INDEX OF HADAMARD MULTIPLICATION BY POSITIVE MATRICES II

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Abstract

For each $n \times n$ positive semidefinite matrix A we define the minimal index $I(A) = \max\{\lambda \geq 0 : A \circ B \succeq \lambda B$ for all $B \succeq 0\}$ and, for each norm N, the N-index $I_N(A) = \min\{N(A \circ B) : B \succeq 0$ and $N(B) = 1\}$, where $A \circ B = [a_{ij} b_{ij}]$ is the Hadamard or Schur product of $A = [a_{ij}]$ and $B = [b_{ij}]$ and $B \succeq 0$ means that B is a positive semidefinite matrix. A comparison between these indexes is done, for different choices of the norm N. As an application we find, for each bounded invertible selfadjoint operator S on a Hilbert space, the best constant $M(S)$ such that $\|STS + S^{-1}TS^{-1}\| \geq M(S)\|T\|$ for all $T \succeq 0$.

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1 Introduction

Given $A = [a_{ij}], B = [b_{ij}] \in M_n = M_n(\mathbb{C})$, the algebra of $n \times n$ matrices over $\mathbb{C}$, denote by $A \circ B$ the Hadamard product $[a_{ij}b_{ij}]$. In this paper $A \succeq 0$ means that A is positive semidefinite ; $P_n = \{A \in M_n : A \succeq 0\}$ denotes the set of positive semidefinite matrices.

Every $A \in M_n$ defines a linear map $\Phi_A : M_n \to M_n$ given by $\Phi_A(B) = A \circ B$, for $B \in M_n$. By Schur's product theorem [22] (see also [14, 7.5.3]) $A \circ B \in P_n$ if $A, B \in P_n$, so that Φ_A is a positive linear map. Actually it is completely positive, i.e., the inflation map $\Phi_A^{(m)}$, which acts entrywise as Φ_A on $M_m(M_n)$, is positive for all $m \in \mathbb{N}$; see [20, Prop. 1.2]. In [23], the second author studied conditions under which

$$\max\{\lambda \geq 0 : \Phi_A(B) \succeq \lambda B, \ \forall\ B \in P_n\} = \inf\{\|\Phi_A(B)\| : B \in P_n,\ \|B\| = 1\}.$$

The problem comes from the theory of conditional expectations. A *conditional expectation* on a C*-algebra $\mathcal{A}$ is a norm one projection $\mathcal{E} : \mathcal{A} \to \mathcal{A}$ such that $\mathcal{E}(\mathcal{A})$ is a sub-C*-algebra of $\mathcal{A}$. Every conditional expectation $\mathcal{E}$ satisfies the condition

$$\sup\{\lambda \geq 0 : \|\mathcal{E}(a)\| \geq \lambda \|a\|,\ \forall\ a \in \mathcal{A}^+\} = \sup\{\lambda \geq 0 : \mathcal{E}(a) \geq \lambda a,\ \forall\ a \in \mathcal{A}^+\}, \quad (1)$$

where $\mathcal{A}^+ = \{c \in \mathcal{A} : c \succeq 0\}$. The inverse of this number is called the *index* of $\mathcal{E}$ and it is useful in the classification of inclusions of subalgebras of C*-algebras. Note that a conditional expectation is completely positive. If $\mathcal{E} : \mathcal{A} \to \mathcal{A}$ is a completely positive map that is not a conditional expectation, (1) fails in general and the problem arises of characterizing those $\mathcal{E}$ such that (1) holds.

For $A \in P_n$ define the *minimal index* $I(A) = \max\{\lambda \geq 0 : A \circ B \succeq \lambda B,\ \forall\ B \in P_n\}$ and the *N-index* $I_N(A) = \max\{\lambda \geq 0 : N(A \circ B) \geq \lambda N(B),\ \forall\ B \in P_n\}$ for any given norm N on M_n. We are mainly concerned with Schatten norms $\|\cdot\|_p$ for $p = 1, 2$, and ∞; we use the shorter notations I_1, I_2, and I_{sp} for $I_{\|\cdot\|_1}$, $I_{\|\cdot\|_2}$, and $I_{\|\cdot\|_\infty}$, respectively. I_{sp} is called the *spectral index* .

If $\mathcal{A} = M_n$, every conditional expectation $\mathcal{E}$ has the form $\mathcal{E}(C) = U\Phi_A(U^*CU)U^*$, where $U \in M_n$ is unitary and $A \in P_n$ is a direct sum of matrices whose diagonal entries are all equal to one. In this case, $\text{Ind}(\mathcal{E})^{-1} = 1/k = I_{sp}(A) = I(A)$, where k is the number of diagonal blocks of A. We remove the inverse in our definition of minimal and N-index in order to avoid complications when the index is zero.

For references on the norm of Φ_A, see [2], [3], [9], [10], [11], [17], [19], [20] and references included therein. There is an extensive bibliography about the index of conditional expectations; see [21] and its references. For a deep study of the index theory of completely positive maps on operator algebras, see [5] and [12].

This paper compares these notions of index and investigates how to compute them. The results obtained are useful in the study of certain operator inequalities. Recall that if $L(\mathcal{H})$ is the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ and $S \in L(\mathcal{H})$ is a selfadjoint invertible operator, then

$$\|STS + S^{-1}TS^{-1}\| \geq 2\|T\|$$

for all $T \in L(\mathcal{H})$ [4]. It is natural to ask whether 2 is the best constant for each fixed S. Using a reduction to the finite dimensional case and a criterion for computing $I_{sp}(B)$ for matrices $B \in P_n$ such that $B \geq 0$, in terms of the principal submatrices of B (see Corollary 4.6), we are able to find for each S, the best constant $M(S)$ such that $\|STS + S^{-1}TS^{-1}\| \geq M(S)\|T\|$ for all $T \succeq 0$.

