ON THE DISTRIBUTION $\left[\delta^{(\ell)}\left(P_+^s\right)\right]^m$

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ABSTRACT. The purpose of this Note is the proof of the formula $\left[\delta^{(\ell)}\left(P_+^s\right)\right]^m$ (cf. formula (4,13)). Here δ es the Dirac-delta function and P_+^s is defined by (1,2). We note that the formula (4.13) is a generalization of the one-dimensional formula $\left[\delta^{(k)}(x^n)\right]^m$ due to A.P. Khapalyuk (cf. [6]). To arrive at our desired formula (4.13), we obtain several others interesting results. These conclusions are, under some conditions, the following formulas,

$\delta(P^2)=2P^{-1}\cdot\delta(P)$, (2,3), $\delta^{(k)}(G)\cdot G^{-k-1}=-\frac{1}{2}\frac{k!}{(-1)^k(2k+1)!}\delta^{(2k+1)}(G)$, (2,5),

$\delta(G)\cdot G^{-1}=-\frac{1}{2}\delta'(G)$, (2,6), $\delta(P^s)=s\left(P^{-1}\right)^{s-1}\delta(P)$, (2,7), $\delta\left(P^2\right)=-\delta'(P)$, (2,8),

$\delta(P^s)=\frac{(-1)^k s}{k!}P^{k+1-s}\delta^{(k)}(P)$, (2,14), $P^{k+1-s}\cdot\delta^{(k)}(P)=\frac{(-1)^{k+1-s}k!}{(s-1)!}\delta^{(s-1)}(P)$, (2,15),

$P_+^{k+1-s}\cdot\delta^{(k)}(P)=\frac{1}{2}\frac{(-1)^{k+1-s}}{(s-1)!}k!\delta^{(s-1)}(P_+)$, (2,16),

$\delta(P^s)=\frac{s}{(-1)^{s-1}(s-1)!}\delta^{(s-1)}(P)$, (2,17), $\delta\left(P_+^s\right)=\frac{1}{2}\frac{s}{(-1)^{s-1}(s-1)!}\delta^{(s-1)}(P_+)$, (2,18),

$\delta^{(\ell)}(P^s)=\frac{s}{(-1)^s(s-1)!}\delta^{(\ell+s-1)}(P)$, (3,4), $\delta^{(\ell)}(P_+^s)=\frac{1}{2}\frac{s}{(-1)^{s-1}(s-1)!}\delta^{(\ell+s-1)}(P_+)$, (3,5),

1. Introduction

We remark that this paragraph 1 is only an abstract of the Gelfand-Shilov conclusions (cf. [1]).

Let $x=(x_1,x_2,\ldots,x_n)$ be a point of the n-dimensional Euclidean space $I\!R^n$.

Consider a non-degenerate quadratic form in n variables of the form

$$P=P(x)=x_1^2+\ldots+x_p^2-x_{p+1}^2-\ldots-x_{p+q}^2\ , \tag{1,1}$$

where $n=p+q$.

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The hypersurface $P=0$ is a hypercone with a singular point (the vertex) at the origin.

We define (cf.[1], page 253, formula (2)) the generalized function P_+^λ, where λ is a complex number, by

$$(P_+^\lambda, \varphi) = \int_{P>0} P^\lambda(x)\varphi(x)dx \ ; \tag{1,2}$$

here $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$.

For $\operatorname{Re}\lambda \geq 0$, this integral converges and is an analytic function of λ. Analytic continuation to $\operatorname{Re}\lambda < 0$ can be used to extend the definition of (P_+^λ, φ).

From [1], formula (13), p.254, we have

$$(P_+^\lambda, \varphi) = \int_0^\infty u^{\lambda+\frac{1}{2}(p+q)-1} G(\lambda, u)du \ , \tag{1.3}$$

where

$$G(\lambda, u) = \frac{1}{4}\int_0^1 (1-t)^\lambda t^{\frac{1}{2}(q-2)} \psi_1(u, tu)dt \ , \tag{1,4}$$

here

$$\psi_1(u, tu) = \psi(r, s) \ , \tag{1,5}$$

and

$$\psi(r, s) = \int \varphi d\Omega^{(p)} d\Omega^{(q)} \ ; \tag{1,6}$$

$d\Omega^{(p)}$ and $d\Omega^{(q)}$ are the elements of surface area on the unit sphere in $\mathbb{R}_p$ and $\mathbb{R}_q$, respectively.

