and so $\mu_{m_1}(\lambda)>\mu_{m_2}(\lambda)$.■

In order to make explicit the dependence on L, let us denote by $S_{L,\lambda,m}$ the operator $S_{\lambda,m}$ defined at the beginning of this section. We will denote also by $\mu_{m,L}$ the function μ_m. Let L_0 be the operator defined by $L_0u=\frac{\partial u}{\partial t}-div\left(A\nabla u\right)+\langle b,\nabla u\rangle$. We have

Lemma 3.6. *Suppose $a_0\neq 0$. Then $\mu_{m,L}(\lambda)>\mu_{m,L_0}(\lambda)$ for all $\lambda\in\mathbb{R}$.*

Proof. Suppose that $\lambda\geq 0$. Let $\Phi\in W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}\left(\partial\Omega\times\mathbb{R}\right)$, with $\Phi>0$, Let $k>\|m\|_\infty$, let $u=S_{L,\lambda,m}\Phi$, let $u_0=S_{L_0,\lambda,m}\Phi$ and let $v=u-u_0$. Then $L_0v=-a_0u<0$ on $\Omega\times\mathbb{R}$, $\left(\lambda\left(k-m\right)+1\right)v+\langle A\nabla v,\nu\rangle=0$ on $\partial\Omega\times(0,T)$ and $v(x,t)$ T periodic in t. Thus the maximum principle gives $v\leq 0$. So $S_{L,\lambda,m}<S_{L_0,\lambda,m}$. This implies $\mu_{m,L}(\lambda)>\mu_{m,L_0}(\lambda)$. Since $\mu_{m,L}(\lambda)=\mu_{-m,L}(-\lambda)$ (and similarly for L_0), the case $\lambda<0$ reduces to the previous

Remark 3.7. Suppose that $a_0 = 0$. Let $k \in \mathbb{R}$, $k > \sum_{1 \le j \le N} \|b_j\|_\infty$, let

$$S_k : W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times \mathbb{R}) \to W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times \mathbb{R})$$

be defined by (2.14) and (2.15) taking there $b_0 = k$ and let ρ_k be its spectral radius. Since $\Phi = 1$ is a positive eigenfunction associated to the eigenvalue $\frac{1}{k}$, the Krein Rutman Theorem asserts that $\rho_k = \frac{1}{k}$. Thus, also by Krein Rutman, there exists a positive eigenvector Ψ for the adjoint operator S_k^* satisfying $S_k^*\Psi = \Psi$. Moreover, such a Ψ is unique up a multiplicative constant.■

Lemma 3.8. *Suppose that $a_0 = 0$ and let S_k, Ψ be as in remark 3.7. Then $\mu_m'(0) = -\frac{\langle \Psi, m\rangle}{\langle \Psi, 1\rangle}$.*

Proof. For $\lambda \in \mathbb{R}$, let u_λ be a solution of (3.4) such that $\lambda \to u_\lambda$ is real analytic and such that $u_0 = 1$.

$$\begin{cases} Lu_\lambda = 0 \text{ on } \Omega \times \mathbb{R} \\ ku_\lambda + \langle A\nabla u_\lambda, \nu\rangle = (\lambda m + \mu_m(\lambda) + k)\, u_\lambda \text{ on } \partial\Omega \times \mathbb{R} \\ u_\lambda(x,t)\ T \text{ periodic in } t \end{cases}$$

we get $u_\lambda = \lambda S_k(m u_\lambda) + (\mu_m(\lambda) + k) S_k u_\lambda$ and so

$$\lambda \langle \Psi, m u_\lambda\rangle + \mu_m(\lambda) \langle \Psi, u_\lambda\rangle = 0.$$

Taking the derivative with respect to λ at $\lambda = 0$ and using that $\mu_m(0) = 0$ and that $u_0 = 1$, the lemma follows.■

For $\Phi, f \in W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times \mathbb{R})$, let $\langle i(\Phi), f\rangle = \int_{\partial\Omega \times (0,T)} \Phi f$. So $i(\Phi) \in \left(W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times \mathbb{R})\right)'$. We have

