RANDOM DIFFEOMORPHIMS AND THE INTEGRATION OF THE CLASSICAL NAVIER-STOKES EQUATIONS

by

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Summary: We give in closed implicit form the random representations for the solutions of the classical Navier-Stokes equation for an incompressible fluid in a smooth manifold isometrically immersed in Euclidean space, and in particular in Euclidean space. For this we apply the methods of Stochastic Differential Geometry, starting by giving a geometrical basis for the construction of diffusion processes of differential forms. We apply this to write in a new way the Navier-Stokes equation, which we find to be a non-linear diffusion equation determined by a non-riemannian geometry. We find this equation to be equivalent to a linear diffusion equation for the vorticity and the Poisson-de Rham equation for the velocity with source given by the vorticity. We solve the initial problem for the vorticity and the Dirichlet problem for the velocity. We finally give the representations for the solutions of NS for euclidean space.

Keywords: incompressible fluid, diffusion processes on smooth manifolds, Riemann-Cartan-Weyl connections, vorticity, Ito formula for differential forms, stochastic differential equations, stochastic analysis, heat equation, Poisson-de Rham equation.

MSC numbers 60J60, 60H10, 35Q30, 58G03, 58G32, 76M35.

PACS numbers 03.40G, 02.40, 02.50, 02.90.

1 Introduction.

In this article we shall present random representations for the velocity of an incompressible fluid on a smooth manifold M which is isometrically embedded in Euclidean space (e.g.: spheres, torii, euclidean domains, etc.) satisfying the covariant Navier-Stokes equations (NS in the following)

$$\partial_t u - \nu \triangle_1 u + \mathcal{L}_u u = -dp, \tag{1}$$

with the incompressibility and initial conditions

$$\delta u = 0, u(0) = u_0, \tag{2}$$

respectively, for the 1-form $u(x,t)$ (the velocity) and a function p (the pressure) both defined on $M \times R$.(A detailed description of the operators appearing in $(1,2)$ will be given below.)

Representations of the solutions of this problem are not known, with a notable exception. If one considers the vorticity $\Omega_t = du_t (= curl u_t$ in 3D) it is known in the case of

M being the Euclidean space, that from (NS) one obtains the evolution equation for the vorticity

$$(\partial_t - \nu \nabla^2)\Omega_t + \mathcal{L}_{u_t}\Omega_t = 0. \tag{3}$$

which is usually written as

$$(\partial_t - \nu \nabla^2)\Omega_t + (u.\nabla)\Omega_t = ((\Omega_t).\nabla)u_t. \tag{4}$$

In this equation we have a *stretching* term $(\Omega_t.\nabla)u_t$ which since in the case of 2D the vorticity is a (pseudo)-scalar, it vanishes completely. Thus in 2D equation (3) is nothing else that the heat equation for *scalars* which of course we know how to solve. In the case of 3D, this stretching term is the source of all the difficulties associated to NS, i.e. global existance of solutions and regularity, poses an additional problem since it is very involved to deal with to yield a Feynman-Kac representation for the vorticity, at least not in a non-invariant framework; cf. Busnello and Flandoli [8]. In fact, as we shall see in the invariant setting presented in this article, it will turn to be a straightforward application of the Ito formula for differential forms defined on smooth manifolds (the velocity is a 1-form and the vorticity a 2-form).

From the knowledge of the representation for the vorticity one obtains a closed-form expression for the velocity in 2D by solving the Poisson-equation

$$\nabla^2 \psi_t = -\Omega_t, \tag{5}$$

where $\psi(t,x) = \psi_t(x)$ is the (scalar) stream- function related to the velocity by $\Omega_t = \nabla \times \psi$. The solution is well known:

$$\psi(x,t) = \int L(x-z)\Omega_t(z)dz, \tag{6}$$

where $L(x) = \frac{-1}{2\pi} log|x|, x \in R^2$ so that we finally obtain

$$u(x,t) = \int K(x-z)\Omega(z,t)dz, \tag{7}$$

where the kernel K is defined by

$$K(x) = \frac{1}{2\pi}\frac{(-x_1, x_2)}{|x|^2}, x \in R^2. \tag{8}$$

This approach is the basis for the so-called random vortex method, and its numerical implementation has lead to a momentous advancement in the numerical integration of NS [1,2]. We note that due to the singularity of the kernel in (7) (also valid for the Poisson kernel in 3D) the velocity diverges for two point-vortices which become arbitrarily near. Then, when considering an ansatz for the vorticity equation given by the linear combination of n point-vortices, one is lead to smooth the kernel and to study further the convergence of this to a solution of NS (cf. [2, 6]). The same singular feature appears in 3D, which if one would be able to compute the vorticity would have to carry out a similar regularization of the kernel. Thus we are lead to take a different approach: we shall give a random integration of the Poisson equation, which does not demand any regularization.

There is a method which allows for these extensions to be carried out, not only in Euclidean space but as well in smooth manifolds. It stems from Stochastic Differential Geometry, i.e. the gauge theory of Brownian processes in smooth manifolds and Euclidean space [1] developed in the pioneering works by Ito [15], Elworthy and Eells [13], P. Malliavin [11] and further elaborated by Elworthy [12], Ikeda and Watanabe [14], Rogers and Williams [3], and P. Meyer [4].[2] In this approach a stochastic calculus for differential forms became only recently available in the works by Elworthy [21] and Kunita [19], in the context of the theory of random flows on smooth manifolds. The approach to the solution of NS founded on the invariant theory of diffusion processes, is based in the following observation. Any diffusion process, even one taking place in an Euclidean domain, is characterized by a certain differential geometry (and still, by a gauge theoretical structure, a connection); this geometry is intrinsic to the process, and can be furthermore related to a geometrical structure of the manifold on which the diffusion process takes place. (For a recent elaboration of the role of certain connections in determining the properties of measures on diffusion processes we urge the reader to confront Elworthy, Li and LeJan [40].)

Indeed, the diffusion tensor provides a metric (which in Riemannian manifolds, coincides with the metric given by the square of this tensor, and in Euclidean space gives a non-trivial metric), and the dual (through this metric) of a torsion 1-form describes the drift. These two structures, are sinthetized into a linear connection (of Riemann-Cartan-Weyl), which determines completely *all* the probabilistic features of the diffusion of processes of scalars as well as differential forms (completely skew-symmetric covariant tensors). In fact the diffusion of differential forms takes place along the diffusion sample paths of the scalar fields which is determined by the laplacian operator (acting on scalars) determined by these connections; in other words, the diffusions of differential forms are determined by the diffusion of scalars. In the construction of the solution of NS we shall characterize the diffusion of the fluid particles (a random Lagrangian representation of the *viscous* fluid), and this in turn will determine the representation for the diffusion of the vorticity (a differential 2-form) along these random paths, and finally the representation of the velocity of the fluid along a driftless scalar diffusion process.

In this article, we shall start by presenting the basic gauge-theoretical structures of Stochastic Differential Geometry, to proceed to the presentation of the corresponding diffusion processes of scalars and differential forms, and further give the Ito formula of stochastic calculus on manifolds. Finally, we shall derive the covariant Navier-Stokes equations, and shall prove that they are equivalent to a system given by the heat equation for the vorticity and the covariant Poisson-de Rham equation for the velocity. We shall then apply our basic constructions to integrate these equations, giving thus implicit representations for NS, first on manifolds and then on euclidean domains. Since several areas of mathematics are used, a brief Appendix reviewing several notions of differential

[1] As we shall see below, a diffusion process -even in Euclidean space- is equivalent to a gauge-theoretical structure which stems from an extension of general relativity to include spin and quantum fluctuations [29,34,35,42] and geometrically defined matter fields [20].

[2] There is another approach to turbulence in fluids which requires diffusion processes. It deals instead of the classical Navier-Stokes equation, with its random extension by inclusion of a random noise forcing. This extension has been the subject of numerous research; see [43]. As stressed there by Ya. Sinai: '3D Navier-Stokes systems probably need completely new ideas'

geometry, analysis and probability is provided.

2 Riemann-Cartan-Weyl Geometry of Diffusions

In this section M denotes a smooth compact orientable n-dimensional manifold (without boundary) provided with a linear connection (see [33]) described by a covariant derivative operator ∇ which we assume to be compatible with a given metric g on M, i.e. $\nabla g = 0$. Given a coordinate chart (x^α) $(\alpha = 1, \ldots, n)$ of M, a system of functions on M (the Christoffel symbols of ∇) are defined by $\nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\gamma} = \Gamma(x)^\alpha_{\beta\gamma} \frac{\partial}{\partial x^\alpha}$. The Christoffel coefficients of ∇ can be decomposed as (see Appendix):

$$\Gamma^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{1}{2} K^\alpha_{\beta\gamma}. \tag{9}$$

The first term in (9) stands for the metric Christoffel coefficients of the Levi-Civita connection ∇^g associated to g, i.e. $\left\{ {\alpha \atop \beta\gamma} \right\} = \frac{1}{2}(\frac{\partial}{\partial x^\beta} g_{\nu\gamma} + \frac{\partial}{\partial x^\gamma} g_{\beta\nu} - \frac{\partial}{\partial x^\nu} g_{\beta\gamma}) g^{\alpha\nu}$, and

$$K^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} + S^\alpha_{\beta\gamma} + S^\alpha_{\gamma\beta}, \tag{10}$$

is the cotorsion tensor, with $S^\alpha_{\beta\gamma} = g^{\alpha\nu} g_{\beta\kappa} T^\kappa_{\nu\gamma}$, and $T^\alpha_{\beta\gamma} = (\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta})$ the skew-symmetric torsion tensor. We are interested in (one-half) the Laplacian operator associated to ∇, i.e. the operator acting on smooth functions on M defined as

$$H(\nabla) := 1/2\nabla^2 = 1/2 g^{\alpha\beta} \nabla_\alpha \nabla_\beta. \tag{11}$$

A straightforward computation shows that $H(\nabla)$ only depends in the trace of the torsion tensor and g, since it is

$$H(\nabla) = 1/2\triangle_g + \hat{Q}, \tag{12}$$

with $Q := Q_\beta dx^\beta = T^\nu_{\nu\beta} dx^\beta$ the trace-torsion one-form and where $\hat{Q}$ is the vector field associated to Q via g: $\hat{Q}(f) = g(Q, df)$, for any smooth function f defined on M. Finally, $\triangle_g$ is the Laplace-Beltrami operator of g: $\triangle_g f = \mathrm{div_g}\ \mathrm{grad} f$, $f \in C^\infty(M)$, with $\mathrm{div_g}$ the Riemannian divergence. Thus for any smooth function, we have $\triangle_g f = 1/[det(g)]^{\frac{1}{2}} g^{\alpha\beta} \frac{\partial}{\partial x^\beta}([det(g)]^{\frac{1}{2}} \frac{\partial}{\partial x^\alpha} f)$.