Consequently, (P_+^λ, φ) has two sets of singularities, namely,

$$\lambda = -1, -2, \ldots, -k, \ldots \tag{1,7}$$

and

$$\lambda = -\frac{n}{2}, -\frac{n}{2}-1, \ldots, -\frac{n}{2}-k, \ldots \tag{1,8}$$

Following to Gelfand-Shilov (cf. [1], pp.255-257), we know that when the singular point $\lambda = -k$ belongs to the first set, but not to the second. This is always the case when the dimension $n = p+q$ is odd, but is also true if n is even and $\lambda > -\frac{n}{2}$; we have,

$$\begin{aligned}(P_+^\lambda, \varphi) = \frac{1}{\lambda+k}\int_0^\infty u^{\lambda+\frac{1}{2}(p+q)-1} G_0(u)du \\ + \int_0^\infty u^{\lambda+\frac{1}{2}(p+q)^{-1}} G_1(\lambda, u)du \ ,\end{aligned} \tag{1,9}$$

where

$$G_0(u) = \operatorname*{Res}_{\lambda=-k} G(\lambda, u) \ , \tag{1,10}$$

and $G_1(\lambda, u)$ is regular at $\lambda = -k$.

Similarly, P_-^λ is defined by

$$(P_-^\lambda, \varphi) = \int_{-P>0} (-P(x))^\lambda \varphi(x) dx \ . \tag{1,11}$$

We also have (cf. [1], formulae (10) and (11), p. 249)) that

$$\left(\delta^{(k)}(P), \varphi\right) = \int_0^\infty \left[\left(\frac{\partial}{2s\partial s} \right)^k \left\{ s^{q-2} \frac{\psi(r,s)}{2} \right\} \right]_{s=r} r^{p-1} dr \tag{1,12}$$

and

$$\left(\delta^{(k)}(P), \varphi\right) = (-1)^k \int_0^\infty \left[\left(\frac{\partial}{2s\partial s} \right)^k \left\{ s^{p-2} \frac{\psi(r,s)}{2} \right\} \right]_{r=s} s^{q-1} ds \tag{1,13}$$

where

$$\psi(r, s) = \int \varphi d\Omega^{(p)} d\Omega^{(q)} \ , \tag{1,14}$$

$$r = \left(x_1^2 + \ldots + x_p^2\right)^{1/2} \ , \tag{1,15}$$

$$s = \left(x_{p+1}^2 + \ldots + x_{p+q}^2\right)^{1/2} \ , \tag{1,16}$$

$p + q = n$.

Taking into account the formulas (11) and (11') of [1], p.250, we arrive at the following definitions:

$$\left(\delta_1^{(k)}(P), \varphi\right) = \int_0^\infty \left[\left(\frac{\partial}{2s\partial s} \right)^k \left\{ s^{q-2} \frac{\psi(r,s)}{2} \right\} \right] r^{p-1} dr \tag{1,17}$$

and

$$\left(\delta_2^{(k)}(P), \varphi\right) = (-1)^k \int_0^\infty \left[\left(\frac{\partial}{2s\partial s} \right)^k \left\{ s^{p-2} \frac{\psi(r,s)}{2} \right\} \right]_{r=s} s^{q-1} ds \tag{1,18}$$

where $\psi(r, s)$ is $r^{1-p} s^{1-q}$ multiplied by the integral of φ over the surface $x_1^2 + \ldots + x_p^2 = r^2$, $x_{p+1}^2 + \ldots + x_{p+q}^2 = s^2$, $p + q = n$.

The integrals (1,17) and (1,18) converge and coincide for

$$k < \frac{p + q - 2}{2} \ . \tag{1,19}$$

If, on the other hand,

$$k \geq \frac{p+q-2}{2} \;, \tag{1,20}$$

these integrals must be understood in the sense of their regularization.

In general, $\delta_1^{(k)}(P)$ and $\delta_2^{(k)}(P)$ may not be the generalized function.

