# Oblique projections and abstract splines

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**Abstract**

Given a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$ and a bounded linear operator $A \in L(\mathcal{H})$ which is positive, consider the set of all $A$-selfadjoint projections onto $\mathcal{S}$

$$\mathcal{P}(A,\mathcal{S}) = \{Q \in L(\mathcal{H}) : Q^2 = Q \ , \ \ Q(\mathcal{H}) = \mathcal{S} \ , \ \ AQ = Q^*A\}.$$

In addition, if $\mathcal{H}_1$ is another Hilbert space, $T : \mathcal{H} \to \mathcal{H}_1$ is a bounded linear operator such that $T^*T = A$ and $\xi \in \mathcal{H}$, consider the set of $(T,\mathcal{S})$ spline interpolants to $\xi$:

$$sp\ (T,\mathcal{S},\xi) = \{\eta \in \xi + \mathcal{S} : \|T\eta\| = \min_{\sigma\in\mathcal{S}} \|T(\xi+\sigma)\|\}.$$

A strong relationship exists between $\mathcal{P}(A,\mathcal{S})$ and $sp\ (T,\mathcal{S},\xi)$. In fact, $\mathcal{P}(A,\mathcal{S})$ is not enpty if and only if $sp\ (T,\mathcal{S},\xi)$ is not empty for every $\xi \in \mathcal{H}$. In this case, for any $\xi \in \mathcal{H}\setminus\mathcal{S}$ it holds

$$sp\ (T,\mathcal{S},\xi) = \{(1-Q)\xi : Q \in \mathcal{P}(A,\mathcal{S})\}$$

and for any $\xi \in \mathcal{H}$ the unique vector of $sp\ (T,\mathcal{S},\xi)$ with minimal norm is $(1-P_{A,\mathcal{S}})\xi$, where $P_{A,\mathcal{S}}$ is a distiguished element of $\mathcal{P}(A,\mathcal{S})$. These results offer a generalization to arbitrary operators of several theorems by de Boor, Atteia, Sard and others, which hold for closed range operators.

## 1 Introduction

Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_1$, $T \in L(\mathcal{H},\mathcal{H}_1)$, $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace and $\xi \in \mathcal{H}$, an abstract spline or a $(T,\mathcal{S})$-spline interpolant to $\xi$ is any element of the set

$$sp\ (T,\mathcal{S},\xi) = \{\eta \in \xi + \mathcal{S} : \|T\eta\| = \min_{\sigma\in\mathcal{S}} \|T(\xi+\sigma)\|\}.$$

Observe that $A = T^*T = |T|^2$, as a positive bounded operator on $\mathcal{H}$, defines a semi-inner product $\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ by $\langle\xi,\eta\rangle_A = \langle A\xi,\eta\rangle$, $\xi,\eta \in \mathcal{H}$ and a corresponding seminorm $\|\cdot\|_A : \mathcal{H} \to \mathbb{R}^+$ given by $\|\eta\|_A = \langle\eta,\eta\rangle_A^{1/2} = \langle A\eta,\eta\rangle_A^{1/2} = \|T\eta\|$. Thus, if for any $\eta \in \mathcal{H}$ we consider $d_A(\eta,\mathcal{S}) = \inf_{\sigma\in\mathcal{S}} \|\eta+\sigma\|_A$, then

$$sp\ (T,\mathcal{S},\xi) = \{\eta \in \xi + \mathcal{S};\ \|\eta\|_A = d_A(\xi,\mathcal{S})\}.$$

If $A$ is an invertible operator, then $\langle,\rangle_A$ is a scalar product, $(\mathcal{H},\langle,\rangle_A)$ is a Hilbert space and, by the projection theorem, $d_A(\xi,\mathcal{S}) = \|(I-P_{A,\mathcal{S}})\xi\|_A$ and $sp\ (T,\mathcal{S},\xi) = \{(I-P_{A,\mathcal{S}})\xi\}$, where $P_{A,\mathcal{S}}$ is unique orthogonal projection onto $\mathcal{S}$ which is orthogonal

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to the inner product $\langle , \rangle_A$. However, if $A$ is not invertible then $\| \cdot \|_A$ is or a seminorm or an incomplete norm and we can not use the projection theorem unless we complete the quotient $\mathcal{H}/\ker A$. One of the main goals of this paper is to get a simpler way of describing the set $sp\ (T, \mathcal{S}, \xi)$.

We start with a positive bounded linear operator $A$ on a Hilbert space $\mathcal{H}$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$. The subspace $\mathcal{S}^{\perp_A} = \{\xi : \langle A\xi, \eta \rangle = 0\ \forall \eta \in \mathcal{S}\}$ is called the $A-$ *orthogonal companion* of $\mathcal{S}$. Notice the identities

$$\mathcal{S}^{\perp_A} = A^{-1}(\mathcal{S}^{\perp}) = A(\mathcal{S})^{\perp} = \ker(PA). \tag{1}$$

Instead of defining adjoint operators with respect to $\langle , \rangle_A$, we restrict our discussion to $A$-selfadjoint operators, i.e. $W \in L(\mathcal{H})$ such that $AW = W^*A$. Notice that any such $W$ satisfies $\langle W\xi, \eta \rangle_A = \langle \xi, W\eta \rangle_A$, $\xi, \eta \in \mathcal{H}$.

The pair $(A, \mathcal{S})$ is said to be *compatible* if there exists a projection $Q \in L(\mathcal{H})$ such that $Q(\mathcal{H}) = \mathcal{S}$ and $AQ = Q^*A$. The main result in this paper is the description of the relationship between the set

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in \mathcal{Q} : R(Q) = \mathcal{S}, AQ = Q^*A\}$$

and $sp\ (T, \mathcal{S}, \xi)$, where $T : \mathcal{H} \to \mathcal{H}_1$ is any bounded linear operator such that $T^*T = A$. A relevant point here is that this method allows to tackle the case of operators with non closed range. Thus, several results by Sard [17], Atteia [3], Golomb [10], Shekhtman [18], de Boor [4], Izumino [12], Delvos [8], Deutsch [7] are generalized to any bounded linear operators $T$.

If $(A, \mathcal{S})$ is compatible, there exists a distinguished element $P_{A,\mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$. The study of the map $(A, \mathcal{S}) \to P_{A,\mathcal{S}}$ was initiated by Pasternak-Winiarski [14] at least for invertible $A$. A geometrical description of that map can be found in [2]. In [11] and [6], the inversibility hypothesis on $A$ was removed, opening, in that way, the posibility that $\mathcal{P}(A, \mathcal{S})$ be empty or have many elements. This induces the notion of compatibility of a pair $(A, \mathcal{S})$. This paper is mainly devoted to explore the relationship of the compatibility of $(A, \mathcal{S})$ with the existence of spline interpolants for every $\xi \in \mathcal{H}$. Section 2 contains a short study on compatibility of a pair $(A, \mathcal{S})$. If $(A, \mathcal{S})$ is compatible, the properties of the distinguished element $P_{A,\mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$ are described. In section 3 we show that $(A, \mathcal{S})$ is compatible if and only if $sp\ (T, \mathcal{S}, \xi)$ is not empty for any $\xi \in \mathcal{H}$ and that $sp\ (T, \mathcal{S}, \xi) = \{(1 - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\}$, for any $\xi \in \mathcal{H} \setminus \mathcal{S}$. Moreover, the vector of $sp\ (T, \mathcal{S}, \xi)$ with minimal norm is exactly $(1 - P_{A,\mathcal{S}})\xi$. In section 4 we present some characterizations of $P_{A,\mathcal{S}}$ which are useful for the study of the convergence of $\{P_{A,\mathcal{S}_n}\xi\}$ if $(A, \mathcal{S}_n)$ is compatible for every $n \in \mathbb{N}$ and $\mathcal{S}_n$ decreases to 0. This study is the goal of secction 5. Finally, section 6 includes several examples of compatibility and spline projections.