Lemma 3.9. *Suppose $a_0 = 0$ and let k, S_k, Ψ be as in remark 3.7. Then*

(i): for $f \in W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times \mathbb{R})$ we have $S_k^* f = i\left(v_{|\partial\Omega\times\mathbb{R}}\right)$ where v is the T periodic solution of the problem

$$\begin{cases} \frac{\partial v}{\partial t} + div(A\nabla v) + \langle b, \nabla v\rangle + div(b)\, v = 0 \text{ on } \Omega \times \mathbb{R} \\ \langle A\nabla v, \nu\rangle + (k + \langle b, \nu\rangle)\, v = f \text{ on } \partial\Omega \times \mathbb{R} \\ v(x,t)\ T \text{ periodic in } t \end{cases} \tag{3.6}$$

(ii) $\Psi \in C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega} \times \mathbb{R}\right)$ and $\min_{\overline{\Omega}\times\mathbb{R}} \Psi > 0$. Moreover, Ψ can be characterized as the (unique up a multiplicative constant) solution of the T periodic problem

$$\begin{cases} \frac{\partial \Psi}{\partial t} + div(A\nabla \Psi) + \langle b, \nabla \Psi\rangle + div(b)\, \Psi = 0 \text{ on } \Omega \times \mathbb{R} \\ \langle A\nabla \Psi, \nu\rangle + \langle b, \nu\rangle\, \Psi = 0 \text{ on } \partial\Omega \times \mathbb{R} \\ \Psi(x,t)\ T \text{ periodic in } t. \end{cases}$$

Proof. Observe that, for $f \in W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times \mathbb{R})$, (3.6) has a unique T periodic solution that $\langle i(v), \Phi\rangle = \int_{\partial\Omega\times(0,T)} S(\Phi) f$, i.e., that

$$\int_{\partial\Omega\times(0,T)} v\Phi = \int_{\partial\Omega\times(0,T)} fu, \tag{3.7}$$

where u is the T periodic solution of the problem

$$\begin{cases} \frac{\partial u}{\partial t} - div\,(A\nabla u) + \langle b, \nabla u\rangle = 0 \text{ on } \Omega\times\mathbb{R} \\ ku + \langle A\nabla u, \nu\rangle = \Phi \text{ on } \partial\Omega\times\mathbb{R} \\ u(x,t) \;\; T \text{ periodic in } t \end{cases} \tag{3.8}$$

Multiplying equation (3.6) by u, equation (3.8) by v, adding, and integrating on $\Omega\times(0,T)$ we get

$$0 = \int_{\Omega\times(0,T)} \frac{\partial (uv)}{\partial t}$$

$$+ \int_{\Omega\times(0,T)} \left[div\,(uA\nabla v) - div\,(vA\nabla u) + v\langle b, \nabla u\rangle + u\langle b, \nabla v\rangle + uvdiv\,(b)\right]$$

The first integral vanishes by the periodicity. Taking into account the boundary conditions of (3.6) and (3.8), an application of the divergence theorem gives (3.7). To prove (ii), consider the operator

$$\widetilde{S} : W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times\mathbb{R}) \to W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times\mathbb{R})$$

defined by $\widetilde{S}f = v_{|\partial\Omega\times\mathbb{R}}$ where v is the solution of (3.6). Note that, via the change of variable $t = T - \tau$, Theorem 2.6 gives that $\widetilde{S}$ is a compact and strongly positive operator . Thus $\widetilde{S}$ has a positive spectral radius which is an eigenvalue with a positive T periodic eigenfunction h associated, that, by Theorem 2.5, belongs to $C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega}\times\mathbb{R}\right)$. Moreover, $\min_{\overline{\Omega}\times\mathbb{R}} h > 0$. Let Ψ be as in Remark 3.7. By Lemma 3.9 h is a positive eigenvector for S^* and so, by Krein Rutman, we get $\Psi = ch$ for some positive constant $c > 0$. Thus (ii) holds.■

We set

$$P(m) = \int_0^T \max_{x\in\partial\Omega} m(x,t)\,dt, \tag{3.9}$$

$$N(m) = \int_0^T \min_{x\in\partial\Omega} m(x,t)\,dt \tag{3.10}$$

Proceeding as in [2] it can be shown that if $P(m) > 0$ then there exists a T periodic curve $\Gamma \in C^2(\mathbb{R}, \partial\Omega)$ such that

$$\int_0^T m(\Gamma(t), t)\,dt > 0$$

we fix, from now on, such a Γ.

For $p \in \partial\Omega$, let $T_p(\partial\Omega)$ denotes the tangent space to $\partial\Omega$ at p and let $\exp_p : T_p(\partial\Omega) \to$

satisfying $\sigma_{p,X}(0) = p$, $\frac{d}{ds}(\sigma_{p,X}(s)) = X$. Since $\partial\Omega$ is of class $C^{2+\gamma}$, $\exp_p$ is a well defined map..