Note that the definition of these generalized functions implies that in any case

$$\delta_2^{(k)}(P) = (-1)^k \delta_1^{(k)}(-P) \;. \tag{1,21}$$

Along this paper we shall need the following formulas (cf. [1], pp. 278-279):

$$\delta^{(k)}(P_+) = (-1)^k k! \operatorname*{Res}_{\lambda=-k-1} P_+^{\lambda} \;, \tag{1,22}$$

and

$$\delta^{(k)}(P_-) = (-1)^k k! \operatorname*{Res}_{\lambda=-k-1} P_-^{\lambda} \;. \tag{1,23}$$

For odd n, as well as for even n and $k < \frac{1}{2}n - 1$, we have

$$\delta^{(k)}(P_+) = \delta_1^{(k)}(P) \tag{1,24}$$

and

$$\delta^{(k)}(P_-) = \delta_1^{(k)}(-P) \;, \tag{1,25}$$

while in the case of even dimension and $k \geq \frac{1}{2}n - 1$,

$$\delta^{(k)}(P_+) - \delta_1^{(k)}(P) \tag{1,26}$$

and

$$\delta^{(k)}(P_-) - \delta_1^{(k)}(-P) \tag{1,27}$$

are generalized functions concentrated at the vertex of the $P = 0$ cone.

From equations (4), p.277 and (5)-(6), p.278 of [1], we note that if p and q are both even and if $k \geq \frac{n}{2} - 1$, then

$$(-1)^k \delta^{(k)}(P_+) - \delta^{(k)}(P_-) = a_{q,n,k} L^{k+1-\frac{n}{2}} \delta(x) \;, \tag{1,28}$$

while in all other cases

$$\delta^{(k)}(P_-) = (-1)^k \delta^{(k)}(P_+) \;, \tag{1,29}$$

where

$$L^k = \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \ldots - \frac{\partial^2}{\partial_{p+q}^2} \right)^k \;,$$

$p+q=n$ and

$$a_{q,n,k}=\frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^{k+1-\frac{n}{2}}\left(k+1-\frac{n}{2}\right)!}\ . \tag{1,30}$$

If the dimension n of the space is even and p and q are even, P_+^λ has simple poles at $\lambda=-\frac{n}{2}-k$, where k is a nonnegative integer, where the residues are given by

$$\begin{aligned}\operatorname*{Res}_{\lambda=-\frac{n}{2}-k} P_+^\lambda &= \frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma\left(\frac{n}{2}+k\right)}\delta_1^{\left(\frac{n}{2}+k-1\right)}(P)\\ &+\frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{2^{2k}k!\Gamma\left(\frac{n}{2}+k\right)}L^k\delta(x)\ ,\end{aligned} \tag{1,31}$$

where

$$L^k=\left\{\frac{\partial^2}{\partial x_1^2}+\ldots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\ldots-\frac{\partial^2}{\partial_{p+q}^2}\right\}^k\ , \tag{1,32}$$

$p+q=n\,,\ k=1,2,\ldots.$

If, on the other hand, p and q are odd, P_+^λ has poles of order 2 at $\lambda=-\frac{n}{2}-k$. Let the Laurent expansion of P_+^λ, about this point, be

$$P_+^\lambda=\frac{C_{-2}^{(k)}}{\left(\lambda+\frac{n}{2}+k\right)^2}+\frac{C_{-1}^{(k)}}{\lambda+\frac{n}{2}+k}+\ldots\ . \tag{1,33}$$

Then, the coefficients are given by

$$C_{-2}^{(k)}=\frac{(-1)^{\frac{1}{2}(q+1)}\pi^{\frac{n}{2}-1}}{2^{2k}k!\Gamma\left(\frac{n}{2}+k\right)}L^k\delta(x)\ ,$$

and

$$\begin{aligned}C_{-1}^{(k)} &= \frac{(-1)^{k-1+\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+k\right)}\delta_1^{\left(\frac{n}{2}+k-1\right)}(P)\\ &+\frac{(-1)^{\frac{1}{2}(q+1)}\pi^{\frac{n}{2}-1}\left[\psi\left(\frac{p}{2}\right)-\psi\left(\frac{n}{2}\right)\right]}{2^{2k}k!\Gamma\left(\frac{n}{2}+k\right)}L^k\delta(x)\ ,\end{aligned}$$

where

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}\ , \tag{1,34}$$

here,

$$\psi(k)=-C+1+\frac{1}{2}+\ldots+\frac{1}{k-1}\ , \tag{1,35}$$

and

$$\psi\left(k+\frac{1}{2}\right)=-C-2\ln 2+2\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)\ , \tag{1,36}$$

where C es the Euler's constant,

$$C = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{n} - \log m\right) = 0,577215664 \ .$$