Consider the family of zero-th order differential operators acting on smooth k-forms, i.e. differential forms of degree k $(k = 0, \ldots, n)$ defined on M:

$$H_k(g, Q) := 1/2\triangle_k + \mathcal{L}_{\hat{Q}}, \tag{13}$$

The first summand of the r.h.s. of (13) we have the Hodge operator acting on k-forms:

$$\triangle_k = (d - \delta)^2 = -(d\delta + \delta d), \tag{14}$$

with d and δ the exterior differential and codifferential operators respectively, i.e. δ is the adjoint operator of d defined through the pairing of k-forms on M: $(\omega_1, \omega_2) := \int g(\omega_1, \omega_2) vol_g$, for arbitrary k-forms ω_1, ω_2, where $vol_g(x) = det(g(x))^{\frac{1}{2}} dx$ is the volume density. The last identity in (14) follows from the fact that $d^2 = 0$ so that $\delta^2 = 0$.

Furthermore, the second term in (13) denotes the Lie-derivative with respect to the vectorfield $\hat{Q}$. Recall that the Lie-derivative is independant of the metric:for any smooth vectorfield X on M

$$\mathcal{L}_X = i_X d + d i_X, , \tag{15}$$

where i_X is the interior product with respect to X: for arbitrary vectorfields $X_1, \ldots, X_{k-1}$ and ϕ a k-form defined on M, we have $(i_X\phi)(X_1, \ldots, X_{k-1}) = \phi(X, X_1, \ldots, X_{k-1})$. Then, for f a scalar field, $i_X f = 0$ and

$$\mathcal{L}_X f = (i_X d + d i_X) f = i_X df = g(\tilde{X}, df) = X(f). \tag{16}$$

where $\tilde{X}$ denotes the 1-form associated to a vectorfield X on M via g. We shall need later the following identities between operators acting on smooth k-forms, which follow easily from algebraic manipulation of the definitions:

$$d\triangle_k = \triangle_{k+1} d, \; k = 0, \ldots, n, \tag{17}$$

and

$$\delta\triangle_k = \triangle_{k-1}\delta, \; k = 1, \ldots, n, \tag{18}$$

and finally, for any vectorfield X on M we have that $d\mathcal{L}_X = \mathcal{L}_X d$ and therefore

$$dH_k(g, Q) = H_{k+1}(g, Q)d, \; k = 0, \ldots, n. \tag{19}$$

Let $R : (TM \oplus TM) \oplus TM \to TM$ be the (metric) curvature tensor defined by: $(\nabla^g)^2 Y(v_1, v_2) = (\nabla^g)^2 Y(v_2, v_1) + R(v_1, v_2)Y(x)$. From the Weitzenbock formula [14] we have

$$\triangle_1\phi(v) = \text{trace } (\nabla^g)^2\phi(-,-)(v) - Ric_x(v, \hat{\phi}_x),$$

for $v \in T_xM$ and $Ric_x(v_1, v_2) = \text{trace } < R(-, v_1)v_2, - >_x$, is the Ricci curvature 2-form. Since $\triangle_0 = (\nabla^g)^2 = \triangle_g$, we see that from the family defined in (13) we retrieve for scalar fields ($k = 0$) the operator $H(\nabla)$ defined in (13).

Proposition 1. Assume that g is non-degenerate. There is a one-to-one mapping

$$\nabla \rightsquigarrow H_k(g, Q) = 1/2\triangle_k + \mathcal{L}_{\hat{Q}}$$

between the space of g-compatible affine connections ∇ with Christoffel coefficients of the form

$$\Gamma^{\alpha}_{\beta\gamma} = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} + \frac{2}{(n-1)} \{\delta^{\alpha}_{\beta} \, Q_{\gamma} \; - \; g_{\beta\gamma} \, Q^{\alpha}\} \tag{20}$$

and the space of elliptic second order differential operators on k-forms ($k = 0, \ldots, n$) with zero potential term.

The connections defined in (20) are called Riemann-Cartan-Weyl (RCW for short) connections [10,18,20,29]. The naming after Weyl of the trace-torsion is motivated by the fact that these geometries can be introduced through scale transformations which extend the Weyl transformations in the first ever conceived gauge theory. They are fundamental to the description of the motion of relativistic spinning particles in exterior gravitational fields [35].

3 Riemann-Cartan-Weyl Diffusions of Differential forms

In this section we shall extend the correspondance of Proposition 1 to a correspondance between RCW connections and diffusion processes of k-forms ($k = 0, \ldots, n$) having $H_k(g, Q)$ as infinitesimal generators (i.g. for short, in the following).Thus, naturally we shall call these processes as RCW diffusion processes.

In the following we shall further assume that $Q = Q(\tau, x)$ is a time-dependant 1-form. The stochastic flow associated to the diffusion generated by $H_0(g, Q)$ has for sample paths the continuous curves $\tau \mapsto x_\tau \in M$ satisfying the Ito invariant non-degenerate s.d.e. (stochastic differential equation)

$$dx(\tau) = X(x(\tau))dW(\tau) + \hat{Q}(\tau, x(\tau))d\tau. \tag{21}$$

In this expression, $X : M \times R^m \to TM$ is such that $X(x) : R^m \to TM$ is linear for any $x \in M$, so that we write $X(x) = (X_i^\alpha(x))$ $(1 \leq \alpha \leq n,\ 1 \leq i \leq m)$ which satisfies $X_i^\alpha X_i^\beta = g^{\alpha\beta}$, and $\{W(\tau), \tau \geq 0\}$ is a standard Wiener process on R^m. Here τ denotes the time-evolution parameter of the diffusion (in a relativistic setting it should not be confused with the time variable), and for simplicity we shall assume always that $\tau \geq 0$. Consider the canonical Wiener space Ω of continuous maps $\omega : R \to R^m, \omega(0) = 0$, with the canonical realization of the Wiener process $W(\tau)(\omega) = \omega(\tau)$. The (stochastic) flow of the s.d.e. (21) is a mapping

$$F_\tau : M \times \Omega \to M, \quad \tau \geq 0, \tag{22}$$

such that for each $\omega \in \Omega$, the mapping $F_.(. \,, \omega) : [0, \infty) \times M \to M$, is continuous and such that $\{F_\tau(x) : \tau \geq 0\}$ is a solution of equation (21) with $F_0(x) = x$, for any $x \in M$.

Let us assume in the following that the components X_i^α, $\hat{Q}^\alpha$, $\alpha, \beta = 1, \ldots, n$ of the vectorfields X and $\hat{Q}$ on M in (21) are predictable functions which further belong to $C_b^{m,\epsilon}$ $(0 < \epsilon < 1, m$ a non-negative integer), the space of Hoelder bounded continuous functions of degree $m \geq 1$ and exponent ϵ, and also that $\hat{Q}^\alpha(\tau) \in L^1(R)$, for any $\alpha = 1, \ldots, n$. With these regularity conditions, if we further assume that $x(\tau)$ is a semimartingale on a probability space $(\Omega, \mathcal{F}, P)$, then it follows from Kunita [19] that the flow of (21) has a modification (which with abuse of notation we denote as)

$$F_\tau(\omega) : M \to M, \quad F_\tau(\omega)(x) = F_\tau(x, \omega), \tag{23}$$

which is a diffeomorphism of class C^m, almost surely for $\tau \geq 0$ and $\omega \in \Omega$. We would like to point out that a similar result follows from working with Sobolev space conditions instead of Hoelder continuity. Indeed, assume that the components of X and $\hat{Q}$, $X_i^\alpha \in H^{s+2}(M)$ and $\hat{Q}^\beta \in H^{s+1}(M)$, $1 \leq i \leq m$, $1 \leq \beta \leq n$, where the Sobolev space $H^s(M) = W^{2,s}(M)$ with $s > \frac{n}{2} + m$, $m \geq 1$. Then, the flow of (3) for fixed ω defines amorphism in $H^s(M, M)$, and hence by the Sobolev embedding theorem, a diffeomorphism in $C^m(M, M)$

Remarks 1. In the differential geometric approach -pioneered by V. Arnold- for integrating NS on a smooth manifold as a perturbation (due to the diffusion term we shall present below) of the geodesic flow in the group of volume preserving diffeomorphisms of M (as the solution of the Euler equation), it was proved that under the above regularity conditions on the initial velocity, the solution flow of NS defines a diffeomorphism in M of class C^m; see Ebin and Marsden [9], and for the Euler equation [32]. The difference of this classical differential geometry approach [32] with the one we shall present in the following, is to integrate NS through a time-dependant *random* diffeomorphism associated with a RCW connection. As wellknown, these regularity conditions are basic in the usual functional analytical treatment of NS pioneered by Leray (see Temam [7]), and they are further related to the multifractal structure of turbulence [27]. This diffeomorphism property of random flows is fundamental for the construction of their ergodic theory (provided an invariant measure for the processes exists), and in particular, of quantum mechanics amd non-linear non-equilibrium thermodynamics [10,17,18,42].