In this paper $L(\mathcal{H})$ is the algebra of all linear bounded operators on the Hilbert space $\mathcal{H}$ and $L(\mathcal{H})^+$ is the subset of $L(\mathcal{H})$ of all selfadjoint positive (i.e. non negative definite) operators. For every $C \in L(\mathcal{H})$ its range is denoted by $R(C)$. If $R(C)$ is closed, then $C^\dagger$ denotes the Moore-Penrose pseudoinverse of $C$. The orthogonal projections onto a closed subspace $\mathcal{S}$ is denoted by $P_{\mathcal{S}}$. The direct sum of subspaces $\mathcal{S}$ and $\mathcal{T}$ is denoted $\mathcal{S}\dot{+}\mathcal{T}$. Finally, $\mathcal{S}\ominus\mathcal{T}$ denotes $\mathcal{S}\cap\mathcal{T}^{\perp}$.

## 2 $A$-selfadjoint projections

Throughout this paper $\mathcal{S}$ denotes a closed subspace of $\mathcal{H}$ and $A$ is a fixed operator in $L(\mathcal{H})^{+}$. Recall that $\mathcal{S}^{\perp_A} = A^{-1}(\mathcal{S}^{\perp})$. It is easy to see that a projection $Q$ belongs to $\mathcal{P}(A,\mathcal{S})$ if and only if $R(Q)=\mathcal{S}$ and $\ker Q \subseteq A^{-1}(\mathcal{S}^{\perp})$. Then

$$\text{the pair } (A,\mathcal{S}) \text{ is compatible} \quad \text{if and only if} \quad \mathcal{S}+A^{-1}(\mathcal{S}^{\perp})=\mathcal{H}. \tag{2}$$

In this case, $\mathcal{P}(A,\mathcal{S})$ has a single element if and only if $\ker A\cap\mathcal{S}=\{0\}$, because

$$\mathcal{S}\cap A^{-1}(\mathcal{S}^{\perp}) = \ker A\cap\mathcal{S}. \tag{3}$$

If $(A,\mathcal{S})$ is compatible, then there is a distinguished element in $\mathcal{P}(A,\mathcal{S})$, namely the unique projection $P_{A,\mathcal{S}}$ onto $\mathcal{S}$ with kernel $A^{-1}(\mathcal{S}^{\perp})\ominus(\ker A\cap\mathcal{S})$. The elements of $\mathcal{P}(A,\mathcal{S})$ can be parametrized by the set of relative suplements of $\ker A\cap\mathcal{S}$ into $A^{-1}(\mathcal{S}^{\perp})$.

The set $\mathcal{P}(A,\mathcal{S})$ can also be characterized using the matrix operator decomposition induced by the orthogonal projection $P=P_{\mathcal{S}}$. Under this representation $A$ has a matrix form

$$A=\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \tag{4}$$

where $a\in L(\mathcal{S})^{+}$, $b\in L(\mathcal{S}^{\perp},\mathcal{S})$ and $c\in L(\mathcal{S}^{\perp})^{+}$. Observe that $P=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $PA=\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ and $PAP=\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Every projection $Q$ with range $\mathcal{S}$ has matrix form $Q=\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$ for some $x\in L(\mathcal{S}^{\perp},\mathcal{S})$. It is easy to see that $Q\in\mathcal{P}(A,\mathcal{S})$ if and only if $x$ satisfies the equation $ax=b$. Then

$$\mathcal{P}(A,\mathcal{S}) = \{\ Q=\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} : x\in L(\mathcal{S}^{\perp},\mathcal{S}) \ \text{ and } \ ax=b\} \tag{5}$$

Notice that equation (5) implies that, if $(A,\mathcal{S})$ is compatible, then $R(b)\subseteq R(a)$. As a corollary of a well known theorem of R. G. Douglas, it can be shown that these two conditions are, indeed, equivalent. First, we recall Douglas' theorem [9]:

**Theorem 2.1** *Let $B,C\in L(\mathcal{H})$. Then the following conditions are equivalent:*

1. *$R(B)\subseteq R(C)$.*
2. *There exists a positive number $\lambda$ such that $BB^*\leq\lambda CC^*$.*
3. *There exists $D\in L(\mathcal{H})$ such that $B=CD$.*

*Moreover, there exists a unique operator $D$ which satisfies the conditions*

$$B = CD\ , \quad \ker D = \ker B \quad \text{and} \quad R(D) \subseteq \overline{R(C^*)}.$$

*In this case,* $\|D\|^2 = \inf\{\lambda : BB^* \le \lambda CC^*\}$; *$D$ is called the* **reduced** *solution of the equation $CX = B$. If $R(C)$ is closed, then $D = C^\dagger B$.*

**Corollary 2.2** *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. If $A$ has matrix form as in (4), then $(A, \mathcal{S})$ is compatible if and only if $R(b) \subseteq R(a)$.*

The next theorem describes some properties of $\mathcal{P}(A, \mathcal{S})$ and $P_{A,\mathcal{S}}$. The norm of $P_{A,\mathcal{S}}$ will be computed in section 5.

**Theorem 2.3** *Let $A \in L(\mathcal{H})^+$ with matrix form (4), such that the pair $(A, \mathcal{S})$ is compatible.*

1. *The distinguished projection $P_{A,\mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$ has matrix form*

$$P_{A,\mathcal{S}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix},$$

   *where $d \in L(\mathcal{S}^\perp, \mathcal{S})$ is the reduced solution of the equation $ax = b$.*

2. *$\mathcal{P}(A, \mathcal{S})$ is an affine manifold which can be parametrized as*

$$\mathcal{P}(A, \mathcal{S}) = P_{A,\mathcal{S}} \ + \ L(\mathcal{S}^\perp, \mathcal{N}),$$

   *where $\mathcal{N} = A^{-1}(S^\perp) \cap \mathcal{S} = \ker A \cap \mathcal{S}$ and $L(\mathcal{S}^\perp, \mathcal{N})$ is viewed as a subspace of $L(\mathcal{H})$. A matrix representation of this parametrization is*

$$\mathcal{P}(A, \mathcal{S}) \ni Q = P_{A,\mathcal{S}} + z = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \ominus \mathcal{N} \\ \mathcal{N} \\ \mathcal{S}^\perp \end{matrix} \quad . \tag{6}$$

3. *$P_{A,\mathcal{S}}$ has minimal norm in $\mathcal{P}(A, \mathcal{S})$, i.e. $\|P_{A,\mathcal{S}}\| = \min\{ \|Q\| : Q \in \mathcal{P}(A, \mathcal{S})\}$.*

*Proof.*

1. If $Q = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}$, then $Q \in \mathcal{P}(A, \mathcal{S})$ and $\ker Q \subseteq A^{-1}(\mathcal{S}^\perp)$. Since $P_{A,\mathcal{S}}$ is characterized by the properties $R(P_{A,\mathcal{S}}) = \mathcal{S}$ and $\ker P_{A,\mathcal{S}} = A^{-1}(\mathcal{S}^\perp) \ominus \mathcal{N}$ then, in order to show that $Q = P_{A,\mathcal{S}}$ it suffices to prove that $\ker Q \subseteq \mathcal{N}^\perp$. Let $\xi \in \ker Q$ and write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \mathcal{S}$ and $\xi_2 \in \mathcal{S}^\perp$. Then $0 = Q\xi = \xi_1 + d\xi_2$. If $\eta \in \mathcal{N}$, then $\langle \xi, \eta \rangle = \langle \xi_1, \eta \rangle = -\langle d\xi_2, \eta \rangle = 0$, because, by Theorem 2.1, $R(d) \subseteq \overline{R(a)}$ and, as an operator in $L(\mathcal{S})$, $\ker a = \mathcal{S} \cap \ker PAP = \mathcal{S} \cap \ker A = \mathcal{N}$.