If the dimension of the space is odd, then for p odd and q even, the generalized function P_+^λ has simple poles at $\lambda = -\frac{n}{2} - k$, for k a nonnegative integer, where the residues are given by

$$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k} P_+^\lambda = M_{k,n,q} L^k \delta(x) \ . \tag{1,37}$$

here

$$M_{k,n,q} = \frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{2^{2k}k!\Gamma\left(\frac{n}{2}+k\right)} \ . \tag{1,38}$$

If, on the other hand, p is even and q is odd, P_+^λ is regular, therefore

$$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k} P_+^\lambda = 0 \ . \tag{1,39}$$

In addition to P_+^λ, we can also define the generalized function P_-^λ by

$$(P_-^\lambda, \varphi) = \int_{-P>0} (-P)^\lambda \varphi dx \ . \tag{1,40}$$

All that we have said about P_+^λ remains true also for P_-^λ except that p and q must be interchanged, and in all the formulas $\delta_1^{(k)}(P)$ must be replaced by

$$\delta_1^{(k)}(-P) = (-1)^k \delta_2^{(k)}(P) \ . \tag{1,41}$$

2. The distribution $\delta(P^s)$.

Consider the generalized function $\delta(P^s)$, where P^s is the distribution defined by

$$P^s = [P(x)]^s = \left(x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2\right)^s \ . \tag{2,1}$$

here $p+q=n$ and δ is the measure of Dirac.

We know (cf. [1], formula (1), p. 236) that

$$\delta(PQ) = P^{-1}\delta(Q) + Q^{-1}\delta(P) \ . \tag{2,2}$$

Putting $P = Q$ in (2,2), we have

$$\delta(P^2) = 2P^{-1} \cdot \delta(P) \ , \tag{2,3}$$

where $\delta(P)$ is defined by the formula (1,12) or (1,13).

On the other hand, putting $\ell = k$ in the formula

$$\delta^{(k)}(G) \cdot \left(G^{-1}\right)^{(\ell)} + \delta^{(\ell)}(G) \cdot \left(G^{-1}\right)^{(k)} = \frac{(-1)k!\ell!}{(k+\ell+1)!}\delta^{(k+\ell-1)}(G) \ , \tag{2,4}$$

valid for $k + \ell + 1 < \frac{n}{2} - 1$ (cf. [2], formula (8.1), p. 17), we have

$$\delta^{(k)}(G) \cdot G^{-k-1} = -\frac{1}{2}\frac{k!}{(-1)^k(2k+1)!}\delta^{(2k+1)}(G) \ , \tag{2,5}$$

where

$$\left(G^{-1}\right)^{(\ell)} = (-1)^\ell \ell! G^{-\ell-1}$$

and

$$G = P = P(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 \ ,$$

$p + q = n$.

From (2,5), we arrive at

$$\delta(G) \cdot G^{-1} = -\frac{1}{2}\delta'(G) \ . \tag{2,6}$$

The above formula, for the one dimensional case, was proved by A. González Domínguez and R. Scarfiello (cf. [3]).

Now, by using (2,2), we obtain the formula

$$\delta(P^s) = s\left(P^{-1}\right)^{s-1}\delta(P) \ . \tag{2,7}$$

In particular, putting $s = 2$ in (2,7) and using (2,6), we arrive to

$$\delta\left(P^2\right) = -\delta'(P) \ . \tag{2,8}$$

From (2,7) and (2,8), we have

$$P^{-1}\delta(P) = -\frac{1}{2}\delta(P) \ ,$$

and this formula generalizes the one dimensional result due to A. González Domínguez and R. Scarfiello, (cf. [3]) which says that $\frac{1}{x} \cdot \delta = -\frac{1}{2}\delta'$.

