$W^{2,p}$ estimates for the parabolic Monge-Ampère equation

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1. Introduction

The parabolic Monge-Ampère operator considered in this paper is

$$\mathcal{M}u = -u_t \det D_x^2 u, \tag{1.1}$$

where $u = u(x,t)$ is convex in x and nonincreasing in t, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $D_x^2 u$ denotes the Hessian of u with respect to the variable x. This operator is relevant in the study of deformation of surfaces by Gauss-Kronecker curvature [**Fir74**], [**Tso85a**], and in a maximum principle for parabolic equations [**Tso85b**]. Together with (1.1), N. V. Krylov [**Kry76**] introduced other parabolic versions of the elliptic Monge-Ampère operator, see [**Lie96**, pp. 406–416] for a complete description and related results.

Our purpose in this paper is to establish that solutions u to $\mathcal{M}u = f$ with f positive, continuous, and f_t satisfying certain growth conditions, have second derivatives in L^p, for $0 < p < \infty$. This is the main result in this paper and is precisely stated in section (2), theorem (2.1). These type of interior estimates have been recently established by L. A. Caffarelli [**Caf90a**] for the elliptic Monge-Ampère equation $\det D_x^2 u = f$, and therefore we extend Caffarelli's result to the parabolic case. The origin of these estimates goes back to Pogorelov [**Pog71**] who proved that convex solutions to $\det D^2 u = 1$ on a bounded convex domain Ω with $u = 0$ on $\partial\Omega$ satisfy the L^∞ estimate

$$C_1(\Omega', \Omega) \leq D^2 u(x) \leq C_2(\Omega', \Omega), \tag{1.2}$$

for $x \in \Omega'$, where Ω' is a convex domain with closure contained in Ω, and C_i are positive constants depending only on the domains. The estimate (1.2) plays an important role in the fundamental estimates proved by Caffarelli, and the crucial estimate that leads to (1.2) is that one can bound the Hessian of u by means of its gradient, [**Pog71**, Theorem 2]. In [**GH98**], the parabolic analogue of [**Pog71**, Theorem 2] was used to estalish a generalization of a celebrated theorem by Calabi [**Cal58**]. Such extension plays an important role in the present paper, see theorem (5.2) below. All these results use the recent theory for cross sections of solutions to the Monge-Ampère equation developed in the papers [**Caf90a**], [**Caf91**], [**CG97**], [**CG96**], [**GH99**], and [**Hua99**]. One of the purposes in this paper is to extend several results of this theory to the parabolic setting and the main difficulty for this extension is due to the presence of the time derivative in the definition of $\mathcal{M}$. However, under some conditions on the right hand side f, we prove that u_t is bounded away from zero and $-\infty$. This permit us to introduce an appropriate notion of parabolic cross section, defined by (4.1), that enjoys properties that lead to the desired result.

We mention that $C^{2+\alpha,1+\alpha/2}$ estimates for solutions to $\mathcal{M}u = f$ were obtained in [**WW92**] when f is Lipschitz continuous in x and t.

Throughout the paper we work with classical solutions but all the estimates are independent of the smoothness and depend only on the structural constants.

Each section in the paper contains results that are interested in themselves. The organization is as follows. We begin in section (2) introducing some notation, definitions and the statement of the main result. In section (3), we show that under certain conditions on the right hand side f, one can bound u_t in the interior of the domain by the bounds for the data on the parabolic boundary. This holds for example if f_t is of bounded mean oscillation. Section (4) contains the proofs of the properties of the parabolic cross sections needed in section (6). Section (5) contains the proof of an approximation theorem crucial for the proof of the $W^{2,p}$ estimates. Section (6) contains the proof of the $W^{2,p}$ estimates. Finally, the appendix, section (7), contains the regularity properties of the parabolic convex envelope.

2. Notation, definitions and statement of the main result

If $Q \subset \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$ then we denote

$$Q(t) = \{x : (x,t) \in Q\}.$$

Let $Q \subset \mathbb{R}^{n+1}$ be a bounded set and $t_0 = \inf\{t : Q(t) \neq \emptyset\}$. The *parabolic boundary* of the bounded domain Q is defined by

$$\partial_p Q = \left(\overline{Q}(t_0) \times \{t_0\}\right) \cup \bigcup_{t \in \mathbb{R}} \left(\partial Q(t) \times \{t\}\right),$$

where $\overline{Q}$ denotes the closure of Q and $\partial Q(t)$ denotes the boundary of $Q(t)$. We say that the set $Q \subset \mathbb{R}^{n+1}$ is a *bowl-shaped domain* if $Q(t)$ is convex for each t and $Q(t_1) \subset Q(t_2)$ for $t_1 \leq t_2$.

Let Q be a bowl-shaped domain in $\mathbb{R}^{n+1}$, and $u \in C(\overline{Q})$. A function $u(x,t)$ is *parabolically convex* in Q or *p-convex*, if it is convex in x and nonincreasing in t.

Given $z_0 = (x_0, t_0) \in Q$, $\ell_{z_0}(x)$ is a *supporting affine function, or supporting hyperplane* to $u(\cdot, t_0)$ at $x = x_0$, if $\ell_{z_0}(x) = u(x_0,t_0) + p\cdot(x - x_0)$ and $u(x,t_0) \geq \ell_{z_0}(x)$ for all $x \in Q(t_0)$. When u is regular we have $p = Du(x_0,t_0)$.