In this paper we write $A \geq 0$ for matrices (or vectors) with nonnegative entries. We write $A \succeq B$ or $A \geq B$ if $A - B \succeq 0$ or $A - B \geq 0$, respectively. $R(A)$ is the range of A and $\ker A$ is the kernel of A, where A is thought of as acting on $\mathbb{C}^n$. A^T is the transpose matrix of A, $\overline{A} = [\overline{a}_{ij}]$ is the conjugate matrix of A, and $A^* = \overline{A}^T$. $\rho(A)$ is the spectral radius of A and $A^\dagger$ is the Moore Penrose pseudoinverse of A. Throughout, p denotes the vector $(1, \ldots, 1)^T$ and E denotes the matrix pp^T, which has all its entries equal to 1.

Section 2 contains some elementary characterizations of the minimal index. We prove that, for a given $A \in P_n$, $I(A) > 0$ if and only if $p \in R(A)$; and, in this case, $I(A)^{-1}$ is the spectral radius of $A^\dagger E$.

Section 3 is devoted to a comparison of the minimal index with the spectral index. The main result in this section is the following: if $A \in P_n$, $A \geq 0$, and there exists a vector $u \in A^{-1}(\{p\})$ such that $u \geq 0$, then $I(A) = I_{sp}(A)$. The converse holds if $I(A) \neq 0$, without the hypothesis that $A \geq 0$.

In Section 4 we compare the indexes associated with the spectral and the Frobenius norms. The main result here is that $I_2(A) = I_{sp}(\overline{A} \circ A)^{1/2}$ for every $A \in P_n$. As a consequence of the proof of this result we compute $I_{sp}(B)$ for matrices $B \in P_n$ such that $B \geq 0$, in terms of the principal submatrices of B (see Corollary 4.6). This criterion is the main tool used in Section 5, where we compute the minimal and spectral indexes of $\Lambda = [\lambda_i\lambda_j + \frac{1}{\lambda_i\lambda_j}]$ for any n-tuple of positive numbers $\lambda_1, \ldots, \lambda_n$ and use them to find, for each bounded Hermitian invertible operator S on a Hilbert space $\mathcal{H}$, the number

$$M(S) = \inf\{\|STS + S^{-1}TS^{-1}\| : T \succeq 0,\ \|T\| = 1\}. \tag{2}$$

For example, if $\|S\| \leq 1$, then $M(S) = \|S\|^2 + \|S\|^{-2}$.

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2 Elementary properties of the index

Let us give more detailed definitions:

Definition 2.1 The Hadamard **minimal** index of $A \in P_n$ is

$$\begin{aligned} I(A) &= \max \{ \lambda \geq 0 : A \circ B \succeq \lambda B \quad \forall\, B \in P_n \} \\ &= \max \{ \lambda \geq 0 : (\Phi_A - \lambda\, Id)B \succeq 0 \quad \text{for all} \quad B \in P_n \} \\ &= \max \{ \lambda \geq 0 : A - \lambda E \succeq 0 \}. \end{aligned}$$

The last equality follows from the fact that for $C \in M_n$, the map Φ_C is positive if and only if $C \succeq 0$.

Definition 2.2 Given a norm N in M_n, the Hadamard N-index for $A \in P_n$ is

$$\begin{aligned} I_N(A) &= \max \{ \lambda \geq 0 \ : \ N(A \circ B) \geq \lambda N(B) \quad \forall B \in P_n \} \\ &= \min \{ N(A \circ B) \ : \ B \in P_n \ \text{ and } \ N(B) = 1 \}. \end{aligned}$$

The index associated with the spectral norm $\|\cdot\|$ is denoted by $I_{sp}(\cdot)$; we call it the *spectral index*. The index associated with the Frobenius norm $\|\cdot\|_2$ is denoted by $I_2(\cdot)$.

Example 2.3 Let $A = [a_{ij}]$ and $B = [b_{ij}] \in P_n$. Then, if $\|\cdot\|_1$ denotes the trace norm,

$$\|B\|_1 = tr(B) = \sum_{i=1}^{n} b_{ii} \quad \text{ and } \quad \|A \circ B\|_1 = tr(A \circ B) = \sum_{i=1}^{n} a_{ii} b_{ii}.$$

From these identities it is easy to see that, if $I_1(\cdot)$ denotes the associated index, then $I_1(A) = \min_{1 \leq i \leq n} a_{ii}$ for every $A \in P_n$

Remark 2.4 Estimation of the N-index of a matrix A can be seen as an inequality, namely, $N(A \circ B) \geq I_N(A) N(B)$ for every $B \in P_n$. It would also be interesting to get such inequalities without the restriction $B \succeq 0$ (of course, for matrices A without zero entries). But in this case, the constant involved is the inverse of the norm induced by N of the map Φ_C, where $c_{ij} = a_{ij}^{-1}$. The computation of such norms is well known (see [11], [20] [19], [10], [17], [9]). For the index associated with the Frobenius norm, the computation of an infimum without the restriction $B \succeq 0$ becomes trivial, but with this restriction it is certainly not trivial, as shown in Theorem 4.3.

The minimal index $I(A)$

The index $I(\cdot)$ is called minimal because $I(A) \leq I_N(A)$ for every unitary invariant norm N. Indeed, given $B \in P_n$, then $A \circ B \succeq I(A) B$ and, by Weyl's monotonicity theorem, $s_i(A \circ B) \geq I(A) s_i(B))$, $1 \leq i \leq n$ (where s_i denote the i-th singular value). Therefore $N(A \circ B) \geq I(A) N(B)$ by Ky Fan's dominance theorem; see [15, 3.5.9].