Let us describe the (first) derivative (or *jacobian*) flow of (21), i.e. the stochastic process $\{v(\tau) := T_{x_0}F_\tau(v(0)) \in T_{F_\tau(x_0)}M, v(0) \in T_{x_0}M\}$; here T_zM denotes the tangent space to M at z and $T_{x_0}F_\tau$ is the linear derivative of F_τ at x_0. The process $\{v_\tau, \tau \geq 0\}$ can be described [21] as the solution of the invariant Ito s.d.e. on TM:

$$dv(\tau) = \nabla^g \hat{Q}(\tau, v(\tau))d\tau + \nabla^g X(v(\tau))dW(\tau) \tag{24}$$

If we take U to be an open neighbourhood in M so that the tangent space on U is $TU = U \times R^n$, then $v(\tau) = (x(\tau), \tilde{v}(\tau))$ is described by the system given by integrating (21) and the covariant Ito s.d.e.

$$d\tilde{v}(\tau)(x(\tau)) = \nabla^g X(x(\tau))(\tilde{v}(\tau))dW(\tau) + \nabla^g \hat{Q}(\tau, x(\tau))(\tilde{v}(\tau))d\tau, \tag{25}$$

with initial condition $\tilde{v}(0) = v_0$. Thus, $\{v(\tau) = (x(\tau), \tilde{v}(\tau)), \tau \geq 0\}$ defines a random flow on TM.

4 Riemann-Cartan-Weyl Gradient Diffusions

Assume that there is an isometric immersion of an n-dimensional manifold M into a Euclidean space R^m given by the mapping $f : M \to R^m, f(x) = (f^1(x), \ldots, f^m(x))$. For example, $M = S^n, T^n$, the n-dimensional sphere or torus respectively, and f is an isometric embedding into R^{n+1}, or still $M = R^m$ with f given by the identity map. The existance of such an immersion is proved by the Nash theorem in the compact manifold case, yet the result is known to be valid as well for non compact manifolds [39]. Assume further that $X(x) : R^m \to T_xM$, is the orthogonal projection of R^m onto T_xM the tangent space at x to M, considered as a subset of R^m. Then, if $e_1, \ldots, e_m$ denotes the standard basis of R^m, we have

$$X(x) = X^i(x)e_i, \text{ with } X^i(x) = \text{grad } f^i(x), i = 1, \ldots, m. \tag{26}$$

We are interested in the RCW *gradient* diffusion processes on compact manifolds isometrically immersed in Euclidean space, given by (21) with the diffusion tensor X given by (26). We shall now give the Ito formula for k-forms on compact manifolds which are isometrically immersed in Euclidean space. Recall the k-th exterior product of a vector field v is written as $\Lambda^k v = v \wedge v \wedge \ldots \wedge v$ (k times). We further denote by $C_c^{1,2}(\Lambda^p(R \times M))$ the space of time-dependant p-forms on M continuously differentiable with respect to the time variable and of class C^2 with respect to the M variable and of compact support with its derivatives.

Theorem 1 Ito Formula for $\mathbf{k-forms}$ (Elworthy [21]) : Let M be isometrically immersed in R^m as in (26). Let $V_0 \in \Lambda^k T_{x_0} M$, $0 \leq k \leq n$. Set $V_\tau = \Lambda^k(TF_\tau)(V_0)$. Then $\partial_\tau + H_k(g, \hat{Q})$ is the i.g. (with domain of definition the differential forms of degree k in $C_c^{1,2}(\Lambda^p R \times M)$) of $\{V_\tau : \tau \geq 0\}$.

Remarks 2. Therefore, starting from the flow $\{F_\tau : \tau \geq 0\}$ of the s.d.e. (21) with i.g. given by $\partial_\tau + H_0(g, Q)$, we construct (fibered on it) the derived velocity process $\{v(\tau) : \tau \geq 0\}$ given by (24) (or (21, 25), with the diffusion tensor given by (26), which has $\partial_\tau + H_1(g, Q)$ for i.g.. Finally, if we consider the diffusion processes of differential forms of degree $k \geq 1$, we further get that $\partial_\tau + H_k(g, Q)$ is the i.g. of the process $\{\Lambda^k v(\tau) : \tau \geq 0\}$, on the Grassmannian bundle $\Lambda^k TM$, $(k = 0, \ldots, n)$. Note that consistent with our notation, and since $\Lambda^0(TM) = M$ we have that $\Lambda^0 v(\tau) \equiv x(\tau), \forall \tau \geq 0$. In particular, $\partial_\tau + H_2(g, Q)$ is the i.g. of the stochastic process $\{v(\tau) \wedge v(\tau) : \tau \geq 0\}$ on $TM \wedge TM$.

The Ito formula will turn out to be the key instrument for writing down random representations for the solutions of linear transport equations, and in particular, for the vorticity as well as for the velocity of an incompressible fluid.

Consider in a smooth manifold M isometrically immersed in Euclidean space, the following initial value problem: We want to solve

$$\frac{\partial}{\partial \tau}\beta = H_p(g, Q)\beta_\tau, \tag{27}$$

with given

$$\beta(0, x) = \beta_0(x), \tag{28}$$

for an arbitrary time-dependant p-form β defined on M which belongs to $C_c^{1,2}(\Lambda^p(R \times M))$. Then, the formal solution of this problem is as follows (cf. p. 256, [37]): Consider the stochastic differential equation

$$dx_s^{\tau,x} = (2\nu)^{\frac{1}{2}} X(x_s^\tau) dW_s + \hat{Q}(\tau - s, x_s^\tau) ds, x_0^{\tau,x} = x. \tag{29}$$

and the derived velocity process $\{v_s^{\tau,v(x)}, v_0^{\tau,x,v(x)} = v(x), 0 \leq s \leq \tau\}$ which in a coordinate system we write as $v_s^{\tau,x,v(x)} = (x_s^\tau, \tilde{v}_s^{\tau,x,v(x)})$ verifying (29) and the s.d.e.

$$d\tilde{v}_s^{\tau,x,v(x)} = (2\nu)^{\frac{1}{2}} \nabla^g X(x_s^{\tau,x})(\tilde{v}_s^{\tau,x,v(x)}) dW_s + \nabla^g \hat{Q}(\tau - s, x_s^\tau)(\tilde{v}_s^{\tau,x,v(x)}) ds, \tilde{v}_0^{\tau,x,v(x)} = v(x) \tag{30}$$

Notice that this system is nothing else than running back in time the jacobian process.

Theorem 2. The formal solution of the initial value problem $(27 - 28)$ is

$$\beta(\tau, x)(\Lambda^p v(x)) = E_x[\beta_0(x_\tau^{\tau,x})(\Lambda^p \tilde{v}_\tau^{\tau,x,v(x)})]. \tag{31}$$

Proof It follows from the Ito formula (cf. [32]).

5 The Navier-Stokes Equation and Riemann-Cartan-Weyl Gradient Diffusions

In the sequel, M is a compact orientable n-manifold (without boundary) provided with a Riemannian metric g. Further, M has a 1-form whose Hodge decomposition is

$$Q(x) = df(x) + u(x), \quad \delta u = -div(\hat{u}) = 0, \tag{32}$$

where f is a scalar field and u is a coclosed 1-form, weakly orthogonal to df, i.e. $\int g(df, u)vol_g = 0$. We shall assume that $u(x, 0) = u(x)$ is the initial velocity 1-form of an incompressible viscous fluid on M, and that we further have a 1-form $Q(x, \tau) = Q_\alpha(x, \tau)dx^\alpha$ whose Hodge decomposition is:

$$Q(x, \tau) = df(x, \tau) + u(x, \tau),$$

with $\delta u_\tau(x) = \delta u(x, \tau) = 0$ (incompressibility condition), and

$$\int g(df_\tau, u_\tau)vol(g) = 0,$$

which further satisfies the evolution equation on $M \times R$ (Eulerian representation of the fluid):

$$\frac{\partial Q_\alpha}{\partial \tau} + \nabla^g_{\hat{u}} Q_\alpha = -Q_\beta \nabla^g_\alpha u^\beta + \nu \triangle_1 Q_\alpha, \tag{33}$$

Here ν is the kinematical viscosity. In the above notations and in the following, all covariant operators act in the M variables only. In the formulation of Fluid Mechanics in Euclidean domains, $Q(x, \tau)$ receives the name of (Buttke) "magnetization variable" [1].

Remarks 3. We recall that to take the Hodge decomposition of the velocity of a viscous fluid is a basic procedure in Fluid Mechanics [1,6,7,9]. We shall see below that Q and in particular u are related to a natural RCW geometry of the incompressible fluid. In the formulation of Quantum Mechanics and of non-linear non-equilibrium thermodynamics stemming from RCW diffusions, we have a Hodge decomposition of the trace-torsion associated to a stationary state; see [10,17,34]. This decomposition allows to associate with the divergenceless term of the trace-torsion a probability current which characterizes the time-invariance symmetry breaking of the diffusion process, and is central to the construction of the ergodic theory of these flows.

Equation (33) is the gauge-invariant form of the NS for the velocity 1-form $u(x, \tau)$. Indeed, if we substitute the Hodge decomposition of $Q(x, \tau) = Q_\tau(x)$ into (33) we obtain,

$$\frac{\partial u}{\partial \tau} + \nabla^g_{\hat{u}_\tau} u_\tau = \nu \triangle_1 u_\tau - d\left(\frac{\partial f}{\partial \tau} + \nabla^g_{\hat{u}_\tau} f + \frac{1}{2}|u_\tau|^2 - \nu \triangle_g f\right). \tag{34}$$

Consider the operator P of projection of 1-forms into co-closed 1-forms: $P\omega = \alpha$ for any one-form ω whose Hodge decomposition is $\omega = df + \alpha$, with $\delta\alpha = 0$. From (18) we get that

$$P\triangle_1 u_\tau = \triangle_1 u_\tau, \tag{35}$$

and further applying P to (34) we finally get the well known covariant NS (with no exterior forces; the gradient of the pressure term disappears by projecting with P [1,9])

$$\frac{\partial u}{\partial \tau} + P[\nabla^g_{\hat{u}_\tau} u_\tau] - \nu \triangle_1 u_\tau = 0. \tag{36}$$

Conversely, starting with equation (34) which is equivalent to NS we obtain (33). Note that Q_τ and u_τ differ by a differential of a function for all times. Multiplication of (34) by $I - P$ (I the identity operator) yields an equation for the evolution of f which is only arbitrary for $\tau = 0$. Now we note that the non-linearity of NS originates from applying P to the term

$$\nabla^g_{\hat{u}_\tau} u_\tau = i_{\hat{u}_\tau} du_\tau,$$

which taking in account (15) can still be written as

$$\mathcal{L}_{\hat{u}_\tau} u_\tau - d i_{\hat{u}_\tau} u_\tau = \mathcal{L}_{\hat{u}_\tau} u_\tau - \frac{1}{2} d(|u_\tau|^2). \tag{37}$$