Lemma 3.10. *For δ positive and small enough, there exists*

$$\Lambda \in C^1\left((-\delta,\delta)^N \times \mathbb{R}, \mathbb{R}^{N+1}\right)$$

such that Λ is a diffeomorphism from $(-\delta,\delta)^N \times \mathbb{R}$ onto an open neighborhood $W_\delta \subset \mathbb{R}^N \times \mathbb{R}$ of the set $\{(T(t), t) : t \in \mathbb{R}\}$ satisfying

(1) $\Lambda\left((-\delta,\delta)^{N-1} \times (0,\delta) \times \mathbb{R}\right) = W_\delta \cap (\Omega \times \mathbb{R})$

(2) $\Lambda\left((-\delta,\delta)^{N-1} \times \{0\} \times \mathbb{R}\right) = W_\delta \cap (\partial\Omega \times \mathbb{R})$

(3) $\Lambda(0,t) = (\Gamma(t), t)$

(4) $\Lambda(.,t)$ *is T periodic in t.*

Moreover, $\Lambda : (-\delta,\delta)^N \times \mathbb{R} \to W_\delta$ and its inverse $\Theta : W_\delta \to (-\delta,\delta)^N \times \mathbb{R}$ are of class $C^{2,1}$ on their respective domains.

Proof. The map $t \to \nu(\Gamma(t))$ is T periodic and belongs to the class $C^{1+\gamma}(\mathbb{R}, \mathbb{R}^N)$. Then there exists a $C^{1+\gamma}$ and T periodic map $t \to A(t)$ from $\mathbb{R}$ into $SO(N)$ such that $A(t)\nu(\Gamma(0)) = \nu(\Gamma(t))$, $t \in \mathbb{R}$. Let $\{X_{1,0}, ..., X_{N-1,0}\}$ be a basis of $T_{\Gamma(0)}(\partial\Omega)$ and let $X_j(t) = A(t) X_{j,0}$, $j = 1, 2, ...N-1$. Thus each X_j is a T periodic map, $X_j \in C^{1+\gamma}(\mathbb{R}, \mathbb{R}^N)$ and for each t, $\{X_1(t), ..., X_{N-1}(t)\}$ is a basis of $T_{\Gamma(t)}(\partial\Omega)$. For δ positive and small enough, and for $(s,t) \in (-\delta,\delta)^N \times \mathbb{R}$, let

$$x(s,t) = \exp_{\Gamma(t)}\left(\sum_{1\le j\le N-1} s_j X_j(t)\right) - s_N \nu\left(\exp_{\Gamma(t)}\left(\sum_{1\le j\le N-1} s_j X_j(t)\right)\right) \tag{3.11}$$

and let

$$\Lambda(s,t) = (x(s,t), t) \tag{3.12}$$

From well known properties of the exponential map it follows easily that, for δ small enough, $(s,t) \to \Lambda(s,t)$ is a $C^{2,1}$ map which satisfies the properties required by the lemma.■

Let $\delta, \Lambda, \Theta, W_\delta$ be as in Lemma 3.10, $\Theta(x,t) = (\Theta_1(x,t), ..., \Theta_{N+1}(x,t))$. Observe that, since Θ_N vanishes identically on $W_\delta \cap (\partial\Omega \times \mathbb{R})$, we have

$$\nabla\Theta_N = -g\nu \qquad \text{on } W_\delta \cap (\partial\Omega \times \mathbb{R}) \tag{3.13}$$

for some $g \in C^1(W_\delta \cap (\partial\Omega \times \mathbb{R}))$ satisfying $g(x,t) \neq 0$ for all $(x,t) \in W_\delta \cap (\partial\Omega \times \mathbb{R})$. Moreover,

$$\Theta'(\Gamma(t), t)\Lambda'(0,t) = Id$$

(where Λ' and Θ' denotes the respective $(N+1) \times (N+1)$ jacobian matrix of Λ and Θ respectively), thus, considering the (N,N) entries in this equality and using (3.13) and that $\frac{\partial\Lambda_N}{\partial s_N}_{|(0,t)} = -\nu(\Gamma(t))$ we get

Lemma 3.11. *Suppose that* $P(m) > 0$. *Then* $\lim_{\lambda\to\infty} \mu_m(\lambda) = -\infty$