Taking into account the following formulae (cf. [7], p. 61)

$$P^{\ell} \cdot \delta^{(k)}(P) = \begin{cases} \dfrac{(-1)^{\ell} k!}{(k-\ell)!}\delta^{(k-\ell)}(P) & \text{if } k \geq \ell, \\ 0 & \text{if } k < \ell, \end{cases} \qquad (2,9)$$

and

$$P_{+}^{\ell} \cdot \delta^{(k)}(P_{+}) = \begin{cases} \dfrac{\frac{1}{2}(-1)^{\ell} k!}{(k-\ell)!}\delta^{(k-\ell)}(P_{+}) & \text{if } k \geq \ell, \\ 0 & \text{if } k < \ell, \end{cases} \qquad (2,10)$$

where $\delta^{(k)}(P)$ is defined by (1,13) and $\delta^{(k)}(P_{+})$ by (1,22), we have

$$\delta\left(P^{s}\right) = \frac{s}{(-1)^{s}(s-1)!}\delta^{(s-1)}(P) \ \text{ for } \ s > 2 \ , \qquad (2,11)$$

and

$$\delta\left(P_{+}^{s}\right) = \frac{1}{2}\frac{s}{(-1)^{s-1}(s-1)!}\delta^{(s-1)}(P_{+}) \ \text{ for } \ s \geq 2 \ . \qquad (2,12)$$

In fact, using the formulae (2,7) and (2,9), we obtain

$$\begin{aligned} \delta\left(P^{s}\right) &= s\left(P^{-1}\right)^{s-1}\left[(-1)P \cdot \delta'(P)\right] \\ &= (-1)sP^{2-s}\delta'(P) \\ &= \frac{(-1)^{2}s}{2}P^{3-s}\delta''(P) \ . \end{aligned} \qquad (2,13)$$

By iterating k-times the above formula (2,13), we arrive to

$$\delta\left(P^{s}\right) = \frac{(-1)^{k}s}{k!}P^{k+1-s}\delta^{(k)}(P) \ . \qquad (2,14)$$

On the other hand, from (2,9) and (2,10), we have,

$$P^{k+1-s} \cdot \delta^{(k)}(P) = \frac{(-1)^{k+1-s}k!}{(s-1)!}\delta^{(s-1)}(P) \ , \qquad (2,15)$$

and

$$P_{+}^{k+1-s} \cdot \delta^{(k)}(P) = \frac{1}{2}\frac{(-1)^{k+1-s}}{(s-1)!}k!\delta^{(s-1)}(P_{+}) \ , \qquad (2,16)$$

where $k+1-s \geq 0$.

From (2,14), using (2,15) and (2,16), we arrive to

$$\delta\left(P^{s}\right) = \frac{s}{(-1)^{s-1}(s-1)!}\delta^{(s-1)}(P) \ \text{ for } \ s > 2 \ , \qquad (2,17)$$

and

$$\delta\left(P_+^s\right)=\frac{1}{2}\frac{s}{(-1)^{s-1}(s-1)!}\delta^{(s-1)}(P_+)\quad\text{for}\quad s\geq 2\ . \tag{2,18}$$

3. The distributions $\delta^{(\ell)}(P^s)$ and $\delta^{\ell}(P_+^s)$.

The distribution $\delta^{(k)}(P)=\frac{\partial^k}{\partial P^k}\delta(P)$ is defined by the formula (cf. [1], form. (8), p. 211):

$$\begin{aligned}\left\langle\delta^{(k)}(P),\varphi\right\rangle&=\left\langle\frac{\partial^k}{\partial P^k}\delta(P),\varphi\right\rangle\\&=(-1)^k\int_{P=0}\left[\frac{\partial^k}{\partial u_1^k}\left\{\varphi_1(u_1,\ldots,u_n)D\binom{2}{u}\right\}\right]du_2\ldots du_n\ .\end{aligned} \tag{3,1}$$

where $\varphi_1(u_1,\ldots,u_n)=\varphi(x_1,x_2,\ldots,x_n)$ and

$$\begin{cases}u_1=P\\u_2=x_2\\\ \vdots\\u_n=x_n\end{cases} \tag{3,2}$$

with the jacobian $D\binom{x}{u}>0$.

We have

$$\begin{aligned}&\left\langle\frac{\partial^\ell}{\partial P^\ell}\left(\frac{\partial^k}{\partial P^k}\delta(P)\right),\varphi\right\rangle\\&=\left\langle\frac{\partial^{\ell+k}}{\partial P^{\ell+k}}\delta(P),\varphi\right\rangle\\&=(-1)^{\ell+k}\int_{P=0}\left[\frac{\partial^{k+\ell}}{\partial u_1^{k+\ell}}\left\{\varphi_1(u_1,\ldots,u_n)D\binom{x}{u}\right\}\right]_{u_1=0}du_2\ldots du_n\\&=\left\langle\delta^{(k+\ell)}(P),\varphi\right\rangle\ .\end{aligned} \tag{3,3}$$