Applying P to (37), we see that the kinetic energy term there disappears and the non-linear term in NS can be written as

$$P[\nabla^g_{\hat{u}_\tau} u_\tau] = P[\mathcal{L}_{\hat{u}_\tau} u_\tau]. \tag{38}$$

Therefore, from (36&38) we see that NS can be written as

$$\frac{\partial u}{\partial \tau} = P[2\nu \triangle_1 - \mathcal{L}_{\hat{u}_\tau}] u_\tau$$

which further taking in account the definition (13) we rewrite in the concise form

$$\frac{\partial u}{\partial \tau} = P H_1(2\nu g, \frac{-1}{2\nu} u_\tau) u_\tau. \tag{39}$$

Remarks 4. Equation (39) is a new covariant way of writing NS. Actually we have found that NS for the velocity of an incompressible fluid is a a non-linear diffusion process determined by a RCW connection. This is a characterization of NS hitherto unknown (cf. [1-9]). This RCW connection which determines NS has $2\nu g$ for the metric, and the time-dependant trace-torsion of this connection is $-u/(2\nu)$. Then, the drift of this process is $2\nu g(\frac{-1}{(2\nu)} u_\tau, -) = g(-u_\tau, -) = -\hat{u}_\tau(-)$, i.e. the vector field conjugate by g to minus the velocity, and thus does not depend explicitly on ν, its dependance on ν being -of course- built-in. Thus, we have a static metric which depends on the kinematical viscosity, and the trace-torsion -initially unnoticed- appeared through a dynamical field given by $-u/(2\nu)$ which in the limit $\nu \to 0$ in which the Euler equations replaces NS, becomes singular while the drift is independant of ν. This characterization applied to the Ito formula will allow for the integration of NS.

Let us introduce a new variable: the vorticity two-form

$$\Omega_\tau = du_\tau. \tag{40}$$

Note that also $\Omega_\tau = dQ_\tau$. Now, if we know Ω_τ for any $\tau \geq 0$, we can obtain u_τ (or still Q_τ) by inverting the definition (40). Namely, applying δ to (40) and taking in account (14) we obtain the Poisson-de Rham equation (would g be hyperbolic, it is the Maxwell-de Rham equation [10a])

$$\triangle_1 u_\tau = -\delta\Omega_\tau. \tag{41}$$

and an identical equation for Q_τ. (Note that if we know Q_τ we can reconstruct f_τ by solving $-\delta Q_\tau = div(\hat{Q}_\tau) = \triangle_g f_\tau$, for any τ.) From the Weitzenbock formula we can write (41) showing the coupling of the Ricci metric curvature to the velocity $u = u_\alpha(x,\tau)dx^\alpha$:

$$(\nabla^g)^2 u_\tau - R_{\alpha\beta} u_\tau^\beta dx^\alpha = -\delta\Omega_\tau. \tag{42}$$

with $R_\alpha^\beta(g) = R_{\mu\alpha}{}^{\mu\beta}(g)$, the Ricci (metric) curvature tensor. Thus, the vorticity Ω_τ is a source for the velocity one-form u_τ, for all τ; in the case that M is an euclidean domain, (41) is integrated to give the Biot-Savart law of Fluid Mechanics as describen in the Introduction.

Now, apply d to (39) and further (Hodge) decompose $L_{-\hat{u}_\tau} u_\tau = \alpha_\tau + dp_\tau$ (with p_τ the pressure at time τ); in account that

$$dPL_{-\hat{u}_\tau} u_\tau = d\alpha_\tau = d(\alpha_\tau + dp_\tau) = dL_{-\hat{u}_\tau} u_\tau = L_{-\hat{u}_\tau} du_\tau = L_{-\hat{u}_\tau}\Omega_\tau,$$

and that from (17) we have that $d\triangle_1 u_\tau = \triangle_2\Omega_\tau$, we therefore obtain the linear evolution equation

$$\frac{\partial\Omega_\tau}{\partial\tau} = H_2(2\nu g, \frac{-1}{2\nu} u_\tau)\Omega_\tau. \tag{43}$$

Theorem 3. Given a compact orientable Riemannian manifold with metric g, the Navier-Stokes equation (39) for an incompressible fluid with velocity one-form $u = u(\tau, x)$ such that $\delta u_\tau = 0$, assuming sufficiently regular conditions, is equivalent to a linear diffusion process for the vorticity given by (43) with u_τ satisfying the Poisson-de Rham equation (41). The RCW connection on M generating this process is determined by the metric $2\nu g$ and a trace-torsion 1-form given by $-u/2\nu$.

6 Integration of the Navier-Stokes equation for the vorticity:

In the following we assume additional conditions on M, namely that it is isometrically immersed in an Euclidean space, so that the diffusion tensor is given in terms of the immersion f by $X = \nabla f$.

Let u denote a solution of (39) and consider the flow $\{F_\tau : \tau \geq 0\}$) of the s.d.e. whose i.g. is $\frac{\partial}{\partial\tau} + H_0(2\nu g, \frac{-1}{2\nu} u)$; from (21) and Theorem 1 we know that this is the flow defined by integrating the non-autonomous Ito s.d.e.

$$dx(\tau) = [2\nu]^{\frac{1}{2}} X(x(\tau))dW(\tau) - \hat{u}(\tau, x(\tau))d\tau, x(0) = x, 0 \leq \tau. \tag{44}$$

We shall assume in the following that X and $\hat{u}_\tau$ have the regularity conditions stated in Section 3 so that the random flow of (44) is a diffeomorphism of M of class C^m.

Theorem 4. Equation (44) is a random Lagrangian representation for the fluid particles positions, i.e. $x(\tau)$ is the random position of the particles of the incompressible fluid whose velocity obeys (39).

Remarks 5. Assuming thus a continuum of particles, we obtain an *exact* representation of the flow of a *exact* solution of NS. Notice that when we set $\nu = 0$, the classical flow of the velocity yields the solution of the *Euler* equation for *inviscid* fluids.

Proof. Consider the derived velocity flow $\{v(\tau) = T_{x_0}F_\tau(v_0) = (x(\tau), \tilde{v}(\tau)) : \tilde{v}(\tau) \in T_{x(\tau)}M, \tau \geq 0\}$ on TM; this process is given by (44) and the process with initial velocity $\tilde{v}(0) = v_0 \in T_{x_0}M$:

$$d\tilde{v}(\tau) = [2\nu]^{\frac{1}{2}}\nabla^g X(x(\tau))(\tilde{v}(\tau))dW(\tau) - \nabla^g \hat{u}(\tau, x(\tau))(\tilde{v}^\tau)d\tau. \tag{45}$$

for any $0 \leq \tau$. From the Ito formula for 1-forms we know that $\frac{\partial}{\partial\tau} + H_1(2\nu g, -\frac{1}{2\nu}u_\tau)$ is the backward i.g. of $\{v(\tau), \tau \geq 0\}$. From the Ito formula for 2-forms we conclude that $\frac{\partial}{\partial\tau} + H_2(2\nu g, -\frac{1}{2\nu}u_\tau)$ is the backward i.g. of the stochastic process $\{v(\tau) \wedge v(\tau), \tau \geq 0\}$ on $TM \wedge TM$. Noting that this process projects by the tangent bundle projection on M on the process given by (44), this concludes with the proof of our assertion.

Remark 6: Note that the drift of $\{\tilde{v}_\tau : \tau \geq 0\}$ is minus the deformation tensor of the fluid. This will have a crucial role in the solutions of NS.

6.1 Initial Value Problem for the Vorticity

Let us find the form of the strong solution (whenever it exists) of the initial value problem for $\Omega(\tau, x)$ satisfying (43) with initial condition $\Omega(0, x) = \Omega_0(x) = du_0(x)$.

For each $\tau \geq 0$ consider the s.d.e. (with $s \in [0, \tau]$):

$$dx_s^{\tau,x} = (2\nu)^{\frac{1}{2}}X(x_s^\tau)dW_s - \hat{u}(\tau - s, x_s^\tau)ds, x_0^{\tau,x} = x. \tag{46}$$

and the derived velocity process $\{v_s^{\tau,v(x)}, v_0^{\tau,x,v(x)} = v(x), 0 \leq s \leq \tau\}$ which in a coordinate system we write as $v_s^{\tau,x,v(x)} = (x_s^\tau, \tilde{v}_s^{\tau,x,v(x)})$ verifying (46) and the s.d.e.

$$d\tilde{v}_s^{\tau,x,v(x)} = (2\nu)^{\frac{1}{2}}\nabla^g X(x_s^{\tau,x})(\tilde{v}_s^{\tau,x,v(x)})dW_s - \nabla^g \hat{u}(\tau - s, x_s^\tau)(\tilde{v}_s^{\tau,x,v(x)})ds, \tilde{v}_0^{\tau,x,v(x)} = v(x) \tag{47}$$

Theorem 5. If there is a $C^{1,2}$ (i.e. continuously differentiable in the time variable $\tau \in [0, T]$, and of class C^2 in the space variable) solution $\tilde{\Omega}_\tau(x)$ of the initial value problem, it is

$$\tilde{\Omega}_\tau(\Lambda^2 v(x)) = E_x[\Omega_0(x_\tau^{\tau,x})(\Lambda^2\tilde{v}_\tau^{\tau,x,v(x)})] \tag{48}$$

where E_x denotes the expectation value with respect to the measure on the process $\{x_\tau^{\tau,x} : \tau \geq 0\}$.

Proof. It is evident from Theorems 1 and 2.

Note that from uniqueness of the representation, it follows that $\Omega_\tau = du_\tau$, for any $\tau \in [0, T]$, and then Ω_τ is a closed 2-form (i.e. $d\Omega_\tau = 0$).

We can give appropiate analytical conditions for which expression (48) yields a unique weak solution of the initial value problem. We shall present them elsewhere.