Proof. Let $\delta, \Lambda, \Theta, W_\delta$ be as in Lemma 3.10. Let $Q_{T,\delta} = (-\delta,\delta)^{N-1} \times [0,\delta) \times (0,T)$ and let $D_{T,\delta} = \Lambda(Q_{T,\delta}) \subset \overline{\Omega} \times (0,T)$. If $f : D_\delta \to \mathbb{R}$ (respectively $f : D_\delta \cap (\partial\Omega \times \mathbb{R}) \to \mathbb{R}$) let $f^{\#} : Q_{T,\delta} \to \mathbb{R}$ (resp. $f^{\#} : (-\delta,\delta)^{N-1} \times \{0\} \times (0,T) \to \mathbb{R}$) be defined by $f^{\#} = f \circ \Lambda$.

For $\lambda > 0$, let $u_\lambda \in C^{2+\gamma,1+\frac{\gamma}{2}}\left(\overline{\Omega} \times \mathbb{R}\right)$ be a positive T periodic solution of (3.4), since $u_\lambda = u_\lambda^{\#} \circ \Theta$ on D_δ, the equation $Lu_\lambda = 0$ on D_δ gives

$$\frac{\partial u_\lambda^{\#}}{\partial t} - div\left(A^{\#}\nabla u_\lambda^{\#}\right) + \left\langle b^{\#}, \nabla u_\lambda^{\#}\right\rangle + a_0^{\#} u_\lambda^{\#} = 0 \text{ on } Q_{T,\delta} \tag{3.15}$$

where $A^{\#}$ is the $N \times N$ symmetric and positive matrix whose (i,j) entry is

$$a_{ij}^{\#}(s,t) = \sum_{1\le l.r\le N} a_{lr}(\Lambda(s,t)) \frac{\partial\Theta_i}{\partial x_l}(\Lambda(s,t)) \frac{\partial\Theta_j}{\partial x_r}(\Lambda(s,t)) \tag{3.16}$$

and where $b^{\#} = \left(b_1^{\#}, ..., b_N^{\#}\right)$ with each $b_j^{\#}$ belonging to $C(Q_{T,\delta}, \mathbb{R})$ and independent of λ.

If $\nu(x,t) = (\nu_1(x,t), ..., \nu_N(x,t))$, the boundary condition

$$\langle A\nabla u_\lambda, \nu\rangle = \lambda m u_\lambda + \mu_m(\lambda) u_\lambda \qquad \text{on } (\partial\Omega \times (0,T)) \cap D_\delta$$

transforms into

$$\sum_{1\le i,j,l\le N} a_{ij}(\Lambda(s,t)) \frac{\partial u_\lambda^{\#}}{\partial s_l}(s,t) \frac{\partial\Theta_l}{\partial x_j}(\Lambda(s,t)) \nu_i(\Lambda(s,t)) \tag{3.17}$$
$$= \lambda m^{\#}(s,t) u_\lambda^{\#}(s,t) + \mu_m(\lambda) u_\lambda^{\#}(s,t)$$

for all $(s,t) \in (-\delta,\delta)^{N-1} \times \{0\} \times (0,T)$.

Let g be given by (3.13). Taking into account (3.16) and (3.13), from (3.17) we get

$$\left\langle A^{\#}\nabla u_\lambda^{\#}, e_N\right\rangle = -\lambda m^{\#} g^{\#} u_\lambda^{\#} - \mu_m(\lambda) g^{\#} u_\lambda^{\#} \tag{3.18}$$
$$\text{on } (-\delta,\delta)^{N-1} \times \{0\} \times (0,T)$$

where ∇ denotes the gradient in the variables $s_1, ..., s_N$ and $e_N = (0, ..., 0, 1)$.

Observe that $\int_0^T m^{\#}(0,t)\,dt = \int_0^T m(\Lambda(0,t))\,dt = \int_0^T m(\Gamma(t),t)\,dt > 0$ and thus, by (3.15) $g^{\#}(0,t) = 1$. Since $m^{\#}$ and $g^{\#}$ are continuous on $(-\delta,\delta)^{N-1} \times \{0\} \times \mathbb{R}$ we have, for η small enough

$$\int_0^T \int_{\{\sigma\in\mathbb{R}^{N-1}:|\sigma|<\eta\}} \left(m^{\#} g^{\#}\right)(\sigma,0,t)\,d\sigma dt > 0 \tag{3.19}$$

and

$$\int_0^T \int_{\{\sigma\in\mathbb{R}^{N-1}:|\sigma|<\eta\}} g^{\#}(\sigma,0,t)\,d\sigma dt > 0 \tag{3.20}$$