2. Let $Q = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}$ with $y \in L(\mathcal{S}^\perp, \mathcal{S})$ and let $d \in L(\mathcal{S}^\perp, \mathcal{S})$ be the reduced solution of the equation $ax = b$. Then $Q \in \mathcal{P}(A, \mathcal{S})$ if and only if $ay = b$. Therefore, if

$z = y - d$, then $Q \in \mathcal{P}(A, \mathcal{S})$ if and only if $Q = P_{A,\mathcal{S}} + z$ and $R(z) \subseteq \ker a = \mathcal{N}$. Concerning the matrix representation (6), recall that $R(d) \subseteq \overline{R(a)} = (\ker a)^\perp = \mathcal{S} \ominus \mathcal{N}$. Therefore

$$Q = P_{A,\mathcal{S}} + z = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \ominus \mathcal{N} \\ \mathcal{N} \\ \mathcal{S}^\perp \end{matrix} .$$

3. If $Q \in \mathcal{P}(A, \mathcal{S})$ has the matrix form given in equation (6), then

$$\|Q\|^2 = \|QQ^*\| = 1 + \left\| \begin{pmatrix} 0 & 0 & d \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right\|^2 \geq 1 + \|d\|^2 = \|P_{A,\mathcal{S}}\|^2.$$

**Remark 2.4** Under additional hypothesis on $A$, another characterizations of compatibility can be used. We mention a sample of these, taken from [6]:

1. If $A$ is injective then the following conditions are equivalent:
   (a) The pair $(A, \mathcal{S})$ is compatible.
   (b) $\mathcal{S}^\perp \subseteq R(A + \lambda(1 - P))$ for some (and then for any) $\lambda > 0$.
   (c) $P(\ \overline{A(\mathcal{S})}\ ) = \mathcal{S}$.
2. If $A$ has closed range then the following conditions are equivalent:
   (a) The pair $(A, \mathcal{S})$ is compatible.
   (b) $R(PAP)$ is closed.
   (c) $\mathcal{S} + \ker A$ is closed.
3. If $R(PAP)$ is closed (or, equivalently, if $R(PA^{1/2})$ or $A^{1/2}(\mathcal{S})$ are closed), then $(A, \mathcal{S})$ is compatible. Indeed, using the matrix form (4), the positivity of $A$ implies that $R(b) \subseteq R(a^{1/2})$ (see, e.g., [1]). If $R(PAP) = R(a)$ is closed, then $R(b) \subseteq R(a^{1/2}) = R(a)$ so that $(A, \mathcal{S})$ is compatible, by Corollary 2.2.

## 3 Splines and $A$-selfadjoint projections

In this section we characterize the existence of splines in terms of the existence of $A$-selfadjoint projections. The first result extends a theorem of Izumino [12] to operators whose ranges are not necessarily closed.

**Proposition 3.1** *Let $T \in L(\mathcal{H}, \mathcal{H}_1)$, $A = T^*T \in L(\mathcal{H})$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. Then, for any $\xi \in \mathcal{H}$,*

$$sp\ (T, \mathcal{S}, \xi) = (\xi + \ \mathcal{S}) \cap \mathcal{S}^{\perp_A}.$$

*In particular, $sp\ (T, \mathcal{S}, \xi)$ is an affine manifold of $L(\mathcal{H})$ and, if $\eta \in sp\ (T, \mathcal{S}, \xi)$, then $sp\ (T, \mathcal{S}, \xi) = \eta + \ker T \cap \mathcal{S}$.*

*Proof.* Suppose that $\eta \in (\xi + \mathcal{S}) \cap A^{-1}(\mathcal{S}^{\perp})$ and $\sigma \in \mathcal{S}$. Then $\langle A\eta, \sigma\rangle = \langle A\sigma, \eta\rangle = 0$ and

$$\|T(\eta+\sigma)\|^2 = \langle A(\eta+\sigma), \eta+\sigma\rangle = \langle A\eta, \eta\rangle + \langle A\sigma, \sigma\rangle \geq \langle A\eta, \eta\rangle = \|T\eta\|^2.$$

Therefore $\eta \in sp\ (T, \mathcal{S}, \xi)$. Conversely, if $\eta \in sp\ (T, \mathcal{S}, \xi)$ and $\sigma \in \mathcal{S}$, then, for any $t \in \mathbb{R}$,

$$\begin{aligned}\|T\eta\|^2 \leq \|T(\eta+t\sigma)\|^2 &= \langle A(\eta+t\sigma), \eta+t\sigma\rangle = \langle A\eta, \eta\rangle + t^2\langle A\sigma, \sigma\rangle + 2t \text{ Re } \langle A\eta, \sigma\rangle \\ &= \|T\eta\|^2 + t^2\langle A\sigma, \sigma\rangle + 2t \text{ Re } \langle A\eta, \sigma\rangle,\end{aligned}$$

therefore $t^2\langle A\sigma, \sigma\rangle + 2t \text{ Re } \langle A\eta, \sigma\rangle \geq 0$ for all $t \in \mathbb{R}$ and a standard argument shows that $\langle A\eta, \sigma\rangle = 0$ and then $\eta \in (\xi + \mathcal{S}) \cap A^{-1}(\mathcal{S}^{\perp})$. ∎

**Theorem 3.2** *Let $T \in L(\mathcal{H}, \mathcal{H}_1)$, $A = T^*T \in L(\mathcal{H})$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace.*

1. *If $\xi \in \mathcal{H}$, $sp\ (T, \mathcal{S}, \xi)$ is not empty $\iff \xi \in \mathcal{S} + A^{-1}(\mathcal{S}^{\perp})$.*
2. *The following conditions are equivalent:*
   *(a) $sp\ (T, \mathcal{S}, \xi)$ is not empty for every $\xi \in \mathcal{H}$.*
   *(b) $\mathcal{S} + A^{-1}(\mathcal{S}^{\perp}) = \mathcal{H}$.*
   *(c) The pair $(A, \mathcal{S})$ is compatible.*
3. *If $(A, \mathcal{S})$ is compatible and $\xi \in \mathcal{H} \setminus \mathcal{S}$, it holds $sp\ (T, \mathcal{S}, \xi) = \{(I - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\}$.*
4. *If $(A, \mathcal{S})$ is compatible, then for every $\xi \in \mathcal{H}$, $(I - P_{A,\mathcal{S}})\xi$ is the unique vector in $sp\ (T, \mathcal{S}, \xi)$ with minimal norm.*

*Proof.* The first assertion follows directly from Proposition 3.1. Indeed, if $\eta \in sp\ (T, \mathcal{S}, \xi)$ and $\eta = \xi + \sigma$ with $\sigma \in \mathcal{S}$, then $\xi = -\sigma + \eta \in \mathcal{S} + A^{-1}(\mathcal{S}^{\perp})$; the converse implication is similar. The second assertion follows from the first one and equation (2). In order to prove the third item, let $\xi \in \mathcal{H}$ and $Q \in \mathcal{P}(A, \mathcal{S})$. Then, by Proposition 3.1 and equation (2),

$$(I - Q)\xi = \xi - Q\xi \in (\xi + \mathcal{S}) \cap \ker Q \subseteq (\xi + \mathcal{S}) \cap A^{-1}(\mathcal{S}^{\perp}) = sp\ (T, \mathcal{S}, \xi).$$

Conversely, let $\eta \in sp\ (T, \mathcal{S}, \xi)$ and $\sigma \in \mathcal{S}$ such that $\xi = \sigma + \eta$. We are looking for some $Q \in \mathcal{P}(A, \mathcal{S})$ such that $Q\xi = \sigma$. Let $\eta_1 = (I - P_{A,\mathcal{S}})\xi$ and $\sigma_1 = \xi - \eta_1 = P_{A,\mathcal{S}}\xi \in \mathcal{S}$. Then, by Proposition 3.1,

$$\sigma - \sigma_1 = \eta_1 - \eta \in \mathcal{S} \cap A^{-1}(\mathcal{S}^{\perp}) = \ker A \cap \mathcal{S}.$$

If $\xi = \sigma_2 + \rho$ with $\sigma_2 \in \mathcal{S}$ and $0 \neq \rho \in \mathcal{S}^{\perp}$, choose $z \in L(\mathcal{S}^{\perp}, \ker A \cap \mathcal{S})$ ($\subseteq L(\mathcal{H})$) such that $z(\rho) = \sigma - \sigma_1$. By Theorem 2.3, $Q = P_{A,\mathcal{S}} + z \in \mathcal{P}(A, \mathcal{S})$ and clearly $Q\xi = \sigma$.