Given $B, C \succeq 0$ the following relation holds:

$$\max\{\alpha \geq 0 : \alpha C \preceq B\} = \|C^{1/2} B^{\dagger} C^{1/2}\|^{-1} = \rho(B^{\dagger} C)^{-1} \tag{3}$$

In fact, if B is non singular, (3) follows from [14, 7.7.3]. If B has rank $r < n$, there exist a unitary matrix U and $\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$ such that Λ_1 is an r-by-r invertible matrix and $B = U \Lambda U^*$. If $\alpha \geq 0$ and $B \succeq \alpha C$ then, setting $D = \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^* & D_{22} \end{bmatrix} = U^* C U$, we get $\begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} = \Lambda \succeq \alpha D$ so that $D_{22} = 0$ and, then, $D_{12} = 0$. Therefore $\Lambda_1 \succeq \alpha D_{11}$ and, by the non singular case, $\rho(\Lambda_1^{-1} D_{11}) \leq \alpha$. The result follows by observing that $\rho(\Lambda_1^{-1} D_{11}) = \rho(B^\dagger C)$. Observe also that the block structure of D and the invertibility of Λ_1 imply the inclusion $R(C) \subset R(B)$.

Taking $B = A$ and $C = E$ in (3) we get $I(A) = \max\{\alpha \geq 0 : A \succeq \alpha E\} = \rho(A^\dagger E)^{-1}$ for every $A \in P_n$ such that $p \in R(A)$. This proves part of the following result.

Proposition 2.5 *Let $A \in P_n$. Then $I(A) \neq 0$ if and only if the vector p belongs to $R(A)$. In this case, for any vector y such that $Ay = p$, we have*

$$I(A) = \rho(A^\dagger E)^{-1} = \langle y, p \rangle^{-1} = (\sum_{i=1}^{n} y_i)^{-1} . \tag{4}$$

Proof. By definition, $I(A) \neq 0$ if and only if there exists $\alpha > 0$ such that $A \succeq \alpha E$. By the comments following (3), this means that $R(E) \subset R(A)$ or, since p spans $R(E)$, that $p \in R(A)$. Finally, $I(A)^{-1} = \rho(A^\dagger E) = \rho(A^\dagger p p^T) = \rho(p^T A^\dagger p) = p^T A^\dagger p = \langle A^\dagger p, p \rangle$, and $A(A^\dagger p) = p$. If y is any vector such that $Ay = p$ then $y - A^\dagger p \in \ker A = R(A)^\perp$, so $\langle y, p \rangle = \langle A^\dagger p, p \rangle$.

Proposition 2.6 *Let $A \in P_n$. Then $I(A) = \min \{ \langle z, Az \rangle : \sum_{i=1}^n z_i = 1 \}$.*

Proof. If $\langle z, p \rangle = 1$, then $\langle z, Az \rangle \geq I(A) \langle z, Ez \rangle = I(A)\ z^* p p^* z = I(A) \langle z, p \rangle^2 = I(A)$. If $p \in R(A)$, let $x \in \mathbb{C}^n$ be such that $Ax = p$. Then $z = I(A)x$ satisfies $\langle z, p \rangle = I(A) \langle x, p \rangle = 1$ and $\langle z, Az \rangle = I(A)\langle z, p \rangle = I(A)$ by Proposition 2.5. If $p \notin R(A) = (\ker A)^\perp$, then there exists $z \in \ker A$ such that $\langle z, p \rangle = 1$ and $\langle z, Az \rangle = 0 = I(A)$. ∎

Remark 2.7 Using proposition (3.9) of [23] and the results of this section, it is easy to see that, for all $A \in P_n$ and $m \in \mathbb{N}$, the inflation matrix $A^{(m)} = E_m \otimes A$ (where $E_m \in P_m$ has all its entries equal to 1) satisfies $I_{sp}(A^{(m)}) = I_{sp}(A)$ and $I(A^{(m)}) = I(A)$. Note that the inflation map $\Phi_A^{(m)} = \Phi_{A^{(m)}}$. Therefore the indexes of Φ_A are invariant under inflations and are invariants of Φ_A as a completely positive map.

$I_N(A)$ for general norms

Let $A \in P_n$ and let N be a norm in M_n. If $I_N(A) = 0$, there is some positive semidefinite matrix C such that $N(C) = 1$ (so $C \neq 0$) and $N(A \circ C) = 0$. But then $A \circ C = 0$, so $c_{ij} = 0$ whenever $a_{ij} \neq 0$. If all $a_{ii} \neq 0$, then all $c_{ii} = 0$, which forces $C = 0$. This contradiction shows that if all $a_{ii} \neq 0$, then $I_N(A) > 0$. Conversely, if some $a_{ii} = 0$ just take $C = E_{ii}/N(E_{ii})$, so $I(A) \leq N(A \circ C) = 0$. Thus,

$$I_N(A) > 0 \quad \text{if and only if all} \quad a_{ii} > 0. \tag{5}$$

Let $J \subseteq \{1, 2, \ldots, n\}$ and let A_J denote the principal submatrix of A associated with J. Then, minimality ensures that

$$I_N(A) \leq I_N(A_J). \tag{6}$$

Remark 2.8 Let $A \in P_n$. Then, it can be shown that for every unitary invariant norm N, the following properties hold:

1. If A has rank one, then $I_N(A) = \min_{1 \leq i \leq n} A_{ii}$.
2. If A is positive and diagonal, then $I_N(D) = N'(D^{-1})^{-1} = \Phi'(a_{11}^{-1}, \ldots, a_{nn}^{-1})^{-1}$, where N' is the dual norm of N and Φ' is the symmetric gauge function on $\mathbb{R}^n$ associated with N'; see [7, Chapter IV].