7 Integration of the Poisson-de Rham equation

In (44) we have that u_τ verifies (41), for every $\tau \geq\in [0,T]$, which in account of (13) we can rewrite as

$$H_1(g,0)u_\tau = -\frac{1}{2}\delta\Omega_\tau, \text{ for any } \tau \geq 0. \tag{49}$$

Let us assume that Ω_τ is $C^{1,2}$ for every $\tau \in [0,T]$, and is given by the expression (48). Consider the autonomous s.d.e. generated by $H_0(g,0) = \frac{1}{2}\triangle_g$:

$$dx_s^{g,x} = X(x_s^g)dW_s, x_0^{g,x} = x. \tag{50}$$

We shall solve the Dirichlet problem in an open set U (of a partition of unity) of M given by (48) with the boundary condition $u_\tau \equiv \phi$ on ∂U, with ϕ a given 1-form such that $\delta\phi = 0$. Then one can 'glue' the solutions and use the strong Markov property to obtain a global solution (cf. [25]). Consider the derived velocity process $v^g(s) = (x^g(s), \tilde{v}^g(s))$ on TM, with $\tilde{v}^g(s) \in T_{x^g(s)}M$, whose i.g. is $H_1(g,0)$ (this follows from Theorem 1), i.e.

$$d\tilde{v}_s^{g,x,v(x)}(x_s^g) = \nabla^g X(x_s^{g,x})(\tilde{v}_s^{g,x,v(x)})dW_s, \tilde{v}_0^{g,x,v(x)} = v(x) \in T_xM. \tag{51}$$

Notice that equations $(50, 51)$ are obtained by taking $u \equiv 0$ in equations $(46, 47)$ and further rescaling by $(2\nu)^{-\frac{1}{2}}$.

Then if u_τ is a C^2 solution of (49) for any fixed τ, applying to it the Ito formula, we obtain that

$$M_s := u_\tau(x_s^g)(\tilde{v}_s^g) + \frac{1}{2}\int_0^s \delta\Omega_\tau(x_r^g)(\tilde{v}_r^g)dr$$

is a local martingale in $[0,\tau_e)$, where τ_e is the first-exit time of U, i.e. $\tau_e = inf\{\tau : x_\tau^g \notin U\}$. Then if u_τ and $\delta\Omega_\tau$ are bounded, then for $s < \tau_e$ we get $|M_s| \leq \| u_\tau \|_\infty + \tau_e/2 \| \delta\Omega_\tau \|_\infty$. Assume now that $E_x^B\tau_e < \infty$ for any $x \in U$, where E^B denotes the expectation value with respect to the measure on the paths of $\{x_\tau^{g,x} : \tau \geq 0\}$; then M_s is a uniformly integrable martingale on $[0,\tau_e)$. Further, since $u_\tau = \phi$ on ∂U, it further satisfies

$$lim_{s\uparrow\tau_e} M_s = \phi(x_{\tau_e}^{g,x})(\tilde{v}_{\tau_e}^{g,x,v(x)}) + 1/2\int_0^{\tau_e} \delta\Omega_\tau(x_r^{g,x})(\tilde{v}_r^{g,x,v(x)})dr, \tag{52}$$

so that

$$M_s \equiv E_x^B[(\phi^g(x_{\tau_e}^{g,x})(\tilde{v}_{\tau_e}^{g,x,v(x)}) + 1/2\int_0^{\tau_e} \delta\Omega_\tau(x_r^{g,x})(\tilde{v}_r^{g,x,v(x)})dr)|\mathcal{F}_s] \tag{53}$$

where $\{\mathcal{F}_s : s \geq 0\}$ is an increasing filtration adapted to $\{x_s^{g,x} : s \geq 0\}$. Taking $s = 0$, since $(x_0^{g,x}, \tilde{v}_0^{g,x,v(x)}) = (x, \tilde{v}(x))$, if $\delta\Omega_\tau$ is bounded, we then obtain if there is a C^2 solution of the Dirichlet problem is given by the identity:

$$\tilde{u}_\tau(x)(v(x)) = E_x^B[\phi(x_{\tau_e}^{g,x})(\tilde{v}_{\tau_e}^{g,x,v(x)}) + \int_0^{\tau_e} \frac{1}{2}\delta\Omega_\tau(x^{g,x})(\tilde{v}^{g,x,v(x)}(s))ds] \tag{54}$$

where the expectation value is with respect to $p^g(s,x,y)$ the transition density of the s.d.e. (50), i.e.the fundamental solution of the heat equation on M:

$$\partial_\tau p(y) = 1/2\triangle_g p(y) \tag{55}$$

with $p(s,x,-) = \delta_x$ as $s \downarrow 0$.

Now, (49) is solvable in the weak sense since the equation for k-forms: $H_k(g,0) \equiv \triangle_k \mu = \beta$ for given β and μ being k-forms ($k = 0,\ldots,n$), has weak solutions if and only if β is orthogonal to every harmonic k-form: i.e. $(\beta,\phi) = \int g(\beta,\phi)vol_g = 0$, for every k-form ϕ on M such that $\triangle_k \phi = 0$ [38]. Indeed, if ϕ is an harmonic form, then $d\phi = 0$, and then $(\delta\Omega,\phi) = (\Omega, d\phi) = 0$.) We can state the following result (cf. Prop.1.6 and page 307, in vol. I, [39]).

Theorem 6. Assume $u \in H^1$, i.e. a 1-form belonging to the Sobolev space H^1. If $\delta\Omega \in H^{k-1}$, then $u \in H^{k+1}$, for $k \geq 0$.

8 Solutions of NS on euclidean space

In the case that M is euclidean space, the solution of NS is easily obtained from the solution in the general case. In this case the immersion f of M is realized by the identity mapping, i.e. $f(x) = x, \forall x \in M$. Hence the diffusion tensor $X = I$, so that the metric g is also the identity. For this case we shall assume that the velocity vanishes at infinity, i.e. $u_t \to 0$ as $|x| \to \infty$. (This allows us to carry out the application of the general solution, in spite of the non-compacity of space). Furthermore, $\tau_e = \infty$. The solution for the vorticity equation results as follows. We have the s.d.e. (see (46))

$$dx_s^{\tau,x} = -u(\tau - s, x_s^{\tau,x})ds + (2\nu)^{\frac{1}{2}} dW_s, x_0^{\tau,x} = x, s \in [0,\tau]. \tag{56}$$

The derived process is given by the solution of the o.d.e. (since in (47) we have $\nabla X \equiv 0$)

$$d\tilde{v}_s^{\tau,x,v(x)} = -\nabla u(\tau - s, x_s^{\tau,x})(\tilde{v}_s^{\tau,x,v(x)})ds, v_0^{\tau,v(x)} = v(x) \in T_x R^n, s \in [0,\tau], \tag{57}$$

Now for $n = 3$ we have that the vorticity $\Omega(\tau,x)$ is a 2-form on R^3, or still by duality has an adjoint 1-form ([38]), or still a function, which with abuse of notation we still write as $\tilde{\Omega}(\tau,.) : R^3 \to R^3$, which from (48) we can write as

$$\tilde{\Omega}(\tau,x) = E_x[\tilde{v}_\tau^{\tau,x,I}\Omega_0(x_\tau^{\tau,x})], \tag{58}$$

where E_x denotes the expectation value with respect to the measure (if it exists) on $\{x_\tau^{\tau,x} : \tau \geq 0\}$, for all $x \in R^3$, and in the r.h.s. of (56) we have matrix multiplication Thus, in this case, we have that the deformation tensor acts on the initial vorticity along the random paths. This action is the one that for $3D$ might produce the singularity of the solution.

In the case of R^2, the vorticity can be thought as a pseudoscalar, since $\Omega_\tau(x) = \tilde{\Omega}_\tau(x)dx^1 \wedge dx^2$, with $\tilde{\Omega}_\tau : R^2 \to R$, and being the curvature identically equal to zero, the vorticity equation is (a *scalar* diffusion equation)

$$\frac{\partial \tilde{\Omega}_\tau}{\partial \tau} = H_0(2\nu I, \frac{-1}{2\nu}u_\tau)\tilde{\Omega}_\tau \tag{59}$$

so that for $\tilde{\Omega}_0 = \tilde{\Omega}$ given, the solution of the initial value problem is

$$\tilde{\Omega}(\tau,x) = E_x[\tilde{\Omega}(x_\tau^{\tau,x})] \tag{60}$$

This solution is qualitatively different from the previous case. Due to a geometrical duality argument, for $2D$ we have factored out completely the derived process in which the action of the deformation tensor on the initial vorticity is present.

Furthermore, the solution of equation (50) is (recall that $X = I$)

$$x_\tau^{g,x} = x + W_\tau, \tag{61}$$

and since $\nabla X = 0$, the derived process (see (51)) is constant

$$v_\tau^{g,x,v(x)} = v(x), \forall \tau \in [0,T]. \tag{62}$$

so that its influence on the velocity of the fluid can be factored out in the representation (52). Indeed, we have

$$\begin{aligned} \tilde{u}_\tau(x)(v(x)) &= E_x^B[\int_0^\infty \frac{1}{2}\delta\Omega_\tau(x+W_s)(\tilde{v}^{g,x,v(x)}(s))ds] \\ &= E_x^B[\int_0^\infty \frac{1}{2}\delta\Omega_\tau(x+W_s)ds(v(x))] \end{aligned}$$

for any tangent vector $v(x)$ at x, and in particular (we take $v(x) = I$) we obtain

$$\tilde{u}_\tau(x) = E_x^B[\int_0^\infty \frac{1}{2}\delta\Omega_\tau(x+W_s)ds]. \tag{63}$$

In this expression we know from (55) that the expectation value is taken with respect to the standard Gaussian function, $p^g(s,x,y) = (4\pi s)^{\frac{-n}{2}} exp(-\frac{|x-y|^2}{4s})$.