The minimality of $\|(1 - P_{A,\mathcal{S}})\xi\|$ is proved as follows. If $\xi \in \mathcal{S}$ then $(I - P_{A,\mathcal{S}})\xi = 0$, which must be minimal. If $\xi \notin \mathcal{S}$, let $\xi = \sigma_2 + \rho$ with $\sigma_2 \in \mathcal{S}$ and $0 \neq \rho \in \mathcal{S}^{\perp}$. By

Theorem 2.3, any $Q \in \mathcal{P}(A, \mathcal{S})$ has the form $Q = P_{A,\mathcal{S}} + z$, with $z \in L(\mathcal{S}^\perp, \ker A \cap \mathcal{S})$ $(\subseteq L(\mathcal{H}))$. Recall that $R(P_{A,\mathcal{S}}) = \mathcal{S} \ominus (\ker A \cap \mathcal{S})$. Therefore

$$\begin{aligned} \|(I-Q)\xi\|^2 &= \|(I-Q)\rho\|^2 = \|\rho - P_{A,\mathcal{S}}(\rho) - z(\rho)\|^2 = \|\rho\|^2 + \|P_{A,\mathcal{S}}(\rho)\|^2 + \|z(\rho)\|^2 \\ &\geq \|\rho\|^2 + \|P_{A,\mathcal{S}}(\rho)\|^2 = \|\rho - P_{A,\mathcal{S}}(\rho)\|^2 = \|(I - P_{A,\mathcal{S}})\xi\|^2 \end{aligned}$$ ∎

**Corollary 3.3** *Let $T \in L(\mathcal{H}, \mathcal{H}_1)$, $A = T^*T \in L(\mathcal{H})$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. Then the following are equivalent:*

1. *sp $(T, \mathcal{S}, \xi)$ has an unique element for every $\xi \in \mathcal{H}$.*
2. *The pair $(A, \mathcal{S})$ is compatible and* $\ker T \cap \mathcal{S} = \{0\}$.

**Remark 3.4** Let $T \in L(\mathcal{H}, \mathcal{H}_1)$, $A = T^*T \in L(\mathcal{H})$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace.

1. If $(A, \mathcal{S})$ is compatible then, by item 4 of Theorem 3.2, the projection $1 - P_{A,\mathcal{S}}$ coincides with the so called *spline projection* for $T$ and $\mathcal{S}$ when $T$ has closed range.
2. If $R(T)$ is closed, then, by Remark 2.4 and Theorem 3.2, *sp* $(T, \mathcal{S}, \xi) \neq \emptyset$ for every $\xi \in \mathcal{H}$ if and only if $\ker T + \mathcal{S}$ is closed. In case that $\ker T \cap \mathcal{S} = \{0\}$, then it is equivalent to the condition that the inclination between $\ker T$ and $\mathcal{S}$ is less than one (see [4] and [7]).
3. If $\xi \in \mathcal{S}$, then *sp* $(T, \mathcal{S}, \xi) = \ker T \cap \mathcal{S}$. On the other hand $(I - Q)\xi = 0$ for every $Q \in \mathcal{P}(A, \mathcal{S})$. So the equality of item 3 of Theorem 3.2 may be false in this case.

# 4 Characterizations of the spline projection $P_{A,\mathcal{S}}$

Fix $A \in L(\mathcal{H})^+$ and a closed subspace $\mathcal{S} \subseteq \mathcal{H}$. As before, we denote $P = P_\mathcal{S}$. In this section two different descriptions of the spline projection $P_{A,\mathcal{S}}$ are given and, as a consequence, we relate $P_{A,\mathcal{S}}$ with the shorted operator (see [1] and Remark 4.4 below).

By Corollary 2.2, it holds that the pair $(A, \mathcal{S})$ is compatible if and only if $R(PA) \subseteq R(PAP)$. In case that $A$ is invertible, it is known (see [2]) that, in the matrix form (4), $a$ is invertible in $L(\mathcal{S})$ and

$$P_{A,\mathcal{S}} = \begin{pmatrix} a^{-1} & 0 \\ 0 & 0 \end{pmatrix} PA = \begin{pmatrix} 1 & a^{-1}b \\ 0 & 0 \end{pmatrix}, \tag{7}$$

because $a^{-1}b$ is the reduced solution of $ax = b$ (see Theorem 2.3). Rewriting (7), we get $(PAP)P_{A,\mathcal{S}} = PA$. Thus, if $A$ is invertible, $P_{A,\mathcal{S}}$ is the reduced solution of the equation $(PAP)X = PA$. Let us consider the general case, in other words, if the pair $(A, \mathcal{S})$ is compatible, let us relate $P_{A,\mathcal{S}}$ with the reduced solution $Q$ of the equation

$$(PAP)X = PA. \tag{8}$$

Observe that, in general, $\overline{R(PAP)}$ is strictly contained in $\mathcal{S}$. Therefore, $R(Q)$ may be smaller that $\mathcal{S} = R(P_{A,\mathcal{S}})$.

**Proposition 4.1** *If the pair $(A, \mathcal{S})$ is compatible, $Q$ is the reduced solution of the equation* (8) *and $\mathcal{N} = \ker A \cap \mathcal{S}$, Then*

$$P_{A,\mathcal{S}} = P_{\mathcal{N}} + Q.$$

*Moreover, $Q$ verifies the following properties:*

1. $Q^2 = Q$, $\ker Q = A^{-1}(\mathcal{S}^{\perp})$ *and* $R(Q) = \mathcal{S} \ominus \mathcal{N}$.
2. $Q$ *is $A$-selfadjoint.*
3. $Q = P_{A,\mathcal{S}\ominus\mathcal{N}}$.

*Proof.* Using the matrix form (4) of $A$, observe that, in $L(\mathcal{S})$, $\ker a = \mathcal{N}$ and $\overline{R(a)} = \overline{R(a^{1/2})} = \mathcal{S} \ominus \mathcal{N}$. Note that $R(Q) \subseteq \overline{R(a)}$. Also $\ker Q = \ker(PA) = A^{-1}(\mathcal{S}^{\perp})$. If $\xi \in \mathcal{S} \ominus \mathcal{N}$, then

$$a(Q\xi) = (PAP)Q\xi = PA\xi = PAP\xi = a(\xi).$$

Since $a$ is injective in $\mathcal{S} \ominus \mathcal{N}$, we can deduce that $Q\xi = \xi$ for all $\xi \in \mathcal{S} \ominus \mathcal{N}$. Now, the compatibility of $(A, \mathcal{S})$ implies that $\mathcal{S} + A^{-1}(\mathcal{S}^{\perp}) = \mathcal{H}$. Also $A^{-1}(\mathcal{S}^{\perp}) \cap \mathcal{S} = \ker A \cap \mathcal{S} = \mathcal{N}$. Therefore $A^{-1}(\mathcal{S}^{\perp}) \dot{+} (\mathcal{S} \ominus \mathcal{N}) = \mathcal{H}$. Then $Q^2 = Q$ and $R(Q) = \mathcal{S} \ominus \mathcal{N}$. Note that

$$\ker Q = A^{-1}(\mathcal{S}^{\perp}) \subseteq A^{-1}((\mathcal{S} \ominus \mathcal{N})^{\perp}) = R(Q)^{\perp_A},$$

so that $Q$ is $A$-selfadjoint by equation (2). On the other hand, $(\mathcal{S} \ominus \mathcal{N}) \cap \ker A = \{0\}$, so that $Q$ is the unique element of $P(A, \mathcal{S} \ominus \mathcal{N})$, by Theorem 2.3. Observe that $R(Q) \subseteq \mathcal{N}^{\perp}$ and $\mathcal{N} \subseteq \ker A \subseteq A^{-1}(\mathcal{S}^{\perp}) = \ker Q$. Therefore $(P_{\mathcal{N}} + Q)^2 = P_{\mathcal{N}} + Q$, $R(P_{\mathcal{N}} + Q) = \mathcal{S}$ and $\ker(P_{\mathcal{N}} + Q) = (A^{-1}(\mathcal{S}^{\perp})) \ominus \mathcal{N}$. These formulae clearly imply that $P_{\mathcal{N}} + Q = P_{A,\mathcal{S}}$ (see Theorem 2.3). ∎