Given $h > 0$, we define

$$Q_h(z_0) = \{(x,t) : u(x,t) \leq \ell_{z_0}(x) + h \text{ and } t \leq t_0\}, \tag{2.1}$$

and

$$S_h(x_0|t_0) = \{x : u(x,t_0) \leq \ell_{z_0}(x) + h\}. \tag{2.2}$$

If $Q \subset \mathbb{R}^{n+1}$ is an open bounded bowl-shaped domain and $u : Q \to \mathbb{R}$ is continuous then the *parabolic normal mapping of* u is the set valued function $\mathcal{P}_u : Q \to \{E : E \subset \mathbb{R}^{n+1}\}$ defined by

$$\mathcal{P}_u(x_0,t_0) =$$
$$\{(p,h) : u(x,t) \geq u(x_0,t_0) + p\cdot(x - x_0),\ \forall x \in Q(t), \text{with } t \leq t_0,\ h = p\cdot x_0 - u(x_0,t_0)\},$$

where $Q(t) = \{x : (x,t) \in Q\}$. If $E \subset Q$ then $\mathcal{P}_u(E) = \cup_{(x,t)\in E}\mathcal{P}_u(x,t)$.

Given a bounded convex domain $\Omega \subset \mathbb{R}^n$ with nonempty interior, let E be the ellipsoid of minimum volume containing Ω with center at the center of mass of Ω. Then there exists an affine transformation T such that $B_{\alpha_n}(0) \subset T(\Omega) \subset B_1(0)$ with $\alpha_n = n^{-3/2}$; see [**Pog78**, p. 90].

The main results in this paper can be summarized in the following theorem. The proof of conclusion (A) is given in section (3), Theorem (3.2); and the proof of (B) is given at the end of section (6).

THEOREM 2.1. *Let u be a parabolically convex solution to $\mathcal{M}u = f$ in the cylinder $Q = \Omega \times (0,T]$ with $u = \phi$ on $\partial_p Q$. Suppose that*

1. *$B_{\alpha_n}(0) \subset \Omega \subset B_1(0)$ convex, $\partial\Omega \in C^{1,\alpha}$ with $\alpha > 1 - \frac{2}{n}$;*
2. *$0 < \lambda \leq f \leq \Lambda$, $f \in C(\bar{Q})$, $f_t \in L^{n+1}(Q)$, and $\exp(A(-f_t)^+) \in L^1(Q)$ for some $A > 0$; and*
3. *$\phi \in C^{2,1}(\bar{Q})$ satisfying $-c_2 \leq \phi_t \leq -c_1$ and $C_1 I \leq D^2\phi \leq C_2 I$ in Q, where*

(A) *There exist positive constants* M_1 *and* M_2*, depending only on the constants above and* $\|f_t\|_{L^{n+1}(Q)}$*, such that*

$$-M_1 \le u_t \le -M_2, \qquad \text{in } Q;$$

(B) *For each* $0 < p < \infty$ *and* $h > 0$ *we have*

$$\iint_{\Omega_h \times (h,T]} \|D_x^2 u(x,t)\|^p \, dxdt \le C,$$

where $\Omega_h = \{x \in \Omega : dist(x, \partial\Omega) > h\}$*, and* C *is a constant that depends only on* p*,* h*,* T*, and the parameters in* (1), (2)*, and* (3).

3. Propagation of the bounds for u_t from the boundary to the interior

Let Ω be a bounded convex domain in $\mathbb{R}^n$, $Q = \Omega \times (0,T)$, and u a parabolically convex function solution to the problem:

$$-u_t \det D^2 u = f \qquad \text{in } Q \tag{3.1}$$

$$u = \phi \qquad \text{on } \partial_p Q. \tag{3.2}$$

The main result of this section is theorem (3.2). We first show that if ϕ_t is bounded on $\partial_p Q$ then the same is true for u_t. This implies together with our assumptions (3.5) and (3.6) on f_t that u_t is bounded away from zero and $-\infty$ in the interior of Q.

Lemma 3.1. *Let* ϕ *be a function defined on* $\bar{Q}$ *such that there exist negative constants* m_1 *and* m_2 *so that*

$$m_1 \le \phi_t \le m_2, \qquad \text{on } \partial\Omega \times [0,T]. \tag{3.3}$$

Assume in addition that $\phi(x,0)$ *is strictly convex in* Ω*, and* $0 < \lambda \le f(x,t) \le \Lambda$ *in* $\bar{Q}$.

There exist negative constants m_1' *and* m_2' *depending only on* m_1, m_2, λ *and* Λ *such that if* u *solves* (3.1) *and* (3.2) *then*

$$m_1' \le u_t \le m_2', \qquad \text{on } \partial_p Q. \tag{3.4}$$

Proof. We have $u_t = \phi_t$ on $\partial\Omega \times [0,T]$. Also $u(x,0) = \phi(x,0)$ on $\bar{\Omega} \times \{0\}$. Hence $\det D^2 u(x,0) = \det D^2\phi(x,0) \approx C$. So $\lambda_1 \le \det D^2 u(x,0) \le \lambda_2$ and by equation (3.1) we are done. □

The following theorem shows global bounds for u_t in Q. Notice that if for example $f_t \in BMO$ then conditions (3.5) and (3.6) hold.

Theorem 3.2. *Assume the hypotheses of Lemma 3.1 and that there exist positive constants* A *and* B *such that*

$$f_t \in L^{n+1}(Q) \tag{3.5}$$

Then there exist negative constants M_1 and M_2, depending only on the constants above and $\|f_t\|_{L^{n+1}(Q)}$, such that

$$M_1 \le u_t \le M_2, \qquad \text{in } Q. \tag{3.7}$$

PROOF. These inequalities will be proved by using auxiliary functions and the following Aleksandrov-Bakelman-Pucci type maximum principle proved by Tso, [**Tso85b**]: if u is a smooth function defined on the cylinder Q then

$$\sup_{Q} u \le \sup_{\partial_p Q} u + C \left(\iint_{\Gamma(u)} |u_t \det D^2 u| \, dxdt \right)^{1/(n+1)}, \tag{3.8}$$

where C is a constant depending only on n, T and the diameter of Ω; and $\Gamma(u)$ is the set

$$\Gamma(u) = \{(x,t) \in Q : u_t(x,t) \ge 0, \text{ and } D^2 u(x,t) \le 0\}.$$

We consider the linearized parabolic Monge-Ampère operator associated with u and defined by

$$L(v) = -\frac{1}{u_t} v_t - \operatorname{trace}\left(\left(D^2 u\right)^{-1} D^2 v \right).$$