Let $\beta \in (0,\eta)$, let $h \in C^\infty(\mathbb{R})$ such that $0 \le h \le 1$, $h(\xi) = 1$ for $\xi \le -\beta$, $h(\xi) = 0$ f

(3.19) and (3.20) it is easy to see that we can pick β small enough such that

$$\int_0^T \int_{\{\sigma\in\mathbb{R}^{N-1}:|\sigma|<\eta\}} \left(G^2 m^{\#} g^{\#}\right)(\sigma,0,t)\, d\sigma dt > 0 \tag{3.21}$$

and

$$\int_0^T \int_{\{\sigma\in\mathbb{R}^{N-1}:|\sigma|<\eta\}} \left(G^2 g^{\#}\right)(\sigma,0,t)\, d\sigma dt > 0 \tag{3.22}$$

Let $B_{\eta,T} = \left\{(s,t)\in\mathbb{R}^N\times(0,T) : |s|<\eta,\ s_N\geq 0\right\}$. We multiply (3.15) by $\frac{G^2}{u_\lambda^{\#}}$ and then, integrating on $B_{\eta,T}$ and taking into account that $u_\lambda^{\#}(.,0) = u_\lambda^{\#}(.,T)$ and that G does not depends on t, we get

$$\int_{B_{\eta,T}} \left[-\frac{G^2}{u_\lambda^{\#}} div\left(A^{\#}\nabla u_\lambda^{\#}\right) + \frac{G^2}{u_\lambda^{\#}}\left\langle b^{\#},\nabla u_\lambda^{\#}\right\rangle + a_0^{\#}G^2\right] = 0 \tag{3.23}$$

Let $v_\lambda^{\#} = -\log u_\lambda^{\#}$. Thus $v_\lambda^{\#}\in C^{2,1}\left(B_{\eta,T}\right)$. A computation gives that

$$\begin{gathered} -\frac{G^2}{u_\lambda^{\#}} div\left(A^{\#}\nabla u_\lambda^{\#}\right) \\ = div\left(G^2 A^{\#}\nabla v_\lambda^{\#}\right) - 2\left\langle A^{\#}G\nabla v_\lambda^{\#},\nabla G\right\rangle - \left\langle A^{\#}G\nabla v_\lambda^{\#}, G\nabla v_\lambda^{\#}\right\rangle \text{ on } B_{\eta,T} \end{gathered}$$

Also,

$$\frac{G^2}{u_\lambda^{\#}}\left\langle b^{\#},\nabla u_\lambda^{\#}\right\rangle = -2\left\langle GA^{\#}\nabla v_\lambda^{\#}, \frac{1}{2}G\left(A^{\#}\right)^{-1}b^{\#}\right\rangle \text{ on } B_{\eta,T}$$

so, from (3.24) the divergence theorem gives

$$\begin{aligned} &\int_0^T \int_{\{\sigma\in\mathbb{R}^{N-1}:|\sigma|<\eta\}} G^2\left\langle A^{\#}\nabla v_\lambda^{\#},\nu\right\rangle \\ &= -2\int_{B_{\eta,T}} \left\langle GA^{\#}\nabla v_\lambda^{\#}, \nabla G + \frac{1}{2}G\left(A^{\#}\right)^{-1}b^{\#}\right\rangle \\ &\quad - \int \left\langle A^{\#}G\nabla v_\lambda^{\#}, G\nabla v_\lambda^{\#}\right\rangle - \int a_0^{\#}G^2 \end{aligned} \tag{3.24}$$

For $w \in \mathbb{R}^N$ and $(s,t) \in B_{\eta,T}$ let $\|w\|_{A^{\#}(s,t)} = \left\langle A^{\#}(s,t)\, w, w \right\rangle$. Taking into account the boundary condition (3.19) and that $G(s) = 0$ for $|s| = \eta$ from (3.24) we get