**Proposition 4.2** *If $(A, \mathcal{S})$ is compatible and $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$, then $R(P_{\mathcal{M}} A^{1/2}) \subseteq R(A^{1/2}P)$. Moreover, equations* (8) *and*

$$(A^{1/2}P)X = P_{\mathcal{M}} A^{1/2} \tag{9}$$

*have the same reduced solution. In particular, if $A^{1/2}(\mathcal{S})$ is closed and* $\ker A \cap \mathcal{S} = \{0\}$, *then*

$$P_{A,\mathcal{S}} = (A^{1/2}P)^{\dagger} P_{\mathcal{M}} A^{1/2} = (A^{1/2}P)^{\dagger} A^{1/2} = (TP)^{\dagger} T \tag{10}$$

*for every $T \in L(\mathcal{H}, \mathcal{H}_1)$ such that $T^*T = A$.*

*Proof.* Denote $B = A^{1/2}$. Recall that $\mathcal{M} = \overline{B(\mathcal{S})} = B^{-1}(\mathcal{S}^{\perp})^{\perp}$. Observe that

$$BP_{\mathcal{M}}B = AP_{A,\mathcal{S}} = APP_{A,\mathcal{S}} : \tag{11}$$

in fact, for $\xi \in \mathcal{H}$, let $\eta = P_{A,\mathcal{S}}\xi$ and $\rho = \xi - \eta \in A^{-1}(\mathcal{S}^{\perp})$; then $B\eta \in \mathcal{M}$ and $B\rho \in B^{-1}(\mathcal{S}^{\perp}) = \mathcal{M}^{\perp}$ . Hence $BP_{\mathcal{M}}B\xi = A\eta = AP_{A,\mathcal{S}}\xi$. By Proposition 4.1, the projection $Q = P_{A,\mathcal{S}} - P_{\mathcal{N}}$ is the reduced solution of the equation $PAPX = PA$. We shall see that $Q$ is the reduced solution of the equation (9). First note that, by equation (11), $BP_{\mathcal{M}}B = (AP)P_{A,\mathcal{S}} = (AP)Q$, so $B(P_{\mathcal{M}}B - BPQ) = 0$. But $R(P_{\mathcal{M}}B - BPQ) \subseteq \overline{R(B)} = (\ker B)^{\perp}$. Hence $Q$ is a solution of (9). Note that $\ker P_{\mathcal{M}}B = B^{-1}(B^{-1}(\mathcal{S}^{\perp})) = A^{-1}(\mathcal{S}^{\perp}) = \ker Q$ by Prop. 4.1. Finally,

$$\overline{R((BP)^*)} = \overline{R(PB)} = \overline{R(PAP)} = \mathcal{S} \ominus \mathcal{N} = R(Q).$$

The first equality of equation (10) follows directly. The second, from the fact that $(A^{1/2}P)^{\dagger}P_{\mathcal{M}} = (A^{1/2}P)^{\dagger}$. The last equality follows easily using the polar decomposition of $T$, because $A^{1/2} = |T|$ ∎

Formula (10), for operators with closed range, is due to Golomb [10].

**Corollary 4.3** *Under the notations of Proposition 4.2, the pair $(A, \mathcal{S})$ is compatible if and only if $R(P_{\mathcal{M}}A^{1/2}) \subseteq R(A^{1/2}P)$.*

*Proof.* Suppose that $R(P_{\mathcal{M}}A^{1/2}) \subseteq R(A^{1/2}P)$. Then, given $\xi \in \mathcal{H}$, there must exists $\sigma \in \mathcal{S}$ such that $P_{\mathcal{M}}A^{1/2}\xi = A^{1/2}\sigma$. Therefore $A^{1/2}(\xi - \sigma) = (1 - P_{\mathcal{M}})A^{1/2}\xi$ and

$$\|A^{1/2}(\xi - \sigma)\| = \|(1 - P_{\mathcal{M}})A^{1/2}\xi\| = d(A^{1/2}\xi, A^{1/2}(\mathcal{S})) = \inf\{\|A^{1/2}(\xi + \tau)\| : \tau \in \mathcal{S}\}. \tag{12}$$

Hence $\xi - \sigma \in sp\ (T, \mathcal{S}, \xi)$ and $sp\ (T, \mathcal{S}, \xi) \neq \emptyset$ for every $\xi \in \mathcal{H}$. This implies compatibility by Theorem 3.2. The converse implication was shown in Proposition 4.2 ∎

**Remark 4.4** If $A \in L(\mathcal{H})^+$ and $\mathcal{S} \in \mathcal{H}$ is a closed subspace, then the set

$$\{X \in L(\mathcal{H})^+ : X \leq A \quad \text{and} \quad R(X) \subseteq \mathcal{S}^{\perp}\}$$

has a maximum (for the natural order relation in $L(\mathcal{H})^+$), which is called the *shorted operator* of $A$ to $\mathcal{S}^{\perp}$. We denote it by $\Sigma(P, A)$. This notion, due to Krein [13] and Anderson-Trapp [1], has many applications to electrical engineering. It is well known (see Pekarev [15]) that

$$\Sigma(P, A) = A^{1/2}P_{\mathcal{T}}A^{1/2},$$

where $\mathcal{T} = A^{-1/2}(\mathcal{S}^{\perp}) = A^{1/2}(\mathcal{S})^{\perp}$. From the proof of Proposition 4.2, it follows that, if $(A, \mathcal{S})$ is compatible, then $A^{1/2}(1 - P_{\mathcal{T}})A^{1/2} = AP_{A,\mathcal{S}}$. Therefore, in this case, $\Sigma(P, A) = A(1 - P_{A,\mathcal{S}})$. More generally, it can be shown that $\Sigma(P, A) = A(1 - Q)$ for every $Q \in \mathcal{P}(A, \mathcal{S})$ (see [6]).

## 5 Convergence of spline projections

This section is devoted to the study of the convergence of abstract splines in the general (i.e. non necesarily closed range) case. Given $A \in L(\mathcal{H})^+$, let us consider a sequence of closed subspaces $\mathcal{S}_n$ such that all pairs $(A, \mathcal{S}_n)$ are compatible. Following de Boor [4] and Izumino [12], it is natural to look for conditions which are equivalent to the fact that $P_{A,\mathcal{S}_n} \to^{SOT} 0$ (i.e. the spline projections converge to $I$), where $\to^{SOT}$ means convergence in the strong operator topology. This problem has a well known solution under the assumption that $R(A)$ is closed (see [4] or [12]). However, in our more general setting, it is possible that the sequence $\{\mathcal{S}_n\}$ decreases to $\{0\}$, while $\|P_{A,\mathcal{S}_n}\|$ tends to infinity (see section 5.7 below). This induces us to consider the following weaker convergence:

**Definition 5.1** Let $A \in L(\mathcal{H})^+$ and $T_n, T \in L(\mathcal{H})$, $n \in \mathbb{N}$. We shall say that the sequence $T_n$ converges $A$-SOT to $T$: $T_n \rightarrow^{A-SOT} T$ if

$$\|(T_n - T)\xi\|_A \rightarrow 0 \quad \text{for every} \quad \xi \in \mathcal{H}.$$

Note that $T_n \rightarrow^{A-SOT} T$ if and only if $A^{1/2}T_n \rightarrow^{SOT} A^{1/2}T$.