[23, Proposition 3.2] tells us that

$$I_{sp}(A) \ = \inf \{ \ I_{sp}(D) : A \preceq D \ \text{ and } D \text{ is diagonal } \}, \tag{7}$$

and one could hope that a similar formula holds for any norm, but it does not. In fact, our Corollary 4.4 says that, for every $A \in P_n$ and the Frobenius norm,

$$\begin{aligned} I_2(A) \ &= \inf\{(\textstyle\sum_1^n D_{ii}^{-2})^{-1/2} : D \text{ is diagonal and } A \circ \bar{A} \preceq D^2\} \\ &= \inf \{ \ I_2(D) \ : \ D \ \text{ is diagonal and } \ A \circ \bar{A} \preceq D^2 \ \}. \end{aligned} \tag{8}$$

Note that the condition $A \circ \bar{A} \preceq D^2$ is strictly less restrictive than $A \preceq D$ (the reverse implication follows from Schur's theorem). Nevertheless, equation (8) allows one to compute the 2-index for every positive semidefinite matrix using only diagonal matrices. We intend to study this type of characterizations of $I_N(A)$ for general norms in a forthcoming paper.

3 $I(A) = I_{sp}(A)$.

In this section we characterize those matrices $A \in P_n$ such that $I(A) = I_{sp}(A)$. In [23] it is shown that for $A = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in P_2$,

$$0 \neq I_{sp}(A) = I(A) \quad \Leftrightarrow \quad b \in \mathbb{R} \ \text{ and } \ 0 \leq b \leq \min\{a, c\} \neq 0. \tag{9}$$

This is easily seen to be equivalent to the conditions

1. $A \geq 0$.
2. There exists a vector $z \geq 0$ such that $Az = (1,1)^T$ (if A is invertible, this means that $A^{-1}(1,1)^T \geq 0$).

We prove that, for positive semidefinite matrices of any size with nonnegative entries, condition 2 is equivalent to the identity $I_{sp}(A) = I(A)$. But first we need two lemmas:

Lemma 3.1 *Let $A \in P_n$ and $L = \{z \in \mathbb{R}^n : \sum_i z_i = 1\}$. Consider the sets*

$$V_1 = \{z \in L : \langle Az, z\rangle = I(A)\} \quad \textit{and} \quad V_2 = \{z \in L : Az = I(A)p\}.$$

Then $V_1 = V_2 \neq \emptyset$. Moreover, any local extreme point of the map $G : L \to \mathbb{R}$ given by $G(z) = \langle Az, z\rangle$, belongs to V_2.

Proof. It is clear that $V_2 \subseteq V_1$. By Proposition 2.6, $I(A) \leq \min\{\langle Av, v\rangle : v \in L\}$. Then the map $G : L \to \mathbb{R}$ given by $G(z) = \langle Az, z\rangle = \sum_{i,j} a_{ij} z_j z_i$, is differentiable and bounded from below. Thus G must have a minimum, which is also a critical point. Let the columns of $X \in M_{n,n-1}$ be a basis for the orthogonal complement of p. Then we seek the unconstrained minimum of

$$\phi(\xi) = G(X\xi + p/n) = \langle A(X\xi + p/n), (X\xi + p/n)\rangle = (X + p/n)^T A(X\xi + p/n)$$

over all $\xi \in \mathbb{R}^{n-1}$. But $\nabla\phi(\xi) = 2X^T A(X\xi + p/n) = 0$ says that at a critical point ξ_0, $Az_0 = \lambda p$ for some λ, where $z_0 \equiv X\xi_0 + p/n \in L$. But, in that case,

$$I(A) \leq \langle Az_0, z_0\rangle = \lambda\langle p, z_0\rangle = \lambda.$$

If $I(A) = 0$ then $\lambda = 0$, because $p \notin R(A)$, by Proposition 2.5. If $I(A) > 0$ then also $\lambda = I(A)$, because $y = \lambda^{-1} z_0$ satisfies $Ay = p$ and

$$\lambda = \langle Az_0, z_0\rangle = \lambda^2\langle Ay, y\rangle = \lambda^2 I(A)^{-1}.$$

So $\xi_0 \in \mathbb{R}^{n-1}$ is a critical point of ϕ if and only if $z_0 = X\xi_0 + p/n \in V_2$. Since each local extreme must be a critical point, this shows that $\emptyset \neq V_1 \subseteq V_2$ and the final assertion is proved ∎

Lemma 3.2 *Let $A \in M_n$, and suppose $x \in \mathbb{C}^n$ with $\|x\| = 1$. Let $y = x \circ \bar{x} = (|x_1|^2, \ldots, |x_n|^2)^T$.*

1. *If $Ay = \lambda p$ for some $\lambda \in \mathbb{C}$, then $(A \circ xx^*)x = \lambda x$.*
2. *Conversely, if all $x_i \neq 0$ and $(A \circ xx^*)x = \lambda x$ for some $\lambda \in \mathbb{C}$, then $Ay = \lambda p$.*

If $A \in P_n$, the eigenvalue λ of the matrix $A \circ xx^*$ associated with the vector x must be $I(A)$ and $Ay = I(A)\, p$.

Proof. Suppose that $Ay = \lambda p$. Then

$$(A \circ xx^*)x = (a_{ij} x_i \bar{x}_j) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} (\sum_j a_{1j}|x_j|^2)\, x_1 \\ \vdots \\ (\sum_j a_{nj}|x_j|^2)\, x_n \end{pmatrix} = \begin{pmatrix} (Ay)_1\, x_1 \\ \vdots \\ (Ay)_n\, x_n \end{pmatrix} = \lambda\, x. \quad (10)$$

(10) shows that if all $x_i \neq 0$ and $(A \circ xx^*)x = \lambda x$, then $Ay = \lambda p$. If $A \in P_n$ and $I(A) = 0$, then $\lambda = 0$ because $p \notin R(A)$. If $I(A) \neq 0$, then $p \in R(A) = (\ker A)^\perp$. So $Ay \neq 0$ because $1 = \|x\|^2 = \langle p, y \rangle \neq 0$. Then $\lambda \neq 0$. If $z = \lambda^{-1} y$, then $Az = p$ and $1 = \langle p, y \rangle = \lambda \langle Az, z \rangle = \lambda\, I(A)^{-1}$, by Proposition 2.5, $\lambda = I(A)$. ∎

Theorem 3.3 *Let $A \in P_n$.*

1. *If $I_{sp}(A) = I(A) \neq 0$, then there exists a vector $u \geq 0$ such that $Au = p$.*
2. *If $A \geq 0$ and there exists a vector $u \geq 0$ such that $Au = p$, then $I_{sp}(A) = I(A)$.*

Proof.