$$\begin{aligned}
&\mu_m(\lambda) \int_0^T \int_{\{\sigma \in \mathbb{R}^{N-1} : |\sigma| < \eta\}} \left(G^2 g^{\#}\right)(\sigma, 0, t)\, d\sigma dt \\
&= -\lambda \int_0^T \int_{\{\sigma \in \mathbb{R}^{N-1} : |\sigma| < \eta\}} \left(G^2 g^{\#} m^{\#}\right)(\sigma, 0, t)\, d\sigma dt \\
&- \int_{B_{\eta,T}} \left\| \left(G \nabla v_\lambda^{\#} + \nabla G + \frac{1}{2} G \left(A^{\#}\right)^{-1} b^{\#} \right)(s,t) \right\|_{A^{\#}(s,t)}^2 ds dt \\
&+ \int_{B_{\eta,T}} \left\| \left(\nabla G + \frac{1}{2} A^{\#} G b^{\#} \right)(s,t) \right\|_{A^{\#}(s,t)}^2 ds dt + \int_{B_{\eta,T}} a_0^{\#} G^2 \\
&\leq -\lambda \int_0^T \int_{\{\sigma \in \mathbb{R}^{N-1} : |\sigma| < \eta\}} \left(G^2 g^{\#} m^{\#}\right)(\sigma, 0, t)\, d\sigma dt \\
&+ \int_{B_{\eta,T}} \left\| \left(\nabla G + \frac{1}{2} A^{\#} G b^{\#} \right)(s,t) \right\|_{A^{\#}(s,t)}^2 ds dt + \int_{B_{\eta,T}} a_0^{\#} G^2 .
\end{aligned}$$

From this inequality, (3.21) and (3.22) the lemma follows.■

4. Principal eigenvalues for periodic parabolic Steklov problems

Let $P(m)$ and $N(m)$ be defined by (3.9) and (3.10) respectively. We have

Theorem 4.1. *Suppose either $a_0 > 0$ and $P(m) > 0$ (respectively $a_0 > 0$ and $N(m) < 0$) or $a_0 = 0, P(m) > 0$ and $\langle \Psi, m \rangle < 0$ (resp. $a_0 = 0$, $N(m) < 0$ and $\langle \Psi, m \rangle > 0$) with Ψ defined as in remark 3.7. Then there exists a unique positive (resp. negative) principal eigenvalue for (1.2) and the associated eigenspace is one dimensional.*

Proof. Suppose $a_0 = 0$ and $P(m) > 0$, $\langle \Psi, m \rangle < 0$. Since $\mu_m(0) = 0$ and, by Lemma 3.8, $\mu_m'(0) > 0$ the existence of a positive principal eigenvalue $\lambda = \lambda_1(m)$ for (1.2) follows from Lemma 3.11. Since μ_m does not vanish identically, the concavity of μ_m gives the uniqueness of the positive principal eigenvalue.

Moreover, if u, v are solutions in $C^{2+\gamma, 1+\frac{\gamma}{2}}\left(\overline{\Omega} \times \mathbb{R}\right)$ for (1.2), then, by the facts stated in remark 3.1, $u = cv$ on $\partial\Omega \times \mathbb{R}$ for some constant c. Since $L(u - cv) = 0$ on $\Omega \times \mathbb{R}$, $u - cv = 0$ on $\partial\Omega \times \mathbb{R}$, and $u - cv$ is T periodic it follows easily from the maximum principle that $u = cv$ on $\Omega \times \mathbb{R}$.

If $a_0 > 0$ then (by remark 3.4) $\mu_m(0) > 0$ and so the existence follows from Lemma 3.11. The other assertions of the theorem follows as in the case $a_0 = 0$. Taking into account that $\mu_m(-\lambda) = \mu_{-m}(\lambda)$ and that $N(m) = -P(-m)$, the assertions about negative principal eigenvalues follow from the previous cases.■

Lemma 4.2. *Suppose that $a_0 = 0$. Then for all $\lambda > 0$ we have*

Proof. We consider first the case $m \geq 0$. Let $\lambda > 0$ and let u_λ be a positive solution of (3.4) normalized by $\|u_\lambda\|_\infty = 1$. From

$$\left\{ \begin{array}{c} \frac{\partial u_\lambda}{\partial t} - div\,(A\nabla u_\lambda) + \langle b, \nabla u_\lambda\rangle = 0 \text{ on } \Omega \times (0,T)\,, \\ \langle A\nabla u_\lambda, \nu\rangle = \lambda m u_\lambda + \mu_m(\lambda)\, u_\lambda \text{ on } \partial\Omega \times (0,T) \\ u_\lambda(.,0) = u_\lambda(.,T) \end{array} \right. \tag{4.1}$$

and since $\langle b, \nabla u_\lambda\rangle = div\,(u_\lambda b) - u_\lambda div\,(b)\,,$ integrating (4.1) on $\Omega \times (0,T)$ and taking into account the periodicity of u_λ and the boundary conditions, the divergence theorem gives

$$\begin{gathered} \mu_m(\lambda) \int_{\partial\Omega\times(0,T)} u_\lambda \\ = -\lambda \int_{\partial\Omega\times(0,T)} m u_\lambda + \int_{\partial\Omega\times(0,T)} u_\lambda \langle b, \nu\rangle - \int_{\Omega\times(0,T)} u_\lambda div\,(b) \end{gathered}$$