We start with the computation of the norm of $P_{A,\mathcal{S}}$ for any compatible pair $(A, \mathcal{S})$. Before that, recall the following formula, due to Ptak [16] (see also [5] and [6]): if $Q_1$ and $Q_2$ are orthogonal projections such that $R(Q_1)\dot{+}R(Q_2) = \mathcal{H}$, then the norm of the unique projection $Q_3$ with $\ker Q_3 = R(Q_1)$ and $R(Q_3) = R(Q_2)$ is

$$\|Q_3\| = (1 - \|Q_1Q_2\|^2)^{-1/2}. \tag{13}$$

**Proposition 5.2** *Let $A \in L(\mathcal{H})^+$ such that the pair $(A, \mathcal{S})$ is compatible. Then*

$$\|P_{A,\mathcal{S}}\|^2 = \inf\{\lambda > 0 : PA^2P \leq \lambda(PAP)^2\}. \tag{14}$$

*If, in addition,* $\ker A \cap \mathcal{S} = \{0\}$, *then*

$$\|P_{A,\mathcal{S}}\| = (1 - \|QP\|^2)^{-1/2}, \tag{15}$$

*where $Q$ denotes the orthogonal projection onto $A^{-1}(\mathcal{S}^\perp)$.*

*Proof.* Let $Q$ be the reduced solution of the equation $(PAP)X = PA$. Then $\|Q\|^2$ equals the infimum of equation (14) by Douglas Theorem. On the other hand, by Proposition 4.1, $\|Q\| = \|P_{A,\mathcal{S}}\|$, showing formula (14). If $\ker A \cap \mathcal{S} = \{0\}$, then Theorem 2.3 assures that $R(P_{A,\mathcal{S}}) = \mathcal{S}$ and $\ker P_{A,\mathcal{S}} = A^{-1}(\mathcal{S}^\perp)$. Therefore (15) follows from Ptak formula (13) ∎

**Remark 5.3** Let $A \in L(\mathcal{H})^+$ such that the pair $(A, \mathcal{S})$ is compatible and $\ker A \cap \mathcal{S} = \{0\}$. Then, if $P_{\ker A}$ is the orthogonal projection onto $\ker A$, then

$$\|P_{A,\mathcal{S}}\| \geq (1 - \|P_{\ker A}P\|^2)^{-1/2}.$$

Indeed, if $Q$ is the projection of equation (15), then $P_{\ker A} \leq Q$, because $\ker A \subseteq A^{-1}(\mathcal{S}^\perp)$. Then $\|P_{\ker A}P\|^2 = \|PP_{\ker A}P\| \leq \|PQP\| = \|QP\|^2$. This inequality, shown by de Boor in [4] in the closed range case, relates the norm of $P_{A,\mathcal{S}}$ with the angle between $\ker A$ and $\mathcal{S}$.

**Proposition 5.4** *Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S}_n$ ($n \in \mathbb{N}$) be closed subspaces such that all pairs $(A, \mathcal{S}_n)$ are compatible. Denote by* $\mathcal{M}_n = \overline{A^{1/2}(\mathcal{S}_n)}$, $n \in \mathbb{N}$.

1. *The following conditions are equivalent:*

(a) $P_{A,\mathcal{S}_n} \to^{A-SOT} 0$.

(b) $\langle AP_{A,\mathcal{S}_n}\xi, \xi\rangle \to 0$, *for every* $\xi \in \mathcal{H}$ *(i.e.* $AP_{A,\mathcal{S}_n} \to^{WOT} 0$ *by polarization).*

(c) $AP_{A,\mathcal{S}_n} \to^{SOT} 0$.

(d) $\Sigma(P_{\mathcal{S}_n}, A) \to^{SOT} A$.

(e) $P_{\mathcal{M}_n} A^{1/2} \to^{SOT} 0$.

2. *If there exists* $C \geq 0$ *such that* $\|P_{A,\mathcal{S}_n}\| \leq C$ *for all* $n \in \mathbb{N}$ *and* $P_{\mathcal{S}_n} A \to^{SOT} 0$, *then* $P_{A,\mathcal{S}_n} \to^{A-SOT} 0$.

3. *If* $P_{A,\mathcal{S}_n} \to^{A-SOT} 0$, *then* $P_{\mathcal{S}_n} A \to^{SOT} 0$.

*Proof.*

1. Because $P^*_{A,\mathcal{S}_n} A = AP_{A,\mathcal{S}_n}$, it is clear that conditions (a), (b) and (c) are equivalent. By Remark 4.4, $\Sigma(P_{\mathcal{S}_n}, A) = A(1 - P_{A,\mathcal{S}_n})$, so that (c) is equivalent to (d). Finally, by Proposition 4.2, we know that $A^{1/2} P_{A,\mathcal{S}_n} = P_{\mathcal{M}_n} A^{1/2}$, and this shows that (a) is equivalent to (e).

2. Suppose that there exists $C \geq 0$ such that $\|P_{A,\mathcal{S}_n}\| \leq C$ for all $n \in \mathbb{N}$ and that $P_{\mathcal{S}_n} A \to^{SOT} 0$. Denote by $P_n = P_{\mathcal{S}_n}$. The fact that $R(P_{A,\mathcal{S}_n}) = R(P_n)$ implies that $P_n P_{A,\mathcal{S}_n} = P_{A,\mathcal{S}_n}$. Therefore, for every $\xi \in \mathcal{H}$,

$$\|P^*_{A,\mathcal{S}_n} A\xi\| = \|P^*_{A,\mathcal{S}_n} P_n A\xi\| \to 0,$$

since $\|P_{A,\mathcal{S}_n}\|$ is bounded. Hence $P^*_{A,\mathcal{S}_n} A = AP_{A,\mathcal{S}_n} \to^{SOT} 0$ so that $P_{A,\mathcal{S}_n} \to^{A-SOT} 0$, by item 1.

3. Suppose that $P_{A,\mathcal{S}_n} \to^{A-SOT} 0$. Then, by item 1, $AP_{A,\mathcal{S}_n} \to^{SOT} 0$. Note that $P_{A,\mathcal{S}_n} P_n = P_n$, so that $P_n P^*_{A,\mathcal{S}_n} = P_n$. Given $\xi \in \mathcal{H}$, we have that

$$\|P_n A\xi\| = \|P_n P^*_{A,\mathcal{S}_n} A\xi\| = \|P_n A P_{A,\mathcal{S}_n}\xi\| \leq \|AP_{A,\mathcal{S}_n}\xi\| \to 0 \qquad \blacksquare$$

**Remark 5.5** With the notations of Proposition 5.4, it follows that $P_{A,\mathcal{S}_n} \to^{A-SOT} 0$ if and only if $A^{1/2}(1 - P_{A,\mathcal{S}_n})\xi \to A^{1/2}\xi$ for every $\xi \in \mathcal{H}$ or, equivalently, the spline interpolants $\xi_n = (1 - P_{A,\mathcal{S}_n})\xi$ satisfy that $T\xi_n \to T\xi$ in $\mathcal{H}_1$, if $T \in L(\mathcal{H}, \mathcal{H}_1)$ and $T^*T = A$. In particular, if $P_{A,\mathcal{S}_n} \to^{A-SOT} 0$, then $\min\{\|T(\xi + \tau)\| : \tau \in \mathcal{S}_n\} = \|T(1 - P_{A,\mathcal{S}_n})\xi\| \to \|T\xi\|$.