Differentiating (3.1) with respect to t yields

$$-(u_t)_t \det D^2 u - u_t \operatorname{trace}\left((D^2 u)^{-1} (\det D^2 u) \, D^2(u_t) \right) = f_t.$$

Consequently,

$$L(u_t) = -\frac{f_t}{f},$$

and if we let

$$v(x,t) = (t+M)^k \, u_t(x,t), \tag{3.9}$$

with $M > 0$, we then have

We first estimate the $\inf_Q u_t$. Set $k = 0$ in (3.9). Then $L(u_t) = -\dfrac{f_t}{f}$. Hence the ABP maximum principle mentioned before applied to $-u_t$ yields

$$\begin{aligned}
\inf_Q u_t - \inf_{\partial_p Q} u_t &\geq -C \left(\iint_{\Gamma(-u_t)} -(u_t)_t \det D^2(u_t)\, dxdt \right)^{1/(n+1)} \\
&= -C \left(\iint_{\Gamma(-u_t)} \frac{-(u_t)_t}{-u_t} (\det D^2 u)^{-1} \det D^2(u_t)\, f\, dxdt \right)^{1/(n+1)} \\
&\geq -C \left(\iint_Q \left| \frac{(u_t)_t}{u_t} + \operatorname{trace}((D^2u)^{-1} D^2(u_t)) \right|^{n+1} f\, dxdt \right)^{1/(n+1)} \\
&= -C \left(\iint_Q |-L(u_t)|^{n+1} f\, dxdt \right)^{1/(n+1)} \\
&= -C \left(\iint_Q \left| \frac{f_t}{f} \right|^{n+1} f\, dxdt \right)^{1/(n+1)} \\
&\approx -C\, \|f_t\|_{L^{n+1}(Q)},
\end{aligned}$$

and consequently the first inequality in (3.7) follows from (3.4).

We now estimate $\sup_Q u_t$. Applying ABP to v defined in (3.9) we get

$$\begin{aligned}
\sup_Q v - \sup_{\partial_p Q} v &\leq C \left(\iint_{\Gamma(v)} -(-v)_t \det D^2(-v)\, dxdt \right)^{1/(n+1)} \\
&= C \left(\iint_{\Gamma(v)} -\frac{(-v)_t}{-u_t} \frac{\det D^2(-v)}{\det D^2 u} f\, dxdt \right)^{1/(n+1)} \\
&\leq C \left(\iint_{\Gamma(v)} \left(\frac{v_t}{-u_t} + \operatorname{trace}((D^2u)^{-1} D^2(-v)) \right)^{n+1} f\, dxdt \right)^{1/(n+1)} \\
&\leq C \left(\iint_Q \left((Lv)^+ \right)^{n+1} f\, dxdt \right)^{1/(n+1)},
\end{aligned}$$

since $\Gamma(v) \subset \{(x,t) : L(v)(x,t) \geq 0\}$. Hence from (3.10) we obtain

$$\begin{aligned}
\sup_Q v &\leq \sup_{\partial_p Q} v + C \left(\iint_Q \left(\left(-\frac{k}{M+t} - \frac{f_t}{f} \right)^+ \right)^{n+1} (t+M)^{k(n+1)} f\, dxdt \right)^{1/(n+1)} \\
&\leq \sup_{\partial_p Q} v + C \left(\iint_Q \left(\left(-\frac{f\,k}{M+t} - f_t \right)^+ \right)^{n+1} (t+M)^{k(n+1)} \frac{1}{f^n}\, dxdt \right)^{1/(n+1)} \\
&\leq \sup v + C \left(\iint \left(\left(-\frac{\lambda\, k}{\cdots} - f_t \right)^+ \right)^{n+1} dxdt \right)^{1/(n+1)} (T+M)^k
\end{aligned}$$

Thus

$$(T+M)^k \sup_Q u_t \leq M^k \sup_{\partial_p Q} u_t + C \left(\iint_Q \left(\left(-\frac{\lambda\, k}{M+T} - f_t \right)^+ \right)^{n+1} dxdt \right)^{1/(n+1)} (T+M)^k.$$

We now estimate

$$\iint_Q \left(\left(-\frac{\lambda\, k}{M+T} - f_t \right)^+ \right)^{n+1} dxdt$$

for k large. We have

$$\begin{aligned}
&\iint_Q \left(\left(-\frac{\lambda\, k}{M+T} - f_t \right)^+ \right)^{n+1} dxdt \\
&= \sum_{j=0}^{\infty} \iint_{(j+1)\frac{\lambda k}{M+T} \leq -f_t < (j+2)\frac{\lambda k}{M+T}} \left(\left(-\frac{\lambda\, k}{M+T} - f_t \right)^+ \right)^{n+1} dxdt \\
&\leq \sum_{j=0}^{\infty} \left((j+1)\frac{\lambda\, k}{M+T} \right)^{n+1} |\{(x,t) \in Q : -f_t(x,t) > (j+1)\frac{\lambda k}{M+T}\}| \\
&\leq \sum_{j=0}^{\infty} \left((j+1)\frac{\lambda\, k}{M+T} \right)^{n+1} \frac{B}{e^{A(j+1)\frac{\lambda\, k}{M+T}}} \qquad \text{by (3.6)} \\
&= B\, e^{-A\frac{\lambda\, k}{M+T}} \left(\frac{\lambda k}{M+T} \right)^{n+1} \sum_{j=0}^{\infty} (j+1)^{n+1} e^{-jA\frac{\lambda k}{M+T}} \\
&\leq C\, e^{-A\frac{\lambda\, k}{M+T}} \left(\frac{\lambda k}{M+T} \right)^{n+1},
\end{aligned}$$

by choosing k large so that $\dfrac{A\lambda k}{M+T} \geq 1$ and $C = \alpha\, B$ with α a universal constant.