Since $m \geq 0$ and $|u_\lambda| \leq 1$ we have $\int_{\partial\Omega\times(0,T)} m u_\lambda \leq P(m)\,|\partial\Omega|\,,$ also $|u_\lambda div\,(b)| \leq \|div\,(b)\|_\infty$ and $|u_\lambda \langle b, \nu\rangle| \leq \|b\|_\infty$. Thus

$$\begin{gathered} T\,|\partial\Omega|\,\mu_m(\lambda) \geq \mu_m(\lambda) \int_{\partial\Omega\times(0,T)} u_\lambda \\ \geq -\lambda P(m)\,|\partial\Omega| - T\,|\partial\Omega|\,\|b\|_\infty - \|div\,(b)\|_\infty\,|\Omega|\,T \end{gathered}$$

so the lemma holds for $m \geq 0$. For the general case, pick $k \in \mathbb{R}$, $k > \|m\|_\infty$ taking into account that $P(m+k) = P(m) + kT$ and that $\mu_{m+k}(\lambda) = \mu_m(\lambda) - k\lambda$ the lemma follows from the previous case applied to $m+k$ instead of m.■

Corollary 4.3. *Suppose $a_0 = 0$. Then $\lim_{\lambda\to\infty} \mu_m'(\lambda) \geq -\frac{P(m)}{T}$.*

Proof. Suppose that $P(m) \neq 0$, then Lemmas 3.11 and 4.2 imply that $\lim_{\lambda\to\infty} \mu_m(\lambda) = \pm\infty$. Also, μ_m is concave, thus there exists $\lim_{\lambda\to\infty} \mu_m'(\lambda)$. Then the L'Hopital rule gives $\lim_{\lambda\to\infty} \mu_m'(\lambda) = \lim_{\lambda\to\infty} \frac{\mu_m(\lambda)}{\lambda} \geq -\frac{P(m)}{T}$, the last inequality by Lemma 4.2. If $P(m) = 0$ and if $\mu_m(\lambda) < 0$ for some $\lambda > 0$ then, since $\mu_m(0) = 0$, the concavity of μ_m implies that $\lim_{\lambda\to\infty} \mu_m(\lambda) = -\infty$ and the above argument applies. If $\mu_m(\lambda) \geq 0$ for all $\lambda > 0$ the concavity implies that $\mu_m'(\lambda) \geq 0$ for all $\lambda > 0$ and so the corollary is also true in this case.■

Lemma 4.4. *Suppose $a_0 = 0$ and let Ψ be as in remark 3.7. Then $P(m) < 0$ implies $\langle \Psi, m\rangle < 0$.*

Proof. Suppose $P(m) < 0$. By corollary 4.3 we have $\lim_{\lambda\to\infty} \mu_m'(\lambda) > 0$ Then, since μ_m is concave we have $\mu_m'(0) > 0$ and so $\langle \Psi, m\rangle < 0$.■

Lemma 4.5. *Suppose $a_0 = 0$. Then μ_m vanishes identically if and only if $P(m) = \langle \Psi, m\rangle = 0$.*

Proof. Suppose that μ_n vanishes identically. Lemma 3.8 gives that $\langle \Psi, m\rangle = 0$. Also, by Lemma 3.11, we have $P(m) \leq 0$. Suppose $P(m) < 0$ and let $\widetilde{m}(t) = \max_{x\in\partial\Omega} m(x,t)\,.$ Since $m \in C(\partial\Omega \times \mathbb{R})$ it follows easily that the $\widetilde{m} \in C[0,T]\,.$ Take ε such that $0 < \varepsilon T <$

2.4, $\langle \Psi, m^* \rangle < 0$. Thus $\mu'_{m^*}(0) > 0$ and then, since $m < m^*$, for λ positive and small enough we have $\mu_m(\lambda) \geq \mu_{m^*}(\lambda) > 0$ contradicting our original assumption.