Therefore, from (2,17) and (2,18), by using (3,3), we finally obtain the following formulae

$$\delta^{(\ell)}(P^s)=\frac{s}{(-1)^s(s-1)!}\delta^{(\ell+s-1)}(P)\quad\text{for}\quad s>2\ , \tag{3,4}$$

and

$$\delta^{(\ell)}(P_+^s)=\frac{1}{2}\frac{s}{(-1)^{s-1}(s-1)!}\delta^{(\ell+s-1)}(P_+)\quad\text{for}\quad s\geq 2\ , \tag{3,5}$$

4. The distribution $\left(\delta^{(\ell)}(P_+^s)\right)^m$.

In this paragraph we need the following result (cf.[5])

$$\delta^{(k)}(P_+)\cdot\delta^{(\ell)}(P_+) = -a_{k,\ell}\delta^{(k+\ell+1)}(P_+)\ , \qquad (4,1)$$

which is valid if p and q are both odd and $0 \le k+\ell-\frac{n}{2}+2 < \frac{n}{2}$.

The constant $a_{k,\ell}$ is defined by

$$a_{k,\ell} = \frac{1}{2}\frac{k!\ell!}{(k+\ell+1)!\left[\psi\left(\frac{p}{2}\right)-\psi\left(\frac{n}{2}\right)\right]}\ , \qquad (4,2)$$

here $\psi(x)$ is given by the formulae (1,34), (1,35) and (1,36).

Other useful expression of (4,1) is given by the formula (cf. [4])

$$\delta^{(k)}(P_+)\cdot\delta^{(\ell)}(P_+) = C_{\ell+1,k+1,q,n}\ L^{k+\ell+2-\frac{n}{2}}\{\delta(x)\}\ , \qquad (4,3)$$

here

$$L^{k+\ell+2-\frac{n}{2}}\delta(x) = \frac{(-1)4^{k+\ell+2-\frac{n}{2}}\left(k+\ell+2-\frac{n}{2}\right)!}{(-1)^{k+\ell+1}(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}\left[\psi\left(\frac{p}{2}\right)-\psi\left(\frac{n}{2}\right)\right]}\delta^{(k+\ell+1)}(P_+)\ , \quad (4,4)$$

if p and q are both odd and $k+\ell+2-\frac{n}{2}\ge 0$, and

$$C_{\ell+1,k+1,q,n} = -\frac{1}{2}\frac{(-1)^{\frac{q-1}{2}}\pi^{\frac{n}{2}-1}k!\ell!(-1)^k(-1)^\ell}{4^{k+\ell+2-\frac{n}{2}}\left(k+\ell+2-\frac{n}{2}\right)!\Gamma(k+\ell+2)}\ . \qquad (4,5)$$

On the other hand, from (3,5), we have

$$\left[\delta^{(\ell)}\left(P_+^s\right)\right]^m = \left(\frac{1}{2}\frac{s}{(-1)^{s-1}(s-1)!}\right)^m\left(\delta^{(\ell+s-1)}\left(P_+\right)\right)^m \quad \text{for}\ \ s\ge 2\ , \qquad (4,6)$$

$m = 1,2,\ldots$.

Now, to obtain $\left(\delta^{(\ell+s-1)}\left(P_+\right)\right)^m$ we shall use the formula (4,1).

In fact, from (4,1), we have

$$\left(\delta^{(\ell+s-1)}\left(P_+\right)\right)^2 = \delta^{(\ell+s-1)}(P_+)\cdot\delta^{(\ell+s-1)}(P_+) = -a_{\ell+s-1,\ell+s-1}\delta^{(2(\ell+s)-1)}(P_+)\ , \qquad (4,7)$$

$$\begin{aligned}\left(\delta^{(\ell+s-1)}\left(P_+\right)\right)^3 &= (-1)a_{\ell+s-1,\ell+s-1}\delta^{(\ell+s-1)}(P_+)\cdot\delta^{(2(s+\ell)-1)}(P_+) \\ &= (-1)^2 a_{\ell+s-1,\ell+s-1},\, a_{\ell+s-1,2(\ell+s)-1}\delta^{(3(\ell+s)-1)}(P_+)\ ,\end{aligned} \qquad (4,8)$$

$$\left(\delta^{(\ell+s-1)}(P_+)\right)^4 = (-1)^3 a_{\ell+s-1,\ell+s-1} \cdot a_{\ell+s-1,2(\ell+s)-1} \cdot a_{\ell+s-1,3(\ell+s)-1}\delta^{(4(\ell+s)-1)}(P_+)\ . \tag{4,9}$$