Therefore from (3.4)

$$\sup_Q u_t \leq -C_0 \left(1 + \frac{T}{M} \right)^{-k} + C\, \frac{\lambda k}{M+T}\, e^{-k\frac{A\lambda}{(M+T)(n+1)}},$$

with $C_0 > 0$, and C depending only on $n, \operatorname{diam}(\Omega), \lambda, \Lambda, A$, and B. We take $M = 1$ and assume $T < 1$. Therefore k can be chosen so that $\dfrac{A\lambda k}{} \geq 1$ independently on

We have

$$
\begin{aligned}
&\sup_Q u_t \\
&\leq -C_0\,(1+T)^{-k}\left(1-\frac{C}{C_0}\,\frac{\lambda k}{(1+T)}\,e^{-k\frac{A\lambda}{(1+T)(n+1)}+k\ln(1+T)}\right)\\
&\leq -C_0\,(1+T)^{-k}\left(1-\frac{C\lambda k}{C_0(1+T)}\,e^{-k\frac{A\lambda}{(1+T)(n+1)}+kT}\right)\\
&\leq -C_0\,(1+T)^{-k}\left(1-\frac{C\lambda k}{C_0}\,e^{-k\frac{A\lambda}{4(n+1)}}\right) \qquad\qquad \text{if } T<T_0=\min\{1,\frac{A\lambda}{4(n+1)}\}\\
&\leq -C_1,
\end{aligned}
$$

if we choose k large (depending only on C, C_0, A, λ and n), and $T < T_0$ independent of C_0.

For general T, we cut $Q = \Omega \times (0, T)$ into a stack of thin slices $Q = \bigcup_{i=0}^{N} \Omega \times (iT_0, (i+1)T_0]$ with $N \approx T/T_0$, and repeat the process above a finite number of times to get the estimate of u_t. Indeed, we have

$$\sup_{\Omega\times(0,T_0]} u_t \leq -C_1.$$

Next,

$$\sup_{\Omega\times(T_0,2T_0]} u_t \leq -C_2,$$

by choosing another k and $v = u_t(t - T_0 + 1)^k$. Continuing in this way,

$$\sup_{\Omega\times((N-1)T_0,NT_0]} u_t \leq -C_N.$$

Therefore the estimate for u_t in Q follows. □

By theorem (3.2), to establish the L^p estimates of $D_x^2 u$ from now onwards we may assume that $m_1 < -u_t < m_2$.

4. Properties of the parabolic sections

Our purpose here is to define a notion of parabolic section that is suitable to establish the $W^{2,p}$ estimates. We could attempt to take as a notion of parabolic section of u the one given by (2.1), but the problem with the sets $Q_h(z_0)$ is that they do not satisfy the shrinking property given in lemma (4.6) and therefore, the type of decomposition given by theorem (4.12) might fail to hold in terms of those sets. This can be fixed by introducing a new definition of parabolic section given by (4.1). These new sections are monotone in h, satisfy the shrinking property and other geometric properties needed to establish a Calderón-Zygmund type decomposition, Let $\delta > 0$ be a small number that will be chosen in a moment. Let us consider the time $t_0 + \delta\, h$. Since $u(x,t)$ is nonincreasing in t we have $u(x, t_0 + \delta\, h) \le u(x, t_0)$ for all x. Let us look at the set

$$S = \{x : u(x, t_0 + \delta\, h) \le \ell_{z_0}(x)\}.$$

This set is non-empty because $x_0 \in S$. Consider

$$\Delta = \min_x \{u(x, t_0 + \delta\, h) - \ell_{z_0}(x)\}.$$

Notice that $\Delta \le 0$. Let $(x_0)^h_{\text{min}}$ be the point where the minimum is attained, that is,

$$\Delta = u((x_0)^h_{\text{min}}, t_0 + \delta\, h) - \ell_{z_0}((x_0)^h_{\text{min}}).$$

We have

$$u(x, t_0 + \delta\, h) - \ell_{z_0}(x) \ge u((x_0)^h_{\text{min}}, t_0 + \delta\, h) - \ell_{z_0}((x_0)^h_{\text{min}}),$$

for all x and therefore

$$u(x, t_0 + \delta\, h) \ge \ell_{z_0}(x) + u((x_0)^h_{\text{min}}, t_0 + \delta\, h) - \ell_{z_0}((x_0)^h_{\text{min}}) := \ell^*_{z_0}(x),$$

for all x. Since $u((x_0)^h_{\text{min}}, t_0 + \delta\, h) = \ell^*_{z_0}((x_0)^h_{\text{min}})$, it follows that $\ell^*_{z_0}(x)$ is a supporting affine function to $u(\cdot, t_0 + \delta\, h)$ at $x = (x_0)^h_{\text{min}}$.

We define the section

$$Q^*_h(z_0) = \{(x,t) : u(x,t) \le \ell^*_{z_0}(x) + h \text{ and } t \le t_0 + \delta\, h\}, \tag{4.1}$$

and notice that

$$Q^*_h(z_0) = Q_h((x_0)^h_{\text{min}}, t_0 + \delta\, h), \tag{4.2}$$

that is, each Q^*_h is a Q_h given by (2.1) at another point with t coordinate slightly larger than t_0. In case, $t_0 + \delta\, h > T$ we replace $t_0 + \delta\, h$ by T.

Remark 4.1 (Location of $(x_0)^h_{\text{min}}$). *We have that*

$$((x_0)^h_{min}, t_0) \in Q_{m_1\, \delta\, h}(x_0, t_0),$$

actually

$$(x_0)^h_{min} \in S_{m_1\, \delta\, h}(x_0 | t_0),$$

where $-u_t \le m_1$.