Let us describe in further detail this solution separately for each dimension. We note first that if $\Omega_\tau \in L^1 \cap C_b^1$ (where C_b^1 means continuously differentiable, bounded with bounded derivatives)

$$E[\delta\Omega_\tau(x+W_s)] = \delta E[\Omega_\tau(x+W_s)] \tag{64}$$

In case $n = 2$, for a 2-form β on M we have $\delta\beta = \delta(\tilde{\beta}dx^1 \wedge dx^2) = -(\partial_2\tilde{\beta}dx^1 - \partial_1\tilde{\beta}dx^2) \equiv -\nabla^\perp\beta$. In case $n = 3$, for a vorticity described by the 1-form (or a vector-valued function) $\tilde{\Omega}_\tau : R^3 \to R^3$ adjoint to the vorticity 2-form Ω_τ, we have that

$$\delta\Omega_\tau = -d\tilde{\Omega}_\tau = -\mathrm{rot}\tilde{\Omega}_\tau. \tag{65}$$

Therefore, we have the following expressions for the velocity: When $n = 2$ we have

$$u_\tau(x) = \int_0^\infty -\frac{1}{2}\nabla^\perp E_x^B[\tilde{\Omega}_\tau(x+W_s)]ds \tag{66}$$

while for $n = 3$ we have

$$u_\tau(x) = \int_0^{\tau_e} \frac{-1}{2} dE_x^B[\tilde{\Omega}_\tau(x+W_s)]ds. \tag{67}$$

Now we can obtain an expression for the velocity which has no derivatives of the vorticity; this follows the basic idea in a construction given by Busnello who starts with the stream function (of an unbounded incompressible fluid) instead of the velocity [8,36]. Consider the semigroup generated by $H_0(I,0) = \frac{1}{2}\triangle$, i.e. $P_s\tilde{\Omega}_\tau(x) = E[\tilde{\Omega}_\tau(x+W_s)]$ (in the case $n = 3$ this means the semigroup given on each component of $\tilde{\Omega}$). From the Elworthy-Bismut formula valid for scalar fields (see [37]) we have that (in the following $e_i, i = 1, 2, 3$ denotes the canonical base in R^2 or R^3)

$$\partial_i P_s \tilde{\Omega}_\tau(x) =< dP_s \tilde{\Omega}(x), e_i >= \frac{1}{s} E_x^B[\tilde{\Omega}(x + W_s) \int_0^s < e_i, dW_r >]$$

$$= \frac{1}{s} E_x^B[\tilde{\Omega}(x + W_s) \int_0^s dW_r^i] = \frac{1}{s} E[\tilde{\Omega}_\tau(x + W_s) W_s^i]. \tag{68}$$

Therefore, for $n = 2$ we have from $(67, 69)$

$$u_\tau(x) = - \int_0^{\tau_e} \frac{1}{2s} E_x^B[\tilde{\Omega}_\tau(x + W_s) W_s^\perp] ds \tag{69}$$

where $W_s^\perp = (W_s^1, W_s^2)^\perp = (W_s^2, -W_s^1)$. Instead, for $n = 3$ we have from $(68, 69)$ that

$$u_\tau(x) = - \int_0^{\tau_e} \frac{1}{2s} E_x^B[\tilde{\Omega}_\tau(x + W_s) \times W_s] ds \tag{70}$$

where $\times$ denotes the vector product and $W = (W^1, W^2, W^3) \in R^3$.

Thus, we have obtained the representations for NS in 2D and 3D. [3]

9 Final Observations

The method of integration applied in the previous section is the extension to differential forms of the method of integration (the so-called martingale problems) of elliptic and parabolic partial differential equations for scalar fields [24,31]. In distinction with the Reynolds approach in Fluid Mechanics, which has the feature of being non-covariant, in the present approach, the invariance by the group of space-diffeomorphims has been the key to integrate the equations, in separating covariantly the fluctuations and drift terms and thus setting the integration in terms of covariant martingale problems. The role of the RCW connection is precisely to yield this separation for the diffusion of scalars and differential forms, and thus the role of the differential geometrical structure is essential. Furthermore, the association between RCW connections and infinitesimal generators of gradient diffusion processes of differential forms, has set the basis for the integration of NS

Notice that in the representations $(48, 54)$, the local dependance on the curvature is built-in (the curvature is defined by second-order derivatives). This dependance might be exhibited through the scalar curvature term in the Onsager-Machlup lagrangian appearing in the path-integral representation of the fundamental solution of the transition densities of equations (48) and (55); this is the Feynman path-integral approach to the Schroedinger equation on manifolds provided with a Riemannian metric [26,28]. We remark that the computation of these path integrals is extremely involved. There is

[3]The same expressions for 2D and 3D were obtained by Busnello and Flandoli [8], following a suggestion by M. Freidlin. The approach of this authors stems from an extension of the Feynman-Kac formula, instead of gauge-theoretical constructions valid for smooth manifolds, and in particular, for Euclidean space.

further a dependance of the solution on the global geometry and topology of M appearing through the Riemannian spectral invariants of M in the short-time asymptotics of these transition densities [23,26,28]. There is another construction of the solution of NS which exhibits the dependance of the solution on the Ricci curvature, and consists in replacing the velocity process on TM by a generalized Hessian flow for the integration for the vorticity and a Ricci flow for the solution of the Poisson-de Rham equation. These alternative constructions [34], allow to integrate NS on an arbitrary compact manifold, lifting thus the restriction to Euclidean submanifolds we have placed in this article. As well, the stochastic differential approach presented in this article, allows as well to solve the kinematic dynamo problem of magnetohydrodynamics, for the passive transport of magnetic fields on smooth compact manifolds [44].

The solution scheme we have presented gives rise to infinite particle random trajectories due to the arbitrariness of the initial point of the Lagrangian paths. This continuous infinite particle solution is *exact* and we have actually computed it explicitly its expression. Actually, to integrate NS we choose a finite set of initial points and we take for Ω_0 a linear combination of 2-forms (or area elements in the 2-dimensional case) supported in balls centered in these points, the so-called many vortices solutions; one can choose the original $f_0(x)$ so that Ω_0 is supported in these balls and these localizations persists in time. Thus the role of the potential term in the Buttke magnetization 1-form in the expression (27) is to push the vortices to be confined on predetermined finite radii balls; see Chorin [1]. This requires that convergence to a solution of NS be proved in addition.

Finally we want to observe that the constructions presented in this article can be extended to the passive transport of differential forms (e.g. a magnetic field) in an incompressible fluid.

10 Appendix

We shall review some basic concepts of the probabilistic, analytical and geometrical realms.

Let $\{\mathcal{F}_\tau : \tau \geq 0\}$ be a family of sub σ-fields of a σ-field $\mathcal{F}$ in a probability space $(\Omega, \mathcal{F}, P)$. It is called a **filtration** of sub σ-fields if it satisfies the following three properties: i)$\mathcal{F}_s \subset \mathcal{F}_\tau$ if $s < \tau$; ii)$\cap_{\epsilon>0}\mathcal{F}_{\tau+\epsilon} = \mathcal{F}_\tau$, and iii) each $\mathcal{F}_\tau$ contains all null sets of $\mathcal{F}$

A stochastic process $x_\tau, \tau \in T$, with T a time-set, say $[0, \infty)$, the interval $[0, T]$ or the real line, is called $(\mathcal{F}_\tau) -$ **adapted** if for each τ, x_τ is $\mathcal{F}_\tau$-measurable. **Predictable sets** are subsets of $[0, \infty) \times R$, which are elements of the smallest σ-algebra relative to which all real $\mathcal{F}_\tau$-adapted, right-continuous processes with left-hand limits are measurable in (τ, ω). A stochastic process $x : [0, \infty) \mapsto S$, where S is a measurable space with σ-algebra $\mathcal{B}$ is called **predictable** if, for any Borel subset $\mathcal{B} \in S$, $\{(\tau, \omega), x(\tau, \omega) \in B\}$ is predictable.

A positive random variable t is called a **stopping time** (with respect to the filtration $\{\mathcal{F}_\tau : \tau \geq 0\}$ if for all $0 \leq \tau, \{t \leq \tau\} \in \mathcal{F}_\tau$. This concept is used to indicate the ocurrence of some random event.

The **conditional expectation** of a real-valued random variable X with respecto to a sub-σ algebra $\mathcal{G}$ of $\mathcal{F}$ is denoted by $E(X|\mathcal{G})$.

Let x_τ be a $(\mathcal{F}_\tau)$-adapted real-valued process such that for each τ, x_τ is integrable. It is called a **martingale** if it satisfies: $E[x_\tau|\mathcal{F}_s] = x_s$ a.s. for any $\tau > s$. Furthermore, x_τ is called a **local martingale** if there exists an increasing sequence of stopping times $\{\tau_n\}$ such that $P(\tau_n < T) \to 0$ as $n \to \infty$, and each stopped time $x_\tau^{\tau_n} \equiv x_{\tau \wedge \tau_n}$ is a martingale, where $\tau \wedge \tau_n = min\{\tau, \tau_n\}$. A martingale is obviously a local martingale (set $\tau_n \equiv T$, for all n. Finally, x_τ is called a **semimartingale** if it can be decomposed as the sum of a localmartingale and a process of bounded variation.

This decomposition (which due to its complicated form we have not written down explicitly in the Ito Formula) sets for the integration of the heat equation and the Dirichlet problem, as done above for the case of differential forms, as the solutions of the so-called **martingale problems** (after Stroock and Varadhan [49]).