1. Observe that $I(A)B \preceq A \circ B \preceq \|A \circ B\|\, I$. By Lemma 2.1 of [23], there exists $x \in \mathbb{R}^n$ such that $\|x\| = 1$ and $I_{sp}(A) = \|A \circ xx^*\|$. So, if $y = x \circ x$, then $\langle y, p \rangle = 1$ and

$$I(A)xx^T \preceq D_x A D_x \preceq I(A)I,$$

which implies that

$$I(A) = I(A)(x^T x)^2 \leq x^T D_x A D_x x = y^T A y \leq I(A) x^T x = I(A).$$

We have $\langle Ay, y \rangle = I(A)$, $y \geq 0$ and $\langle y, p \rangle = 1$. Then, by Lemma 3.1, $Ay = I(A)p$. Take $u = I(A)^{-1} y$.

2. Let u be a nonnegative vector such that $Au = p$. Let $y = I(A)u$ and $x = (y_1^{1/2}, \ldots, y_n^{1/2})^T$. Note that $\|x^2\| = \langle y, p \rangle = 1$. By Lemma 3.2 we know that x is an eigenvector of $A \circ xx^*$ with eigenvalue $I(A)$. Recall that always $I(A) \leq I_{sp}(A)$.

Case 1. Suppose that x has strictly positive entries. Since $A \circ xx^* \geq 0$, it is well known (see Corollary 8.1.30 of [14]) that the eigenvalue $I(A)$ of x must be the spectral radius of $A \circ xx^*$. Since $A \circ xx^* \in P_n$ we deduce that $I(A) = \|A \circ xx^*\| \geq I_{sp}(A)$.

Case 2. Let $J = \{i : x_i \neq 0\}$, A_J the principal submatrix of A determined by the indexes of J and similarly define x_J. Then x_J is an eigenvector of $A_J \circ x_J x_J^*$ with eigenvalue $I(A)$. Note also that $A_J \circ x_J x_J^* \succeq I(A_J) x_J x_J^*$ and $x_J x_J^*(x_J) = \|x_J\|^2 x_J = x_J$. Then

$$0 \leq \langle (A_J \circ x_J x_J^* - I(A_J) x_J x_J^*) x_J, x_J \rangle = I(A) - I(A_J)$$

and, by the definition of I, $I(A_J) = I(A)$. Now, as in Case 1, we can deduce that

$$I(A) = I(A_J) = \|A_J \circ x_J x_J^*\| \geq I_{sp}(A_J) \geq I_{sp}(A),$$

where the last inequality holds by (6). ∎

Corollary 3.4 *Let $A \in P_n$ such that $A \geq 0$ and $I_{sp}(A) = I(A)$. Let u be a nonnegative vector such that $Au = p$ and $y = I(A)\ u$.*

1. *Let $x = (y_1^{1/2}, \ldots, y_n^{1/2})^T$. Then $\|x\| = 1$ and $\|A \circ xx^*\| = I_{sp}(A)$.*
2. *Let $J = \{i : u_i \neq 0\}$ and denote by A_J the principal submatrix of A determined by J. Then $I(A) = I(A_J) = I_{sp}(A_J) = I_{sp}(A)$.*

Proof. This follows from the proof of Theorem 3.3. ∎

Remark 3.5 In Theorem 3.3(2), the hypothesis that $A \geq 0$ is essential. Indeed, consider $A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ and $u = (2,3)^T$. Then $Au = (1,1)^T$ but $1/5 = I(A) \neq I_{sp}(A) = 1$. For $A \in P_n$, we conjecture that $I(A) = I(sp, A) \neq 0$ implies that $A \geq 0$, as in the 2×2 case.

4 $I_{sp}(A)$ and $I_2(A)$

In this section we study the relation between the indexes associated with the spectral and Frobenius norms. In Lemma 2.1 of [23] it is shown that the index $I_{sp}(\cdot)$ is always attained at rank one projections. The index $I_1(\cdot)$ has the same property (see Example 2.3). It is natural to conjecture that the same result holds for any unitary invariant norm N. We show that the conjecture is true for the Frobenius norm:

Proposition 4.1 *Let $A \in P_n$. Then there exists an $x \in \mathbb{C}^n$, such that $\|x\| = 1$ and $I_2(A) = \|A \circ xx^*\|_2$. That is, $I_2(A)$ is attained at a rank one projection.*

Proof. Let $\lambda = \max\{\mu \geq 0 : \|A \circ B\|_2 \geq \mu \|B\|_2$ for all $B \in P_n$ with rank $1\}$. By its definition $\lambda \geq I_2(A)$. Let us prove that $\|A \circ B\|_2 \geq \lambda \|B\|_2$ for all $B \in P_n$. Indeed, for $B \succeq 0$, write $B = \sum_{i=1}^k B_i$, where each B_i has rank one, $B_i \in P_n$, and $B_i B_j = 0$ if $i \neq j$. Then

$$\lambda^2 \|B\|_2^2 = \lambda^2 \sum_{i=1}^k \|B_i\|_2^2 \leq \sum_{i=1}^k \|A \circ B_i\|_2^2.$$