Indeed, we write

$$\begin{aligned}
&u((x_0)^h_{\text{min}}, t_0) - \ell_{z_0}((x_0)^h_{\text{min}}) \\
&= u((x_0)^h_{\text{min}}, t_0) - u((x_0)^h_{\text{min}}, t_0 + \delta\, h) + u((x_0)^h_{\text{min}}, t_0 + \delta\, h) - \ell_{z_0}((x_0)^h_{\text{min}}) \\
&\le u((x_0)^h_{\text{min}}, t_0) - u((x_0)^h_{\text{min}}, t_0 + \delta\, h) \\
&= u_t((x_0)^h_{\text{min}}, \tau)(-\delta\, h) \\
&\le m_1\, \delta\, h.
\end{aligned}$$

We now recall the notion of normalization of the section $Q_h(x_0, t_0)$. Consider $S_h(x_0|t_0)$ given by (2.2), and let T be the affine transformation that normalizes and define the transformation

$$T_p(x,t) = \left(Tx, \frac{t-t_0}{h}\right),$$

with its corresponding inverse

$$T_p^{-1}(y,s) = \left(T^{-1}y, t_0 + s\,h\right).$$

We let

$$v(y,s) = u(T_p^{-1}(y,s)) = u(T^{-1}y, t_0 + s\,h).$$

If $\bar{\ell}_{(Tx_0,0)}(y)$ is a supporting affine function to $v(\cdot,0)$ at $y = Tx_0$, then we have

$$\bar{\ell}_{(Tx_0,0)}(y) = v(Tx_0,0) + Dv(Tx_0,0)\cdot(y - Tx_0).$$

This follows from the fact that $Dv(y,s) = (T^{-1})^t(Du)(T^{-1}y, t_0 + s\,h)$. If we denote

$$Q_h(u;(x_0,t_0)) = \{(x,t) : u(x,t) \le \ell_{z_0}(x) + h \text{ and } t \le t_0\},$$

then we have the following formula

$$T_p\left(Q_h(u;(x_0,t_0))\right) = Q_h(v;(Tx_0,0)).$$

Lemma 4.2. *There exists $\delta > 0$ sufficiently small depending only on m_1, the lower bound of u_t, that is $u_t \ge -m_1$, such that*

(1) *If $h \le H$ then $Q_h^*(z_0) \subset Q_H^*(z_0)$.*
(2) *$Q_{h/2}(z_0) \subset Q_h^*(z_0)$.*

Proof. We begin with 1. Let $(x,t) \in Q_h^*(z_0)$. Then $u(x,t) \le \ell_{z_0}^*(x) + h = \ell_{z_0}(x) + \Delta + h$ and $t \le t_0 + \delta\,h$. Hence

$$\begin{aligned}
u(x,t) - \ell_{z_0}(x) &\le \Delta + h \\
&= u((x_0)^h_{\min}, t_0 + \delta\,h) - \ell_{z_0}((x_0)^h_{\min}) + h \\
&\le u((x_0)^H_{\min}, t_0 + \delta\,h) - \ell_{z_0}((x_0)^H_{\min}) + h \\
&= u((x_0)^H_{\min}, t_0 + \delta\,h) - u((x_0)^H_{\min}, t_0 + \delta\,H) \\
&\qquad + u((x_0)^H_{\min}, t_0 + \delta\,H) - \ell_{z_0}((x_0)^H_{\min}) + h \\
&= u_t((x_0)^H_{\min}, \tau)\,(h - H)\,\delta + h + u((x_0)^H_{\min}, t_0 + \delta\,H) - \ell_{z_0}((x_0)^H_{\min}).
\end{aligned}$$

Now $-m_1 \le u_t \le -m_2$ and $h - H < 0$. Then $-m_1\,(h-H) \ge u_t\,(h-H) \ge -m_2\,(h-H)$. Using this in the previous chain of inequalities yields

$$u(x,t) - \ell_{z_0}(x) \le -m_1\,(h-H)\,\delta + h + u((x_0)^H_{\min}, t_0 + \delta\,H) - \ell_{z_0}((x_0)^H_{\min}).$$

If we pick $\delta \le \dfrac{1}{m_1}$ then $-m_1(h-H)\delta + h \le H$ and we obtain

$$u(x,t) - \ell_{z_0}(x) \le u((x_0)^H_{\min}, t_0 + \delta\,H) - \ell_{z_0}((x_0)^H_{\min}) + H.$$

Therefore

$$u(x,t) \le \ell_{z_0}(x) + u((x_0)^H_{\min}, t_0 + \delta\,H) - \ell_{z_0}((x_0)^H_{\min}) + H = \ell^{**}_{z_0}(x) + H,$$

We now prove 2. Let $(x,t)\in Q_{h/2}(z_0)$. We have $t\le t_0$ and

$$
\begin{aligned}
&u(x,t)-\ell_{z_0}(x)\le \frac{h}{2}\\
&=(u-\ell_{z_0})((x_0)^h_{\min},t_0+\delta\, h)-(u-\ell_{z_0})((x_0)^h_{\min},t_0+\delta\, h)+\\
&\qquad (u-\ell_{z_0})((x_0)^h_{\min},t_0)-(u-\ell_{z_0})((x_0)^h_{\min},t_0)+\frac{h}{2}\\
&\le (u-\ell_{z_0})((x_0)^h_{\min},t_0+\delta\, h)\\
&\qquad -\Big((u-\ell_{z_0})((x_0)^h_{\min},t_0+\delta\, h)-(u-\ell_{z_0})((x_0)^h_{\min},t_0)\Big)+\frac{h}{2}\\
&=(u-\ell_{z_0})((x_0)^h_{\min},t_0+\delta\, h)-\Big(u_t((x_0)^h_{\min},\tau)\,\delta\, h\Big)+\frac{h}{2}\\
&\le (u-\ell_{z_0})((x_0)^h_{\min},t_0+\delta\, h)+m_1\,\delta\, h+\frac{h}{2}.
\end{aligned}
$$

If we now pick δ so that $m_1\,\delta<\dfrac{1}{2}$ we are done. □

4.1. Engulfing property for parabolic sections. We now prove the engulfing property for the sections $Q^*_h(z_0)$.

Lemma 4.3 (Engulfing property). *There exists a constant $\theta>1$ such that for each $z_1\in Q^*_h(z_0)$ we have $Q^*_h(z_0)\subset Q^*_{\theta\, h}(z_1)$.*