**Proposition 5.6** *Let* $A \in L(\mathcal{H})^+$ *and* $\mathcal{S}_2 \subseteq \mathcal{S}_1 \subseteq \mathcal{H}$ *closed subspaces. Suppose that* $(A, \mathcal{S}_1)$ *is compatible. Denote by* $P_i = P_{\mathcal{S}_i}$, $i = 1, 2$ *and* $a_1 = P_1 A P_1 \in L(\mathcal{S}_1)^+$. *Then*

$$(A, \mathcal{S}_2) \text{ is compatible} \quad \text{if and only if} \quad (a_1, \mathcal{S}_2) \text{ is compatible in } L(\mathcal{S}_1).$$

*Proof.* We know that, if $A = \begin{pmatrix} a_1 & b_1 \\ b_1^* & c_1 \end{pmatrix}$, in the matrix decomposition induced by $P_1$, then $R(b_1) \subseteq R(a_1)$. Hence also $R(P_2 b_1) \subseteq R(P_2 a_1)$. If $a_1 = \begin{pmatrix} a_2 & b_2 \\ b_2^* & c_2 \end{pmatrix}$, using now the matrix decomposition induced by $P_2$, then $P_2 a_1 = a_2 + b_2$ and $P_2 A(1-P_2) = b_2 + P_2 b_1$. Hence

$$R(P_2 b_1) \subseteq R(P_2 a_1) = R(a_2) + R(b_2) \quad \text{and} \quad R(P_2 A(1-P_2)) = R(b_2) + R(P_2 b_1).$$

Therefore the pair $(A, \mathcal{S}_2)$ is compatible if and only if $R(P_2 A(1-P_2)) \subseteq R(P_2 A P_2) = R(a_2)$ if and only if $R(b_2) \subseteq R(a_2)$ if and only if the pair $(a_1, \mathcal{S}_2)$ is compatible ∎

**Example 5.7** Let $A \in L(\mathcal{H})^+$ injective but not invertible. With the notations of Proposition 5.6 it is easy to see that $P_1 P_{A,\mathcal{S}_2} P_1 = P_{A,\mathcal{S}_2} P_1 \in \mathcal{P}(a_1, \mathcal{S}_2)$. Note that $a_1$ is injective, so that $\mathcal{P}(a_1, \mathcal{S}_2)$ has a unique element and

$$P_{a_1,\mathcal{S}_2} = P_{A,\mathcal{S}_2} P_1 \quad \Rightarrow \quad \|P_{A,\mathcal{S}_2}\| \geq \|P_{a_1,\mathcal{S}_2}\|. \tag{16}$$

We shall see that there exists a sequence $\mathcal{S}_n$, $n \in \mathbb{N}$, of closed subspaces of $\mathcal{H}$ such that

1. the pair $(A, \mathcal{S}_n)$ is compatible for every $n \in \mathbb{N}$,
2. $\mathcal{S}_{n+1} \subseteq \mathcal{S}_n$ for every $n \in \mathbb{N}$,
3. $\bigcap_{n\geq 1} \mathcal{S}_n = \{0\}$, so that $P_{\mathcal{S}_n} \to^{SOT} 0$,
4. $\|P_{A,\mathcal{S}_n}\| \to \infty$.

In order to prove this fact, we need the following Lemma:

**Lemma 5.8** Let $B \in L(\mathcal{H})^+$ be injective non invertible. Then, for every $\varepsilon > 0$, there exist a closed subspace $\mathcal{S} \subseteq \mathcal{H}$ such that the pair $(B, \mathcal{S})$ is compatible, $P_{\mathcal{S}} B P_{\mathcal{S}}$ is not invertible in $L(\mathcal{S})$ and $\|P_{B,\mathcal{S}}\| \geq \varepsilon^{-1}$.

*Proof.* Let $\eta \in \mathcal{H}$ be a unit vector. Denote by $\xi = B\eta$ and consider the subspace $\mathcal{S} = \{\xi\}^\perp$ and $P = P_{\mathcal{S}}$. It is clear that $\eta \in B^{-1}(\mathcal{S}^\perp)$. First note that $\langle \xi, \eta \rangle = \langle B\eta, \eta \rangle > 0$, so that $\eta \notin \mathcal{S}$. Since $\mathcal{S}$ is an hyperplane, this implies that $\mathcal{S} + B^{-1}(\mathcal{S}^\perp) = \mathcal{H}$ and the pair $(B, \mathcal{S})$ is compatible. Also $PBP$ is not invertible, because $\dim \mathcal{S}^\perp = 1 < \infty$. Note that $B^{-1}(\mathcal{S}^\perp)$ is the subspace generated by $\eta$. Hence, if $Q = P_{B^{-1}(\mathcal{S}^\perp)}$, it is easy to see that $\|PQ\| = \|P\eta\|$. Then, by equation (15),

$$\|P_{B,\mathcal{S}}\| = (1 - \|PQ\|^2)^{-1/2} = (1 - \|P\eta\|^2)^{-1/2} = \|(1-P)\eta\|^{-1}$$

$$\text{and} \qquad \|(1-P)\eta\| = |\langle \eta, \frac{\xi}{\|\xi\|} \rangle| = \frac{\langle \eta, B\eta \rangle}{\|B\eta\|} .$$

So, it suffices to show that there exists a unit vector $\eta$ such that $\langle \eta, B\eta \rangle \leq \varepsilon \|B\eta\|$. Consider $\rho \in \mathcal{H} \setminus R(B^{1/2})$ a unit vector. Let $\rho_n$ be a sequence of unit vectors in $R(B^{1/2})$ such that $\rho_n \to \rho$. Let $\mu_n \in \mathcal{H}$ such that $B^{1/2}\mu_n = \rho_n$, $n \in \mathbb{N}$, and denote by $\xi_n = B^{1/2}\rho_n = B\mu_n$, and $\xi = B^{1/2}\rho$. It is easy to see, using that $B(\mu_n) = \xi_n \to \xi \notin R(B)$, that $\|\mu_n\| \to \infty$. Denote by $\eta_n = \mu_n \|\mu_n\|^{-1}$. Then

$$\frac{\langle \eta_n, B\eta_n \rangle}{\|B\eta_n\|} = \frac{\langle \mu_n, B\mu_n \rangle}{\|\mu_n\|^2 \|B\eta_n\|} = \frac{\|B^{1/2}\mu_n\|^2}{\|\mu_n\| \, \|B\mu_n\|} = \frac{1}{\|\mu_n\| \, \|\xi_n\|} \to 0,$$

because $\xi_n \to \xi \neq 0$ ∎

By an inductive argument, using Lemma 5.8, Proposition 5.6 and equation (16), we can construct a sequence of compatible subspaces $\mathcal{S}_n$, $n \in \mathbb{N}$, such that $\mathcal{S}_{n+1} \subseteq \mathcal{S}_n$ and $\|P_{A,\mathcal{S}_n}\| \to \infty$. We can also get that $P_{\mathcal{S}_n} \to^{SOT} 0$ by interlacing, before constructing the subspace $\mathcal{S}_{n+1}$, a spectral subspace $\mathcal{T}_n$ of $P_{\mathcal{S}_n}AP_{\mathcal{S}_n}$ (as an operator in $L(\mathcal{S}_n)$), in such a way that $P_{\mathcal{T}_n}AP_{\mathcal{T}_n}$ is not invertible and the projections $P_{\mathcal{T}_n} \to^{SOT} 0$ (this can be done recursively by testing the projections $P_{\mathcal{T}_n}$ in the first $n$ elements of a countable dense subset of $\mathcal{H}$), and taking $\mathcal{S}_{n+1}$ as a subspace of $\mathcal{T}_n$. Note that the pairs $(P_{\mathcal{S}_n}AP_{\mathcal{S}_n}, \mathcal{T}_n)$ are clearly compatible, so that also the pairs $(A, \mathcal{T}_n)$ are compatible by Proposition 5.6.

**Remark 5.9** Recall from Remark 4.4 that, if $(A,\mathcal{S})$ is compatible, then $A(1-P_{A,\mathcal{S}}) = \Sigma(P,A)$. Then

$$0 \leq AP_{A,\mathcal{S}} = A - \Sigma(P,A) \leq A.$$

This implies that $\|AP_{A,\mathcal{S}}\| \leq \|A\|$, while $\|P_{A,\mathcal{S}}\|$ can be arbitrarily large.