On the other hand, using $\operatorname{tr}(XY) \geq 0$ for positive semidefinite matrices X and Y,

$$\begin{aligned} \|A \circ B\|_2^2 &= \operatorname{tr}((A \circ B)^*(A \circ B)) = \sum_{ij} \operatorname{tr}(A \circ B_i)^*(A \circ B_j) \\ &\geq \sum_i \operatorname{tr}(A \circ B_i)^*(A \circ B_i) = \sum_i \|A \circ B_i\|_2^2. \end{aligned}$$

∎

Proposition 4.2 *Let $A \in P_n$.*

1. *There exists a nonnegative vector x such that $\|x\| = 1$ and $\|A \circ xx^*\|_2 = I_2(A)$.*

2. *Any such vector x satisfies $(A \circ \bar{A} \circ xx^*)x = I(B_J)x$, where $B = A \circ \bar{A}$ and $J = \{i : x_i \neq 0\}$.*

Proof. Let y be an unit vector such that $\|A \circ yy^*\|_2 = I_2(A)$. Let $x_i = |y_i|$. It is easily checked that $\|x\| = 1$ and $\|A \circ xx^*\|_2 = \|A \circ yy^*\|_2 = I_2(A)$, which proves 4.2(1). Let $B = A \circ \bar{A} \in P_n$. Let y be a nonnegative unit vector and let $z = (y_1^2, \ldots, y_n^2)^T$. Then

$$\|A \circ yy^*\|_2^2 = \sum_{i,j} |a_{ij}|^2 y_i^2 y_j^2 = \sum_{i,j} b_{ij} z_j z_i = \langle Bz, z \rangle$$

and $\sum_1^n z_i = 1$. Then $\|A \circ yy^*\|_2 = I_2(A)$ if and only if $\langle Bz, z\rangle$ is the minimum of the map $G(v) = \langle Bv, v\rangle$ restricted to the simplex $\Delta = \{v \in \mathbb{R}^n : v \geq 0 \text{ and } \sum_1^n v_i = 1\}$. Using Lemma 3.1, we know that if z belongs to the interior Δ° of Δ, then z is a local extremum of G in the plane $L = \{z \in \mathbb{R}^n : \sum_i z_i = 1\}$, so $Bz = I(B)\, p$.

If the vector x of item 1 satisfies $x_i > 0$ for all i, then $z = x \circ x \in \Delta^\circ$ and $Bz = I(B)\, p$. By Lemma 3.2, $(A \circ \bar{A} \circ xx^*)x = I(B)x$, showing 4.2(2) in this case. If some $x_i = 0$, let $J = \{i : x_i \neq 0\}$, let B_J be the principal submatrix of B determined by the indexes of J, and similarly define x_J. Then $I_2(A) = \|A \circ xx^*\|_2 = \|A_J \circ x_J x_J^*\|_2 \geq I_2(A_J)$ and

$$I_2(A) = I_2(A_J) = \|A_J \circ x_J x_J^*\|_2,$$

because the converse inequality always holds by (6). Note that, by its definition, x_J has no zero entries. By the previous case, x_J is an eigenvector of $B_J \circ x_J x_J^*$ with eigenvalue $I(B_J)$. But clearly $B \circ xx^*$ has zeroes outside $J \times J$, so x is an eigenvector of $B \circ xx^*$ if and only if x_J is an eigenvector of $B_J \circ x_J x_J^*$. Note that the eigenvalue of x is always $I(B_J)$. ∎

Theorem 4.3 *Let $A \in P_n$. Then $I_2(A) = I_{sp}(\bar{A} \circ A)^{1/2}$.*

Proof. If $B = \bar{A} \circ A$ and $y \in \mathbb{C}^n$ with $\|y\| = 1$, then

$$\|A \circ yy^*\|_2^2 = \sum_{i,j} |a_{ij}|^2 |y_i|^2 |y_j|^2 = \langle (B \circ yy^*)y, y\rangle \leq \|B \circ yy^*\|.$$

Therefore $I_2(A)^2 \leq I_{sp}(B)$. On the other hand, let x be a nonnegative unit vector such that $I_2(A)^2 = \|A \circ xx^*\|^2$ and $J = \{i : x_i \neq 0\}$. Then, by Proposition 4.2, $(B \circ xx^*)x = I(B_J)x$ and

$$I_2(A)^2 = \|A \circ xx^*\|^2 = \langle (B \circ xx^*)x, x\rangle = I(B_J).$$

But x_J is a unit eigenvector of $B_J \circ x_J x_J^*$ with strictly positive entries. So, by Lemma 3.2, $B_J(x_J \circ x_J) = I(B_J)(1, \ldots, 1)^T$. Suppose that $I_2(A) \neq 0$. Then $I(B_J) \neq 0$, $B_J \geq 0$, the vector $u = I(B_J)^{-1}(x_J \circ x_J)$ has strictly positive entries, and $B_J u = (1, \ldots, 1)^T$. Hence we can apply Theorem 3.3 to B_J and, by (6),

$$I(B_J) = I_{sp}(B_J) \geq I_{sp}(B) \geq I_2(A)^2 = I(B_J).$$

If $I_2(A) = 0$, then (5) ensures that some $a_{ii} = 0$, so $I_{sp}(B) = 0$ by (5). ∎

Corollary 4.4 *Let $A \in P_n$. Then*

$$I_2(A) = \inf \{ \ (\Sigma_1^n d_{ii}^{-2})^{-1/2} : \ D \ \text{ is positive diagonal and } \ A \circ \bar{A} \preceq D^2 \ \}$$

$$= \inf \{ \ I_2(D) \ : \ D \ \text{ is positive diagonal and } \ A \circ \bar{A} \preceq D^2 \ \}.$$