Let D be a domain in Euclidean space R^d and let R^l another Euclidean space (eventually $d = l$). Let $m \in N$; denote $C^m \equiv C^m(D; R^l)$ the set of all maps $f : D \to R^l$ which are m-times continuously differentiable. For the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in N^d$, define

$$D_x^\alpha = \frac{\partial^\alpha}{(\partial x^1)^{\alpha_1} \ldots (\partial x^d)^{\alpha_d}}, \text{with} |\alpha| = \sum_{i=1}^d \alpha_i. \tag{71}$$

Let K be a subset of D. Set

$$||f||_{m,K} = sup_{x \in K} \frac{f(x)}{(1 + |x|)} + \sum_{1 \leq |\alpha| \leq m} sup_{x \in K} |D^\alpha f(x)|. \tag{72}$$

Then $C^m(D; R^l)$ is a Frechet space by seminorms $\{||.||_{m;K}$: K are compacts in $D\}$. When $K = D$ we write $||.||_{m;K}$ as $||.||_K$. Now let δ such that $0 < \delta < 1$. Denote by $C^{m;\delta} \equiv C^{m;\delta}(D; R^l)$ the set of all $f \in C^m$ such that $D^\alpha f, |\alpha| = m$ are δ-Holder continuous. By the seminorms

$$||f||_{m+\delta;K} = ||f||_{m;K} + \sum_{\alpha=m} sup_{x,y \in K, x \neq y} \frac{D^\alpha f(x) - D^\alpha f(y)}{|x - y|^\delta}, \tag{73}$$

is a Frechet space, the so-called **space of $\delta -$ Holder continuous $\mathbf{C^m}$ mappings**. When $D = K$ we write $||.||_{m+\delta;K}$ as $||.||_{m+\delta}$. Denote further as $C_b^{m;\delta}$ the set $\{f \in C^{m;\delta}$: $||f||_{m+\delta} < \infty\}$. A continuous mapping $f(\tau, x)$, $x \in D, \tau \in T\}$ is said to belong to the class $\mathbf{C^{m;\delta}}$ if for every $\tau, f(\tau) \equiv f(\tau, .)$ belongs to $C^{m;\delta}$ and $||f(\tau)||_{m+\delta;K}$ is integrable on T with respect to τ in any compact subset K. If the set K is replaced by D, f is said to belong to the class $\mathbf{C_b^{m;\delta}}$. Furthermore, if $||f(\tau)||_{m+\delta}$ is bounded in τ, it is said to belong to the class $\mathbf{C_{ub}^{m;\delta}}$.

Consider the **canonical Wiener space** Ω of continuous maps $\omega : R \to R^d, \omega(0) = 0$, with the canonical realization of the Wiener process $W(\tau)(\omega) = \omega(\tau)$. Let $\phi_{s,\tau}(x, \omega), s, \tau \in T, x \in R^d$ be a continous R^d-valued random field defined on $(\Omega, \mathcal{F}, P)$. Then, for almost all $\omega \in \Omega$, $\phi_{s,\tau}(\omega) \equiv \phi_{s,\tau}(., \omega)$ defines a continuous map from R^d into itself, for any s, τ. Then, let us assume that $\phi_{s,\tau}$ satisfies the conditions: (i) $\phi_{s,u}(\omega) = \phi_{\tau,u}(\omega)\phi_{s,\tau}(\omega)$, holds for all s, τ, u, where in the r.h.s. of (i) we have the composition of maps, (ii) $\phi_{s.s}(\omega) = id$, for all s, where id denotes the identity map, (iii) $\phi_{s,\tau}(\omega) : R^d \to R^d$, is an onto homeomorphism for all s, τ, and (iv) $\phi_{s,\tau}(x, \omega)$, is an onto homeomorphism with respect to x for all s, τ, and the derivatives are continuous in (s, τ, x).

Let $\phi_{s,\tau}(\omega)^{-1}$ be the inverse map of $\phi_{s,\tau}(\omega)$. Then i) and ii) imply that $\phi_{\tau,s}(\omega) = \phi_{s,\tau}(\omega)^{-1}$. This fact and condition iii) show that $\phi_{s,\tau}(\omega)^{-1}$ is also continuous in (s, τ, x) and further condition iv) implies that $\phi_{s,\tau}(\omega)^{-1}(x)$ is k-times differentiable with respect to x. Hence $\phi_{s,\tau}(\omega) : R^d \to R^d$ is actually a C^k-diffeomorphism of M, for all s, τ, the so-called random diffeomorphic flow. We can regard $\phi_{s,\tau}(\omega)^{-1}(x)$ as a random field with parameter (s, τ, x) which is often denoted as $\phi^{-1}_{s,\tau}(x, \omega)$. Therefore,

$$\phi^{-1}_{s,\tau}(x) = \phi_{\tau,s}(x) \tag{74}$$

holds for all s, τ, x a.s.When we choose the initial time s of $\phi_{s,\tau}(x, \omega)$ to be 0, we shall write $\phi_\tau(x, \omega)$.

Let us finally review several fundamental notions in differential geometry. Let M be a d-dimensional C^∞-manifold. Let $x \in M$. By a **tangent vector at x** we mean a real-valued linear mapping V defined on the set $F(M)$ of smooth functions defined on M such that $V(fg) = V(f)g + fV(g)$, for any $f, g \in F(M)$. The set of tangent vectors at x form a linear space $T_x(M)$ called the tangent space at x, with the rules

$$(V_1 + V_2)(f) = V_1(f) + V_2(f), \&(\lambda V)(f) = \lambda V(f).$$

The dual of this vector space, $T_x(M)^*$ is called the cotangent space at x. Let $(x^1, \ldots, x^d)$ be a local coordinate in a coordinate neighbourhood U of x. Every $f \in F(M)$ is expressed on U as a C^∞-function $f(x^1, \ldots, x^d)$. Then, $f \mapsto (\frac{\partial f}{\partial x^i})(x)$ is a tangent vector at x for every $i = 1, \ldots, n$; this is denoted by $(\frac{\partial}{\partial x^i})_x$. It is easy to see that $\{(\frac{\partial}{\partial x^i})_x, i = 1, \ldots, d\}$ forms a basis for $T_x(M)$. Furthermore the set $\{T_x(M), x \in M\}$ $(\{T_x(M)^*, x \in M\}$ has a structure of smooth $2d$-dimensional manifold, the **tangent (cotangent) manifold** of M, denoted as TM (T^*M). By a vector field we mean a mapping $V : x \in M \mapsto V(x) \in T_x(M)$. V is called a C^∞-vector field if for every $f \in F(M), (Vf)(x) := V(x)f$ is a C^∞-function. Thus V is a C^∞-vector field if and only if V is a linear mapping of $F(m)$ into $F(M)$ such that $V(fg) = V(f)g + fV(g)$. Similarly one can define vector fields with the regularity conditions one chooses for the space of functions defined on M. The set of all vector fields is denoted as $\Xi(M)$.

A **tensor of type (p,q)** at $x \in M$ is an element in the tensor product $T_x(M)^p_q = T_x(M) \otimes \ldots \otimes T_x(M) \otimes T_x(M)^* \otimes \ldots \otimes T_x(M)^*$, where we are taking p-times (q-times) product of the vector space $T_x(M)$ $(T_x(M)^*)$ given by the the tangent (cotangent) space at x to M, is the linear space formed by all multilinear mappings $u : T_x(M)^* \times \ldots \times T_x(M)^* \times T_x(M) \times \ldots \times T_x(M) \to R$; here we are taking the p (q)-product of $T_x(M)^*$ ($T_x(M)$ respectively). Choosing a local coordinate $(x^1, \ldots, x^d)$ of M, we have a basis $(\frac{\partial}{\partial x^1})_x, \ldots, (\frac{\partial}{\partial x^d})_x$ in $T_x(M)$; its dual basis of $T_x(M)^*$ is denoted by $(dx^1)_x, \ldots, (dx^d)_x$. We denote by $(\frac{\partial}{\partial x^1})_x \otimes \ldots \otimes (\frac{\partial}{\partial x^d})_x \otimes (dx^1)_x \otimes \ldots \otimes (dx^d)_x$ the linear element $u \in T_x(M)^p_q$ such that

$$u((dx^{k_1})_x, \ldots, (dx^{k_d})_x, (\frac{\partial}{\partial x^{j_1}})_x, \ldots, (\frac{\partial}{\partial x^{j_q}})_x) = \delta^{k_1}_{i_1} \ldots \delta^{k_p}_{i_p} \delta^{j_1}_{l_1} \ldots \delta^{j_q}_{l_q} \tag{75}$$

for every $k_1, \ldots, k_p, l_1, \ldots, l_q$. Clearly the system

$$\{(\frac{\partial}{\partial x^{i_1}})_x \otimes \ldots \otimes (\frac{\partial}{\partial x^{i_p}})_x \otimes (dx^{j_1})_x \otimes \ldots \otimes (dx^{j_q})_x\}, i_1, \ldots, \imath_p, j_1, \ldots, j_q = 1, \ldots, d, \quad (76)$$

forms a basis of $T_x(M)^p_q$. A C^k-**tensor field of type** (p,q) is a mapping

$$x \to u(x) \in T_x(M)^p_q, u : M \mapsto T_x(M)^p_q \quad (77)$$

whose components $u^{i_1 \ldots i_p}_{j_1 \ldots j_q}(x)$ with respect to the above basis in $T_x(M)^p_q$ are C^k in every coordinate neighbourhood. The $u^{i_1 \ldots i_p}_{j_1 \ldots j_q}(x)$ obey the usual rule under a coordinate transformation $(x^1, \ldots, x^d) \to (\tilde{x}^1, \ldots, \tilde{x}^d)$:

$$\tilde{u}^{i_1 \ldots i_p}_{j_1 \ldots j_q}(x) = u^{k_1 \ldots k_p}_{l_1 \ldots l_q}(x) \frac{\partial \tilde{x}^{l_1}}{\partial x^{k_1}} \cdots \frac{\partial \tilde{x}^{l_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{l_q}}{\partial \tilde{x}^{j_q}} \quad (78)$$

Here and in the following, repeated indices means summation over them. Conversely, if a system of C^k-functions $u^{i_1 \ldots i_p}_{j_1 \ldots j_q}(x)$ is defined in every coordinate neighbourhood and satisfies (50), then there exists a unique (p,q)-tensor field whose components coincide with it. Thus, a **(1,0)-tensor field** is a vector field. A $(0,1)$ is called a **differential 1-form**. Generally, a **differential p-form** is a $(0,p)$-tensor which is alternate, i.e. its components satisfy

$$u_{\sigma(i_1) \ldots \sigma(i_p)}(x) = sgn(\sigma) u_{i_1 \ldots i_p} \quad (79)$$

for every permutation σ. If we set $dx^{i_1} \wedge \ldots \wedge dx^{i_p} = \frac{1}{p!} \sum_\sigma sgn(\sigma) dx^{\sigma(i_1)} \otimes \ldots \otimes dx^{\sigma(i_p)}$, then a p-form is expressed as

$$u(x) = u_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} = p! \sum_{i_1 < \ldots < i_p} u_{i_1 \ldots i_p}(x) dx^{i_1} \wedge \ldots dx^{i_p}. \quad (80)$$