# 6 Some examples

In this section we present several examples opairs $(A,\mathcal{S})$ which are not compatible and pairs $(A,\mathcal{S})$ which are compatible and such that the spline projector $P_{A,\mathcal{S}}$ can be explicitely computed. Observe that Example 6.4 can not be studied under the closed range hypothesis, considered by Atteia, de Boor and Izumino.

**Example 6.1** Let $A \in L(\mathcal{H})^+$ and

$$M = \begin{pmatrix} A & A^{1/2} \\ A^{1/2} & I \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix} \in L(\mathcal{H} \oplus \mathcal{H})^+$$

Denote by $\mathcal{S} = \mathcal{H} \oplus \{0\}$ and by $N = \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix}$. Since $M = N^*N$, then $\ker M = \ker N = \{\xi \oplus -A^{1/2}\xi : \xi \in \mathcal{H}\}$ which is the graph of $-A^{1/2}$. Note that $R(N) = (R(A^{1/2}) + R(I)) \oplus \{0\} = \mathcal{S}$, so that $R(M)$ is also closed. If $A$ is injective with non closed range, then $(M,\mathcal{S})$ is not compatible (because $R(A)$ is properly included in $R(A^{1/2})$). Observe that this implies that the inclination between $\mathcal{S}$ and $\ker M$ is one, cf. [4].

**Remark 6.2** Let $P \in \mathcal{P}$, $R(P) = \mathcal{S}$ and $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in L(\mathcal{H})^+$. It is well known that the positivity of $A$ implies that $R(b) \subseteq R(a^{1/2})$. Therefore, if $\dim \mathcal{S} < \infty$ then the pair $(A,\mathcal{S})$ is compatible: in fact in this case $R(a) = R(PAP)$ must be closed, so $R(b) \subseteq R(a^{1/2}) = R(a)$ and Corollary 2.2 can be applied. On the other hand, if $\dim \mathcal{S}^\perp < \infty$ and $R(A)$ is closed then, by Remark 2.4, $(A,\mathcal{S})$ is compatible. However, if $R(A)$ is not closed, then the pair $(A,\mathcal{S})$ can be non compatible:

**Proposition 6.3** *Let $P \in \mathcal{P}$, $R(P) = \mathcal{S}$ and $A \in L(\mathcal{H})^+$. Suppose that $A$ is injective non invertible and $\dim \mathcal{S}^\perp < \infty$. Then $(A, \mathcal{S})$ is compatible if and only if $\mathcal{S}^\perp \subseteq R(A)$.*

*Proof.* By equation (2), $(A, \mathcal{S})$ is compatible if and only if $A^{-1}(\mathcal{S}^\perp) + \mathcal{S} = \mathcal{H}$. Since $A$ is injective, equation (3) says that $A^{-1}(\mathcal{S}^\perp) \cap \mathcal{S} = \{0\}$. Now the result becomes clear because $\dim A^{-1}(\mathcal{S}^\perp) = \dim(\mathcal{S}^\perp \cap R(A))$ ∎

**Example 6.4** Let $T \in L(\mathcal{H}, L^2(\Pi))$ given by $Te_m = \frac{e^{i(m+1)t}}{m}$, where $e_m$ ($m \in \mathbb{N}$) is an orthonormal basis of $\mathcal{H}$. Then $A = T^*T$ is given by $Ae_m = \frac{e_m}{m^2}$, which is injective non invertible. Let $\xi_1, \ldots, \xi_n \in R(A)$, denote by $\mathcal{S} = \{\xi_1, \ldots, \xi_n\}^\perp$ and $P = P_\mathcal{S}$. If $\xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \ldots, \xi_i^{(m)}, \ldots)$, $1 \le i \le n$, denote by

$$\eta_i = (\xi_i^{(1)}, 4\xi_i^{(2)}, \ldots, m^2\xi_i^{(m)}, \ldots) \in \mathcal{H}\ , \quad 1 \le i \le n,$$

and $Q$ the orthogonal projection onto the subspace $\mathcal{T}$ generated by $\eta_1, \ldots, \eta_n$. It is clear that $\mathcal{T} = A^{-1}(\mathcal{S}^\perp)$. Then $(A, \mathcal{S})$ is compatible and $P_{A,\mathcal{S}}$ is the projection onto $\mathcal{S}$ with kernel $\mathcal{T}$. Therefore (cf [16] or [5]) $\|PQ\| < 1$,

$$P_{A,\mathcal{S}} = (1 - QP)^{-1}(1 - Q) = \sum_{k=0}^{\infty} (QP)^k (1 - Q)$$

and $\|P_{A,\mathcal{S}}\| = \|1 - P_{A,\mathcal{S}}\| = (1 - \|PQ\|^2)^{-1/2}$.

**Remark 6.5** Let $B \in L(\mathcal{H})^+$ be injective non invertible. Let $\xi \in \mathcal{H}$ a unit vector, $\mathcal{S} = \{\xi\}^\perp$, $P = P_\mathcal{S}$ and $P_\xi = 1 - P$. Let $B = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ in terms of $P$. By Proposition 6.3, $(B, \mathcal{S})$ is compatible if and only if $\xi \in R(B)$. Note that the sequence $\xi_n$ (in $R(B)$) of Lemma 5.8 converges to $\xi \notin R(B)$. This is, precisely, the fact which implies that $\|P_{B,\{\xi_n\}^\perp}\|$ converges to infinity.

**Example 6.6** Fix $\mathcal{S}$ a closed subspace of $\mathcal{H}$ and consider the set

$$\mathcal{A}_\mathcal{S} = \{A \in L(\mathcal{H})^+ : \text{ the pair } (A, \mathcal{S}) \text{ is compatible } \}$$

and the map $\alpha : \mathcal{A}_\mathcal{S} \to \mathcal{Q}$ given by $\alpha(A) = P_{A,\mathcal{S}}$. We shall see that $\alpha$ is not continuous. Indeed, let $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, and suppose that $R(b) = R(a)$ is a closed subspace $\mathcal{M}$ properly included in $\mathcal{S}$. Denote by $\mathcal{N} = \mathcal{S} \ominus \mathcal{M}$ and consider the projection $P_\mathcal{N}$ and some element $u \in L(\mathcal{S}^\perp, \mathcal{N}) \subseteq L(\mathcal{H})$, $u \neq 0$. Consider, for every $n \in \mathbb{N}$,

$$A_n = A + \tfrac{1}{n}\,(P_\mathcal{N} + u)^*(P_\mathcal{N} + u) = A + \tfrac{1}{n} = \begin{pmatrix} 1 & 0 & u \\ 0 & 0 & 0 \\ u^* & 0 & u^*u \end{pmatrix} \begin{matrix} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^\perp \end{matrix}$$

$$= \begin{pmatrix} \frac{1}{n} & 0 & \frac{1}{n}\,u \\ 0 & a & b \\ \frac{1}{n}\,u^* & b^* & c + \frac{1}{n}\,u^*u \end{pmatrix} \ge A \ge 0.$$

It is clear that $A_n \to A$. Note that $a$ is invertible in $L(\mathcal{M})$. Then, by Theorem 2.3,

$$P_{A,\mathcal{S}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1}b \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^\perp \end{matrix},$$

Also $a + \frac{1}{n}P_{\mathcal{N}}$ is invertible in $L(\mathcal{S})$, for every $n \in \mathbb{N}$. Then

$$P_{A_n,\mathcal{S}} = \begin{pmatrix} n & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{n} & 0 & \frac{1}{n}\, u \\ 0 & a & b \\ \frac{1}{n}\, u^* & b^* & c + \frac{1}{n}\, u^*u \end{pmatrix} = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & a^{-1}b \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^\perp \end{matrix}$$

for all $n \in \mathbb{N}$. Therefore $\alpha(A_n) = P_{A_n,\mathcal{S}} \not\to P_{A,\mathcal{S}} = \alpha(A)$. Note that the sequence $\alpha(A_n)$ converges (actually, it is constant) to $P_{A,\mathcal{S}} + u$, which belongs to $\mathcal{P}(A, \mathcal{S})$ by Theorem 2.3.

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