Proof. This is a direct consequence of Theorem 4.3 and Proposition 3.2 of [23]. ∎

Remark 4.5 In Theorem 4.3 we get information about $A \in P_n$ using $B = \bar{A} \circ A$. But it can also be used to get information about any $B \in P_n$ with $B \geq 0$, using $A = (b_{ij}^{1/2})$. Unfortunately it may certainly happen that $A \notin P_n$. Nevertheless this obstruction can be removed in the following way: given a selfadjoint (but not necessarily positive semidefinite) matrix $A \in M_n$, consider the index

$$I_2(A) = \min\{ \ \|A \circ xx^*\|_2 \ : \ \|x\| = 1 \ \},$$

which, by Proposition 4.1, is consistent with Definition 2.2 when $A \succeq 0$. A careful inspection of the proofs of Proposition 4.2 and Theorem 4.3 shows that they remain true using this new index if the condition "$A \in P_n$" is replaced by "$A = A^*$ and $B = \bar{A} \circ A \in P_n$". Note that Lemma 3.1, Lemma 3.2, and Theorem 3.3 are applied only to the positive semidefinite matrix B or its principal submatrices. The inequality $I_2(A) \leq I_2(A_J)$ in (6) (which is also used in the proofs) remains valid for this new index. This observation is useful in order to avoid the unpleasant condition "$A = (b_{ij}^{1/2}) \in P_n$" in the following result.

Corollary 4.6 *Suppose $B \in P_n$ and $B \geq 0$. Then there exists a subset J_0 of $\{1, 2, \ldots, n\}$ such that $I_{sp}(B) = I_{sp}(B_{J_0}) = I(B_{J_0})$. Therefore*

$$I_{sp}(B) = \min\{ \ I_{sp}(B_J) \ : \ I_{sp}(B_J) = I(B_J) \ \}.$$

If $A = (b_{ij}^{1/2})$ (which may be not positive semidefinite), then J_0 can be characterized as $J_0 = \{i : x_i \neq 0\}$ for any unit vector x such that $I_2(A) = \|A \circ xx^*\|_2$. Also $I_{sp}(B) = \|B \circ xx^*\| = \langle By, y\rangle$, where $y = (|x_1|^2, \ldots, |x_n|^2)^T$.

Proof. Use Remark 4.5 and the proof of Theorem 4.3. ∎

5 An Operator Inequality

In this section we compute the indexes of a particular class of matrices and, as an appplication, we get a new operator inequality, closely related to the inequality proved in [8]; see also [4], [18].

Let $x = (\lambda_1, \ldots, \lambda_n)^T \in \mathbb{R}_+^n$, $S = \{\lambda_1, \ldots, \lambda_n\}$, and

$$\Lambda = \Lambda_x = (\lambda_i\lambda_j + \frac{1}{\lambda_i\lambda_j} \ )_{ij} \in P_n.$$

Observe that Λ has rank 1 or 2.

5.1 Computation of $I(\Lambda)$

1. If all λ_i are equal, then $\Lambda = (\lambda_1^2 + \lambda_1^{-2})\ E$ and $I(\Lambda) = \lambda_1^2 + \lambda_1^{-2}$.
2. If $\#S > 1$, then the range of Λ is spanned by the independent vectors $x = (\lambda_1, \ldots, \lambda_n)^T$ and $y = (\lambda_1^{-1}, \ldots, \lambda_n^{-1})^T$, because $\Lambda = xx^* + yy^* = [xy][xy]^*$ has rank 2.
3. If $\#S = 2$, say $S = \{\lambda, \mu\}$, then $p = ax + by$, with $a = (\lambda+\mu)^{-1}$ and $b = \lambda\mu(\lambda+\mu)^{-1}$. If a vector z satisfies $\Lambda z = p$, then

$$p = \Lambda z = (xx^* + yy^*)z = \ \langle z, x\rangle x + \langle z, y\rangle y.$$

Therefore

$$I(\Lambda) = \ \langle z, p\rangle^{-1} = (\langle z, x\rangle^2 + \langle z, y\rangle^2)^{-1} = \frac{(\lambda+\mu)^2}{1+\lambda^2\mu^2} = I(\Lambda_0),$$

where the last equality is shown in Remark 4.3 of [23].
4. If $\#S > 2$, it is easy to see that p is not in the subspace spanned by x and y. Then $I(\Lambda)$ must be zero by Proposition 2.5 .

Note that $I(\Lambda) \neq 0$ if and only $\#S \leq 2$.

5.2 Computation of $I_{sp}(\Lambda)$

We shall compute $I_{sp}(\Lambda)$ using Corollary 4.6 and therefore use the principal submatrices of Λ, which are matrices of the same type. Let $J \subset \{1, 2, \ldots, n\}$, let $S_J = \{\lambda_j : j \in J\}$, and let x_J be the induced vector. Then $\Lambda_J = \Lambda_{x_J}$ and so $I_{sp}(\Lambda_J) \neq 0$. Suppose that $I_{sp}(\Lambda_J) = I(\Lambda_J)$. Then $\#S_J \leq 2$ by 5.1. If $\#S_J = 2$, let $i_1, i_2 \in J$ be such that $\lambda_{i_1} \neq \lambda_{i_2}$. By Theorem 3.3 there exists a vector $y \in \mathbb{R}^J$ such that $y \geq 0$ and $\Lambda_J\ y = p_J$. Let $z_1 = \Sigma\{y_k : k \in J \text{ and } \lambda_k = \lambda_{i_1}\} \geq 0$ and $z_2 = \Sigma\{y_j : j \in J \text{ and } \lambda_j = \lambda_{i_2}\} \geq 0$. Easy computations show that $\Lambda_{\{i_1,i_2\}}(z_1, z_2)^T = (1, 1)^T$. Then, by Theorem 3.3 and 5.1,

$$I_{sp}(\Lambda_J) = I(\Lambda_J) = \frac{(\lambda_{i_1} + \lambda_{i_2})^2}{1 + \lambda_{i_1}^2\lambda_{i_2}^2} = I(\Lambda_{\{i_1,i_2\}}) = I_{sp}(\Lambda_{\{i_1,i_2\}}).$$