The **exterior product** $\alpha \wedge \beta$ of a p-form α and a q-form β is a $(p+q)$-form defined by

$$(\alpha \wedge \beta)(x) = \alpha_{k_1 \ldots k_p}(x) \beta_{k_{p+1} \ldots k_{p+q}}(x) dx^{k_1} \wedge \ldots \wedge dx^{k_{p+q}} \quad (81)$$

The **exterior differential** $d\alpha$ of a p-form α is a $(p+1)$-form defined by

$$(d\alpha)(x) = \frac{\partial \alpha_{k_1 \ldots k_p}}{\partial x^i} dx^i \wedge dx^{k_1} \wedge \ldots \wedge dx^{k_p}. \quad (82)$$

By an **affine** (also called in the physics literature [10, 46] as a **Riemann – Cartan**) connection ∇ we mean a rule which associates to every vector field $X \in \Xi(M)$ a linear mapping $\nabla_X : \Xi(M) \to \Xi(M)$ having the following properties: (i) $\nabla_X Y$ is bilinear in X and Y; (ii) $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$, and (iii) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$. The operator ∇_X is called **covariant differentiation with respect to X**, and ∇ is also called a **covariant derivative**. A system of functions (the so-called **Christoffel coefficients** or components of ∇) $\{\Gamma^i_{jk}\}$ is defined in a coordinate neighbourhood by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k} \quad (83)$$

In a local coordinate system, $\nabla_X Y$ may be expressed as

$$\nabla_X Y = [X^i(x)Y^j(x)\Gamma^k_{ij}(x) + X^l(x)\frac{\partial Y^k}{\partial x^l}]\frac{\partial}{\partial x^k} \tag{84}$$

where $X = X^i(x)\frac{\partial}{\partial x^i}$ and $Y = Y^i(x)\frac{\partial}{\partial x^i}$. The components of ∇ obey the following transformation rule under a coordinate transformation $x \to \tilde{x}$:

$$\tilde{\Gamma}^k_{ij} = \frac{\partial x^p}{\partial \tilde{x}^i}\frac{\partial x^q}{\partial \tilde{x}^j}\frac{\partial \tilde{x}^k}{\partial x^r}\Gamma^r_{pq} + \frac{\partial^2 x^r}{\partial \tilde{x}^l \partial \tilde{x}^j}\frac{\partial \tilde{x}^k}{\partial x^r} \tag{85}$$

Conversely, any system of functions with appropiate regularity conditions Γ^k_{ij} defined in each coordinate neighbourhood and satisfying (66) determines an affine connection, as above.

Given a tensor field $u(x) = (u^{i_1 \dots i_p}_{j_1 \dots j_q}(x)$ of type (p,q), we can define a tensor field $(\nabla u)(x) = (u^{i_1 \dots i_p}_{j_1 \dots j_q;k}(x))$ of type $(p, q+1)$ by the rule

$$\begin{aligned} u^{i_1 \dots i_p}_{j_1 \dots j_q;k}(x) \ &:= \nabla_{\frac{\partial}{\partial x^k}} u^{i_1 \dots i_p}_{j_1 \dots j_q}(x) = \tfrac{\partial}{\partial x^k} u^{i_1 \dots i_p}_{j_1 \dots j_q}(x) + \Sigma^p_{\alpha=1} \Gamma^{i_\alpha}_{kl}(x) u^{i_1 i_2 \dots \hat{l} \dots i_p}_{j_1 j_2 \dots j_q}(x) \\ & \qquad - \Sigma^q_{\beta=1} \Gamma^m_{k,j_\beta}(x) u^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots m \dots j_q}(x) \end{aligned} \tag{86}$$

where $\hat{m}$ and $\hat{l}$ means that these indices have replaced j_β and i_α respectively. By the transformation rules of the coefficients of ∇, it is easy to see that $\nabla u(x)$ is a tensor field; it is usually called the **covariant derivative of u(x)**. For $x = X^i \frac{\partial}{\partial x^i} \in \Xi(m)$, the (p,q)-tensor field $\nabla_X u$ defined by

$$(\nabla_X u)^{i_1 \dots i_p}_{j_1 \dots j_q} = X^k u^{i_1 \dots i_p}_{j_1 \dots j_k;k} \tag{87}$$

is called the covariant derivative of $u(x)$ in the direction X. Note that if $u = Y \in \Xi(M)$, the above definition coincides with the original $\nabla_X Y$.

For an affine connection $\nabla = \{\Gamma^k_{ij}\}$, $T^k_{ij} := \Gamma^k_{ij} - \Gamma^k_{ji}$ are the components of a tensor T of type $(1,2)$; it is called the **torsion tensor**. An intrinsic definition of the torsion tensor T is

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], X, Y \in \Xi(M) \tag{88}$$

A non-null torsion expresses the non-closure of infinitesimal parallelograms in M and thus is associated with a dislocation tensor (cf.[23]). An affine connection is called **symmetric** or **torsion free** if the torsion tensor is zero, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$.

A C^∞-manifold M is called a **Riemannian manifold** if a tensor field $g = (g_{ij})$ of type $(0,2)$ is given on M such that: (i) g is symmetric, i.e.: $g_{ij}(x) = g_{ji}(x)$; (ii): g is positive definite, i.e. $g_{ij}(x)\phi i^i \phi^j$ for all $x \in M$ and non-zero $\phi \in R^d$; g is called the **Riemannian metric (tensor field)**. It defines an inner product on each tangent space $T_x(M)$ by

$$< X, Y >:= g(X,Y) = g_{ij}(x)X^iY^j, X = X^i(\frac{\partial}{\partial x^i})_x \mathrm{and} Y = Y^i(\frac{\partial}{\partial x^i})_x \tag{89}$$

By duality, the metric g induces a positive-definite symmetric tensor of type $(2,0)$, which with abuse of notation we also denote as g.

An affine connection $\nabla = \{\Gamma^k_{ij}\}$ is called **g-compatible** or **compatible with the Riemannian metric g** if the inner product is preserved during a parallel translation of tangent vectors. That is, for every smooth curve $c(t) \in M$ and tangent vectors $X^i(t)\frac{\partial}{\partial x^i}$ and $Y^i(t)\frac{\partial}{\partial x^i}$ at $c(t)$

$$\frac{dX^i}{dt} + \Gamma^i_{jk}(c(t))\frac{dc^j(t)}{dt}X^k(t) = 0, \text{and} \frac{dY^i}{dt} + \Gamma^i_{jk}(c(t))\frac{dc^j(t)}{dt}Y^k(t) = 0 \tag{90}$$

imply that

$$\frac{d}{dt}(g_{ij}(c(t))X^i(t)Y^j(t)) = 0. \tag{91}$$

From this it is easy to conclude that ∇ is compatible with g if and only if

$$\frac{\partial}{\partial x^k}g_{ij} = g_{lj}\Gamma^l_{ki} + g_{il}\Gamma^l_{kj} \tag{92}$$

for all $i, j = 1, \ldots, d$. An affine connection which is compatible with g and is further symmetric is unique, and given by

$$\Gamma^\alpha_{\beta\gamma} := \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} = \frac{1}{2}(\frac{\partial}{\partial x^\beta}g_{\nu\gamma} + \frac{\partial}{\partial x^\gamma}g_{\beta\nu} - \frac{\partial}{\partial x^\nu}g_{\beta\gamma})g^{\alpha\nu} \tag{93}$$

This connection is called the **Levi-Civita connection**, or still, the Riemannian connection.

The Christoffel coefficients of an affine connection ∇ compatible with g can be decomposed as in (9), i.e.

$$\Gamma^i_{j,k} = \begin{Bmatrix} i \\ jk \end{Bmatrix} + \frac{1}{2}K^i_{jk}. \tag{94}$$

where

$$K^i_{jk} = T^i_{jk} + S^i_{jk} + S^i_{jk}, \tag{95}$$

is the cotorsion tensor, with $S^i_{jk} = g^{i\nu}g_{j\kappa}T^\kappa_{\nu k}$, and $T^i_{jk} = (\Gamma^i_{jk} - \Gamma^i_{kj})$ the skew-symmetric torsion tensor. Indeed, we have from (93) the following identities

$$\frac{\partial}{\partial x^k}g_{ij} - g_{mj}\Gamma^m_{ki} - g_{im}\Gamma^m_{kj} = 0,$$

$$\frac{\partial}{\partial x^j}g_{ik} - g_{mk}\Gamma^m_{ji} - g_{im}\Gamma^m_{jk} = 0,$$

and

$$-\frac{\partial}{\partial x^i}g_{jk} + g_{mk}\Gamma^m_{ij} + g_{jm}\Gamma^m_{ik} = 0.$$

Hence

$$\begin{aligned} \begin{Bmatrix} i \\ jk \end{Bmatrix} &= \tfrac{1}{2}(\tfrac{\partial}{\partial x^j}g_{\nu k} + \tfrac{\partial}{\partial x^k}g_{j\nu} - \tfrac{\partial}{\partial x^\nu}g_{jk})g^{i\nu} \\ &= \tfrac{1}{2}g^{is}g_{mj}(\Gamma^m_{ks} - \Gamma^m_{sk}) + \tfrac{1}{2}g^{i\nu}g_{mk}(\Gamma^m_{j\nu} - \Gamma^m_{\nu j}) + \tfrac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj}) \\ &= -\tfrac{1}{2}(S^i_{jk} + S^i_{kj}) + \tfrac{1}{2}(\Gamma^i_{kj} + \Gamma^i_{jk}) = -\tfrac{1}{2}(S^i_{jk} + S^i_{kj}) + \Gamma^i_{kj} - \tfrac{1}{2}T^i_{jk}, \end{aligned}$$

from which we get (9).

Acknowledgements

The author would like to express his deep gratitude to Prof. K.D. Elworthy (Univ. of Warwick, U.K.) and Prof.I. Simao (Univ. of Lisbon), for their kind invitation to present this work at the Workshop on Evolution Equations, Nonlinear Partial Differential Equations 99, May 17-20, 1999, CMAF-Univ.of Lisbon.

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