Proof. Let $z_1=(x_1,t_1)\in Q^*_h(z_0)$, with $z_0=(x_0,t_0)$. Let $\ell_{z_0}(x)$ be a supporting hyperplane to $u(\cdot,t_0)$ at $x=x_0$, and $(x_0)^h_{\min}$ the point at which the minimum of $u(x,t_0+\delta\, h)-\ell_{z_0}(x)$ is attained. We let

$$\ell^*_{z_0}(x)=\ell_{z_0}(x)+u((x_0)^h_{\min},t_0+\delta\, h)-\ell_{z_0}((x_0)^h_{\min}).$$

We have

$$Q^*_h(z_0)=\{(x,t): u(x,t)\le \ell^*_{z_0}(x)+h \text{ and } t\le t_0+\delta h\},$$

and

$$Q^*_h(z_0)=Q_h(u;((x_0)^h_{\min},t_0+\delta h)).$$

Let

$$T_p(x,t)=\left(Tx,\frac{t-t_0}{h}\right),$$

and

$$v(y,s)=\frac{1}{h}(u-\ell^*_{z_0})(T^{-1}y,t_0+sh).$$

We have

$$T_p(Q^*_h(z_0))=\{(y,s): v(y,s)\le 1 \text{ and } s\le \delta\}.$$

Consider $\ell_{z_1}(x)$ a supporting hyperplane to $u(\cdot,t_1)$ at $x=x_1$, and $(x_1)^{\theta h}_{\min}$ the point at which the minimum of $u(x,t_1+\delta\,\theta\, h)-\ell_{z_1}(x)$ is attained. We let

We have

$$
\begin{aligned}
u(x,t) - \ell^*_{z_1}(x) &= u(x,t) - \ell^*_{z_0}(x) + \ell^*_{z_0}(x) - \ell^*_{z_1}(x)\\
&= h\, v(Tx, \frac{t-t_0}{h}) + \ell^*_{z_0}(x) - \ell^*_{z_1}(x).
\end{aligned}
$$

We have

$$Q^*_{\theta h}(x_1,t_1) = \{(x,t) : u(x,t) \leq \ell^*_{z_1}(x) + \theta h \text{ and } t \leq t_1 + \delta\theta h\},$$

and

$$T_p(Q^*_{\theta h}(x_1,t_1)) = \{(y,s) : v(y,s) + \frac{1}{h}\{\ell^*_{z_0}(x) - \ell^*_{z_1}(x)\} \leq \theta \text{ and } s \leq \frac{t_1-t_0}{h} + \delta h\}.$$

We want to prove that

$$Q^*_h(z_0) \subset Q^*_{\theta h}(z_1), \tag{4.3}$$

for $z_1 \in Q^*_h(z_0)$. The inequality (4.3) is equivalent to

$$T_p(Q^*_h(z_0)) \subset T_p(Q^*_{\theta h}(z_1)), \tag{4.4}$$

with $T_p(z_1) \in T_p(Q^*_h(z_0))$. To show (4.4), let $(y,s) \in T_p(Q^*_h(z_0))$. We have

$$v(y,s) + \frac{1}{h}\{\ell^*_{z_0}(T^{-1}y) - \ell^*_{z_1}(T^{-1}y)\} \leq 1 + \frac{1}{h}\{\ell^*_{z_0}(T^{-1}y) - \ell^*_{z_1}(T^{-1}y)\},$$

and we shall estimate

$$\frac{1}{h}\{\ell^*_{z_0}(T^{-1}y) - \ell^*_{z_1}(T^{-1}y)\}.$$

We write

$$
\begin{aligned}
&\ell^*_{z_0}(T^{-1}y) - \ell^*_{z_1}(T^{-1}y)\\
&= \ell_{z_0}(T^{-1}y) - \ell_{z_1}(T^{-1}y) + u((x_0)^h_{\min}, t_0 + \delta\, h) - \ell_{z_0}((x_0)^h_{\min})\\
&\qquad - \{u((x_1)^{\theta h}_{\min}, t_1 + \delta\,\theta\, h) - \ell_{z_1}((x_1)^{\theta h}_{\min})\}\\
&= A + B - C.
\end{aligned}
$$

To estimate C, we set

$$
\begin{aligned}
0 \geq C &= u((x_1)^{\theta h}_{\min}, t_1 + \delta\,\theta\, h) - \ell_{z_1}((x_1)^{\theta h}_{\min})\\
&= u((x_1)^{\theta h}_{\min}, t_1 + \delta\,\theta\, h) - \ell_{z_1}((x_1)^{\theta h}_{\min}) - (u((x_1)^{\theta h}_{\min}, t_1) + u((x_1)^{\theta h}_{\min}, t_1)\\
&= u_t((x_1)^{\theta h}_{\min}, \tau)\,\delta\,\theta\, h + u((x_1)^{\theta h}_{\min}, t_1) - \ell_{z_1}((x_1)^{\theta h}_{\min}).
\end{aligned}
$$

Since $u_t \geq -m_1$, we obtain

Therefore

$$\begin{aligned}
&\ell^*_{z_0}(T^{-1}y) - \ell^*_{z_1}(T^{-1}y)\\
&\leq \ell_{z_0}(T^{-1}y) - \ell_{z_1}(T^{-1}y) + u((x_0)^h_{\min}, t_0 + \delta\, h) - \ell_{z_0}((x_0)^h_{\min})\\
&\qquad + m_1\,\delta\,\theta\,h - u((x_1)^{\theta h}_{\min}, t_1) + \ell_{z_1}((x_1)^{\theta h}_{\min})\\
&= \ell_{z_0}(T^{-1}y) - \ell_{z_0}((x_0)^h_{\min}) + \ell_{z_1}((x_1)^{\theta h}_{\min}) - \ell_{z_1}(T^{-1}y)\\
&\qquad + u((x_0)^h_{\min}, t_0 + \delta\, h) - u((x_1)^{\theta h}_{\min}, t_1) + m_1\,\delta\,\theta\,h\\
&\leq \ell_{z_0}(T^{-1}y) - \ell_{z_0}((x_0)^h_{\min}) + \ell_{z_1}((x_1)^{\theta h}_{\min}) - \ell_{z_1}(T^{-1}y)\\
&\qquad + u((x_0)^h_{\min}, t_1) - u((x_1)^{\theta h}_{\min}, t_1) + m_1\,\delta\,\theta\,h\\
&= A' + B' + C' + m_1\,\delta\,\theta\,h,
\end{aligned}$$

since $u((x_0)^h_{\min}, t_0 + \delta\, h) \leq u((x_0)^h_{\min}, t_1)$ because $t_1 \leq t_0 + \delta\, h$.