Therefore, in order to compute $I_{sp}(\Lambda)$ using Corollary 4.6, we need to consider only the diagonal entries of Λ and some of the principal submatrices of size 2×2. If $\lambda_i \neq \lambda_j$, (9) ensures that

$$I_{sp}(\Lambda_{\{i,j\}}) = I(\Lambda_{\{i,j\}}) \quad \Leftrightarrow \quad \lambda_i\lambda_j + \frac{1}{\lambda_i\lambda_j} \leq \min\{\lambda_i^2 + \frac{1}{\lambda_i^2}\ ,\ \lambda_j^2 + \frac{1}{\lambda_j^2}\}.$$

If $\lambda_i < \lambda_j$, this condition is equivalent to

$$\lambda_i^2 \leq \frac{1}{\lambda_i\lambda_j} \leq \lambda_j^2. \tag{11}$$

In particular, this implies that $\lambda_i < 1 < \lambda_j$. Then, by Corollary 4.6,

$$I_{sp}(\Lambda) = \min\{M_1, M_2\} \tag{12}$$

where $M_1 = \min_i \lambda_i^2 + \lambda_i^{-2} = \min_i \Lambda_{ii}$ and

$$M_2 = \inf\{\frac{(\lambda_i+\lambda_j)^2}{1+\lambda_i^2\lambda_j^2} \ : \ \lambda_i < 1 < \lambda_j \ \text{ and } \ \lambda_i^2 \le \frac{1}{\lambda_i\lambda_j} \le \lambda_j^2\}.$$

For example, if all $\lambda_i \ge 1$ (or all $\lambda_i \le 1$), then by (11), $I_{sp}(\Lambda) = M_1 = \min_i \lambda_i^2 + \lambda_i^{-2}$. On the other hand, if $\lambda \ne 1$ and $x = (\lambda, \lambda^{-1})^T$, then

$$I_{sp}(\Lambda_x) = M_2 = \frac{\lambda^2+\lambda^{-2}}{2} + 1 < M_1 = \lambda^2 + \lambda^{-2}.$$

Proposition 5.3 *Let $\mathcal{H}$ be a Hilbert space and let S be a bounded selfadjoint invertible operator on $\mathcal{H}$. Let $M(S)$ be the best constant such that*

$$\|STS + S^{-1}TS^{-1}\| \ge M(S)\|T\| \quad \text{for all } \ 0 \preceq T \in L(\mathcal{H}).$$

Then $M(S) = \min\{M_1(S), M_2(S)\}$, where

$$M_1(S) = \min_{\lambda\in\ \sigma(S)} \lambda^2 + \lambda^{-2} \qquad \text{and}$$

$$M_2(S) = \inf\{\frac{(|\lambda|+|\mu|)^2}{1+\lambda^2\mu^2} : \lambda, \mu \in \sigma(S), |\lambda| < |\mu| \text{ and } \lambda^2 \le \frac{1}{|\lambda\mu|} \le \mu^2\}.$$

In particular, if $\|S\| \le 1$ (or $\|S^{-1}\| \le 1$), then

$$M(S) = \|S\|^2 + \|S\|^{-2} \quad (\text{resp. } \ \|S^{-1}\|^2 + \|S^{-1}\|^{-2}).$$

Proof. We follow the same steps as in [8]. By taking the polar decomposition of S, we can assume that $S > 0$, because the unitary part of S is also the unitary part of S^{-1}; it commutes with S and S^{-1} and it preserves norms. Note that we must change $\sigma(S)$ by $\sigma(|S|) = \{|\lambda| : \lambda \in \sigma(S)\}$.

By the spectral theorem, we can assume that $\sigma(S)$ is finite, because S can be approximated in norm by operators S_n such that each $\sigma(S_n)$ is a finite subset of $\sigma(S)$, $\sigma(S_n) \subset \sigma(S_{n+1})$ for all $n \in \mathbb{N}$ and $\cup_n \sigma(S_n)$ is dense in $\sigma(S)$. Then $M(S_n)$ (and $M_i(S_n)$, $i = 1, 2$) converges to $M(S)$ (resp. $M_i(S)$, $i = 1, 2$).

We can suppose also that $\dim \mathcal{H} < \infty$, by choosing an appropriate net of finite rank projections $\{P_F\}_{F\in\mathcal{F}}$ that converges strongly to the identity and replacing S, T by $P_F S P_F$, $P_F T P_F$. Indeed, the net may be choosen in such a way that $SP_F = P_F S$ and $\sigma(P_F S P_F) = \sigma(S)$ for all $F \in \mathcal{F}$. Note that for every $A \in L(H)$, $\|P_F A P_F\|$ converges to $\|A\|$.

Finally, we can suppose that S is diagonal by a unitary change of basis in $\mathbb{C}^n$. In this case, if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of S (with multiplicity) and $x = (\lambda_1, \dots, \lambda_n)^T$, then

$$STS + S^{-1}TS^{-1} = \Lambda_x \circ T.$$

None of our reductions (unitary equivalences and compressions) change the fact that $0 \preceq T$. Now the statement follows from formula (12). If $\|S\| \leq 1$ then $M(S) = M_1(S)$, because $M_2(S)$ is the infimum of the empty set. Clearly $M_1(S)$ is attained at the element $\lambda \in \sigma(S)$ such that $|\lambda| = \|S\|$. ∎

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