By iterating m-times the above formulae, we arrive to

$$\begin{aligned}&\left(\delta^{(\ell+s-1)}(P_+)\right)^m\\ =&(-1)^{m-1}a_{\ell+s-1,\ell+s-1} \cdot a_{\ell+s-1,2(\ell+s)-1)} \cdots a_{\ell+s-1,(m-1)(\ell+s)-1}\delta^{(m(\ell+s)-1)}(P_+)\end{aligned} \tag{4,10}$$

From (4,2), we have

$$\begin{aligned}&a_{\ell+s-1,\ell+s-1} \cdot a_{\ell+s-1,2(\ell+s)-1} \cdot a_{\ell+s-1,3(\ell+s)-1} \cdots a_{\ell+s-1,(m-1)(\ell+s)-1}\\ =&\frac{1}{2}\frac{((\ell+s-1)!)^2}{(2(\ell+s)-1)!\left[\psi\left(\frac{p}{2}\right)-\psi\left(\frac{n}{2}\right)\right]}\\ &\cdot\frac{1}{2}\frac{(\ell+s-1)!(2(\ell+s)-1)!}{(3(\ell+s)-1)!\left[\psi\left(\frac{p}{2}\right)-\psi\left(\frac{n}{2}\right)\right]}\\ &\cdot\frac{1}{2}\frac{(\ell+s-1)!(3(\ell+s)-1)!}{(4(\ell+s)-1)!\left[\psi\left(\frac{p}{2}\right)-\psi\left(\frac{n}{2}\right)\right]}\\ &\cdots\frac{1}{2}\frac{(\ell+s-1)!((m-1)(\ell+s)-1)!}{(m(\ell+s)-1)!\left[\psi\left(\frac{p}{2}\right)-\psi\left(\frac{n}{2}\right)\right]}\\ =&\left(\frac{1}{2}\right)^{m-1}\frac{[(\ell+s-1)!]^m}{\left[\psi\left(\frac{p}{2}\right)-\psi\left(\frac{n}{2}\right)\right]^{m-1}(m(\ell+s)-1)!}\ .\end{aligned} \tag{4,11}$$

From (4,10) and (4,11), we have

$$\begin{aligned}&\left(\delta^{(\ell+s-1)}(P_+)\right)^m\\ =&(-1)^{m-1}\left(\frac{1}{2\left[\psi\left(\frac{p}{2}\right)-\psi\left(\frac{n}{2}\right)\right]}\right)^{m-1}\frac{[(\ell+s-1)!]^m}{(m(\ell+s)-1)!}\delta^{(m(\ell+s)-1)}(P_+).\end{aligned} \tag{4,12}$$

Finally, from (4,6) and (4,12), we arrive to the following formula

$$\left[\delta^{(\ell)}\left(P_+^s\right)\right]^m = \left(\frac{1}{(-1)^{s-1}}\right)^m (-1)^{m-1} \cdot A_{s,p,q,n,\ell,m} \cdot \delta^{(m(\ell+s)-1)}(P_+)\ , \tag{4,13}$$

under the following conditions,

i) p and q are odd numbers, (4,14)

ii) $0 \le m(\ell+s) - \frac{n}{2} + 2 < \frac{n}{2}$. (4,15)

In the formula (4,13), $A_{s,p,q,n,\ell,m}$ is given by

$$A_{s,p,q,n,\ell,m} = \left(\frac{1}{2}\frac{2}{(s-1)!}\right)^m \frac{(\ell+s-1)!}{(m(\ell+s)-1)!} \cdot \left(\frac{1}{2\left[\psi\left(\frac{p}{2}\right)-\psi\left(\frac{n}{2}\right)\right]}\right)^{m-1}, \tag{4,16}$$

for $m = 1, 2, \ldots$.

The formula (4,13), under the conditions (4,14), (4,15) and (4,16), is our desired result and this conclusion finishes our Note.

Remark.

We note that the formula (4,13) is a generalization of the one-dimensional formula $\left[\delta^{(k)}(x^m)\right]^n$ due to A.P. Khapalyuk (cf. [6]).

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