Now

$$\begin{aligned}
A' &= \ell_{z_0}(T^{-1}y) - \ell_{z_0}((x_0)^h_{\min}) = Du(x_0, t_0)\cdot(T^{-1}y - (x_0)^h_{\min})\\
B' &= \ell_{z_1}((x_1)^{\theta h}_{\min}) - \ell_{z_1}(T^{-1}y) = Du(x_1, t_1)\cdot((x_1)^{\theta h}_{\min} - T^{-1}y)\\
C' &= u((x_0)^h_{\min}, t_1) - u((x_1)^{\theta h}_{\min}, t_1) = Du(\xi, t_1)\cdot((x_0)^h_{\min} - (x_1)^{\theta h}_{\min}).
\end{aligned}$$

If at the beginning of the proof we subtract from u the supporting hyperplane $\ell_{z_0}(x)$ then we may assume that $Du(x_0, t_0) = 0$. By definition of v we have

$$u(x,t) = \ell^*_{z_0}(x) + h\, v(Tx, \frac{t - t_0}{h}),$$

and consequently

$$Du(x,t) = h\, T^t Dv(Tx, \frac{t - t_0}{h}).$$

Hence $(T^{-1})^t Du(x,t) = h\, Dv(Tx, \frac{t - t_0}{h})$, and we have

$$\begin{aligned}
A' &= 0\\
B' &= Du(x_1, t_1)\cdot T^{-1}(T((x_1)^{\theta h}_{\min}) - y) = (T^{-1})^t Du(x_1, t_1)\cdot(T((x_1)^{\theta h}_{\min}) - y)\\
&= h\, Dv(Tx_1, \frac{t_1 - t_0}{h})\cdot(T((x_1)^{\theta h}_{\min}) - y)\\
C' &= Du(\xi, t_1)\cdot((x_0)^h_{\min} - (x_1)^{\theta h}_{\min}) = h\, Dv(T\xi, \frac{t_1 - t_0}{h})\cdot(T((x_0)^h_{\min}) - T((x_1)^{\theta h}_{\min})).
\end{aligned}$$

We now pick M sufficiently large and T the affine transformation that normalizes the section $S_{Mh}(x_0|t_0)$. By using the properties of elliptic sections and the convexity of v we can bound Dv, see [**GH00**, Lemma 1.1], and we obtain that $B' \leq L\,h$ and $C' \leq L\,h$. Therefore we get

$$\ell^*_{z_0}(T^{-1}y) - \ell^*_{z_1}(T^{-1}y) \leq 2L\,h + m_1\,\delta\,\theta\,h.$$

If we pick δ so that $m_1\,\delta \leq 1/2$ and next θ such that $2L + \frac{1}{2}\theta \leq \theta$, we then obtain

4.2. Engulfing property at different times. We recall that

$$Q_h(u;z_0)=\{(x,t): u(x,t)\le u(x_0,t_0)+Du(x_0,t_0)\cdot(x-x_0)+h, \qquad t\le t_0\}$$

and

$$S_h(x_0|t_0)=\{x: u(x,t_0)\le u(x_0,t_0)+Du(x_0,t_0)\cdot(x-x_0)+h\}.$$

Consider $S_{2h}(x_0|t_0)$ and let T be the affine transformation normalizing $S_{2h}(x_0|t_0)$, and

$$T_p(x,t)=\left(Tx,\frac{t-t_0}{h}\right).$$

Let $\ell_{z_0}(x)=u(x_0,t_0)+Du(x_0,t_0)\cdot(x-x_0)$ and the function

$$v(y,s)=\frac{1}{h}(u-\ell_{z_0})(T^{-1}y,t_0+s\,h).$$

Then

$$T_p\left(Q_{2h}(u;z_0)\right)=Q_2(v;(Tx_0,0))$$

is normalized. We have $\min_{Q_2} v=v(Tx_0,0)=0$, $v=2$ on $\partial_p Q$ and $|v_t|\approx$ *constant*.

Let $z_1=(x_1,t_1)\in Q_1(v;(Tx_0,0))$, then

$$|Dv(x_1,t_1)|\le\frac{2}{\operatorname{dist}((x_1,t_1),\partial Q_2(t_1))}\le C$$

by properties of the elliptic sections. We claim that

$$S_1(x_1|t_1)\subset S_\theta(x_0|t_0).$$

Indeed, if $x\in S_1(x_1|t_1)$ then $v(x,t_1)\le v(x_1,t_1)+Dv(x_1,t_1)\cdot(x-x_1)+1\le 2+C=\theta$. Since $t_1<t_0$ and $v(x,\cdot)$ is monotonic in t, we have $v(x,t_0)\le\theta$ and hence $x\in S_\theta(x_0|t_0)$ because the supporting hyperplane defining $S_\theta(x_0|t_0)$ equals zero. On the other hand,

$$S_1(x_0|t_0)\subset S_\theta(x_1|t_1).$$

Because, if $x\in S_1(x_0|t_0)$ then $v(x,t_0)<1$ and since $|v_t|\approx$ *constant*, we have $v(x,t_1)<C$. Now $|\ell_{z_1}(x)|=|v(x_1,t_1)+Dv(x_1,t_1)\cdot(x-x_1)|\le C_1$. Hence $(u-\ell_{z_1})(x,t_1)\le C+C_1=\theta$ and so $x\in S_\theta(x_1|t_1)$.