Suppose now that $P(m) = \langle \Psi, m \rangle = 0$. Then $\mu'_m(0) = 0$ and also, by corollary 4.3, $\lim_{\lambda\to\infty} \mu'_m(\lambda) \geq 0$. Then the concavity of μ_m implies that μ'_m vanishes identically on the positive axis, and so, since $\mu_m(0) = 0$ the same is true for μ_m and since μ_m is analytic, vanishes on the whole line.■

Theorem 4.6. *Suppose $a_0 = 0$ and that μ_m does not vanish identically. Then the conditions $P(m) > 0$ and $\langle \Psi, m \rangle < 0$ (respectively $N(m) < 0$ and $\langle \Psi, m \rangle > 0$) are necessary for the existence of a positive (resp. negative) principal eigenvalue for (1.2).*

Proof. Suppose that $\mu_m(\lambda_1) = 0$ for some $\lambda_1 > 0$. Since $\mu_m(0) = 0$ and μ_m is concave we must have $\mu'_m(0) > 0$ and so $\langle \Psi, m \rangle < 0$. To see that $P(m) > 0$ we proceed by contradiction. Suppose $P(m) \leq 0$. Corollary 4.3 implies that $\lim_{\lambda\to\infty} \mu'_m(\lambda) \geq 0$ and so, since μ_m is concave, we have $\mu'_m(\lambda) \geq 0$ for all $\lambda > 0$ and then, since $\mu'_m(0) > 0$, μ_m cannot vanish on the positive axis.■

Theorem 4.7. *Suppose $a_0 > 0$. Then the condition $P(m) > 0$ (respectively $N(m) < 0$) is necessary for the existence of a positive (resp. negative) principal eigenvalue for (1.2).*

Proof. For $\lambda > 0$, by lemmas 3.5 and 3.6 we have $\mu_{m,L}(\lambda) \geq \mu_{\widetilde{m},L}(\lambda) \geq \mu_{\widetilde{m},L_0}(\lambda)$. Suppose that $P(m) \leq 0$. Corollary 4.3 gives $\lim_{\lambda\to\infty} \mu'_{\widetilde{m},L_0}(\lambda) \geq 0$ and so $\mu_{\widetilde{m},L_0}(\lambda) \geq 0$ for all $\lambda > 0$. Since $\mu_m(0) > 0$ the concavity of μ_m implies that μ_m cannot vanish on the positive axis.■

Theorem 4.8. *Let $\lambda \in \mathbb{R}$ such that $\mu_m(\lambda) > 0$. Then for all $h \in W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times \mathbb{R})$ the problem*

$$
\begin{gathered}
Lu = 0 \text{ in } \Omega \times \mathbb{R}, \\
\langle A\nabla u, \nu \rangle = \lambda m u + h \quad \text{on } \partial\Omega \times \mathbb{R} \\
u(x,t) \;\; T \text{ periodic in } t
\end{gathered}
\tag{4.2}
$$

has a unique solution. Moreover $h > 0$ implies that $\min_{\overline{\Omega}\times(0,T)} u > 0$.

Proof. Let k, $S_{\lambda,k,m}$ and $\rho_{\lambda,k,m}$ be as in remark 3.1. Since $\mu_m(\lambda) > 0$ we have $\rho_{\lambda,k,m} < \frac{1}{\lambda k+1}$ and so , since $S_{\lambda,k,m}$ is a strongly positive operator, $\left(\frac{1}{\lambda k+1} I - S_{\lambda,k,m}\right)^{-1}$ is a well defined and positive operator. (4.2) is equivalent to $u = (\lambda k + 1) S_{\lambda,k,m} u + S_{\lambda,k,m} h$, i.e. to

$$
u = \frac{1}{\lambda k + 1} S_{\lambda,k,m} \left(\frac{1}{\lambda k + 1} I - S_{\lambda,k,m} \right)^{-1} h.
$$

So the theorem follows.■

Let $\lambda_1(m)$ (respectively $\lambda_{-1}(m)$) be the positive (resp. negative) principal eigenvalue for the weight m with the convention that $\lambda_1(m) = +\infty$ (resp. $\lambda_{-1}(m) = -\infty$) if there not exists such a principal eigenvalue. From the properties of μ_m we obtain, as immediate

Corollary 4.9. *Assume that* $a_0 > 0$. *Then the interval* $(\lambda_{-1}(m), \lambda_1(m))$ *does not contains eigenvalues for problem (1.2). If* $a_0 = 0$, *the same is true for the intervals* $(\lambda_{-1}(m), 0)$ *and* $(0, \lambda_1(m))$.

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