LEMMA 4.4. *Let u be such that $-u_t\approx$ constant and $-u_t\det D^2u\approx$ constant. Suppose that $z_1=(x_1,t_1)\in Q_h(u;z_0)$. Then*

(1) $S_h(x_1|t_1)\subset S_{\theta h}(x_0|t_0)$.
(2) $S_h(x_0|t_0)\subset S_{\theta h}(x_1|t_1)$.

PROOF. Normalize the section $Q_{2h}(u;z_0)$ as before.

We have $T_p(x,t)=\left(Tx,\dfrac{t-t_0}{h}\right)=(x^*,t^*)$; $u^*(x^*,t^*)=\dfrac{1}{h}(u-\ell_{z_0})(T_p^{-1}(x^*,t^*))$; $Q_2^*(z_0^*)=T_pQ_{2h}(z_0)$, and $Q_1^*(z_0^*)=T_pQ_h(z_0)$. We have $TS_h(x_1|t_1)=S_1^*(x_1^*|t_1^*)$. By the inclusions previously proved we have

$$S_1^*(x_1^*|t_1^*)\subset S_\theta^*(x_0^*,x_0^*)$$

and

$$S_1^*(x_0^*|t_0^*)\subset S_\theta^*(x_1^*,x_1^*).$$

Taking T^{-1} in these inclusions yields the lemma. $\square$

4.3. Shrinking property of parabolic sections.

PROPOSITION 4.5. *Let Q be a normalized bowl-shaped domain in $\mathbb{R}^{n+1}$, and u a parabolically convex function in Q satisfying:*

$$\lambda \le -u_t \det D^2 u \le \Lambda \qquad \text{in } Q,$$
$$u \ge 0 \qquad \text{in } Q,$$
$$\min_Q u = 0,$$
$$-m_1 \le u_t \le -m_2 \qquad \text{in } Q,$$
$$\text{and } u = 1 \text{ on } \partial_p Q..$$

Then

(1) *If $u(z_0) < 1 - \epsilon$ then dist$(z_0, \partial_p Q) \ge C\,\epsilon^{n+1}$ where C is a constant depending only on n, m_1 and Λ.*
(2) *If dist$(z_0, \partial_p Q) \ge \epsilon$ then $u(z_0) \le 1 - m_2\,\epsilon$.*

PROOF. We begin with the proof of 1. Since u is parabolically convex and $u = 1$ on $\partial_p Q$, we have that $u \le 1$ in Q. Let $v(x,t) = u(x,t) - 1$. We have that $\lambda \le -v_t \det D^2 v \le \Lambda$, $\min_Q v = -1$ and $v = 0$ on $\partial_p Q$. By application of Theorem 2.1 from [**GH98**] to the function v we get

$$|v(x_0,t_0)|^{n+1} \le C(n,\Lambda)\,\mathrm{dist}(x_0, \partial(Q \cap \{t_0\})),$$

and since $v(x_0,t_0) \le -\epsilon$, we obtain

$$\mathrm{dist}(x_0, \partial(Q \cap \{t_0\})) \ge C(n,\Lambda)\,\epsilon^{n+1}.$$

Since Q is bowl-shaped, we have $Q \cap \{t_0\} \subset Q \cap \{t\}$ for $t \ge t_0$, and consequently

$$\mathrm{dist}(x_0, \partial(Q \cap \{t\})) \ge C(n,\Lambda)\,\epsilon^{n+1}, \qquad \text{for } t \ge t_0.$$

Let us now assume that $t < t_0$. We write $u(x_0,t) - u(x_0,t_0) = u_t(x_0,\tau)(t - t_0)$, and by the estimates for u_t we get $u(x_0,t) \le u(x_0,t_0) + m_1\,(t_0 - t)$ for $t < t_0$. If in addition $t_0 - c_1\,\epsilon \le t \le t_0$ then $0 \le t_0 - t \le c_1\,\epsilon$ and we get

$$u(x_0,t) \le u(x_0,t_0) + m_1\,c_1\,\epsilon = 1 - \epsilon + m_1\,c_1\,\epsilon = 1 - \frac{\epsilon}{2},$$

by picking $c_1 = \dfrac{1}{2\,m_1}$. Hence

$$\mathrm{dist}(x_0, \partial(Q \cap \{t\})) \ge C(n,\Lambda)\,\epsilon^{n+1}, \qquad \text{for } t_0 - c_1\,\epsilon \le t \le t_0.$$

Let now $z_1 = (x_1,t_1) \in \partial_p Q$ such that

$$\mathrm{dist}(z_0, \partial_p Q) = |z_0 - z_1|.$$

We have $\mathrm{dist}(z_0, \partial_p Q) \ge |x_0 - x_1|$. If $t_1 > t_0$ then

$$|x_0 - x_1| \ge \mathrm{dist}(x_0, \partial(Q \cap \{t_1\})) \ge C\,\epsilon^{n+1}.$$

Also, if $t_0 - c_1\,\epsilon \le t \le t_0$ then

If $t_1 < t_0 - c_1\,\epsilon$ then $\operatorname{dist}(z_0, \partial_p Q) \geq t_0 - t_1 \geq c_1\,\epsilon$. Therefore in any case we obtain the inequality

$$\operatorname{dist}(z_0, \partial_p Q) \geq C(n, m_1, \Lambda)\,\epsilon^{n+1}.$$

We now prove 2. Let t_{bdy} be the time such that $(x_0, t_{\text{bdy}}) \in \partial_p Q$. Now $u(x_0,t_0) - u(x_0, t_{\text{bdy}}) = u_t(x_0,\tau)(t_0 - t_{\text{bdy}})$. We have $t_0 - t_{\text{bdy}} \geq \operatorname{dist}(z_0, \partial_p Q)$ and consequently $u_t(x_0,\tau)(t_0 - t_{\text{bdy}}) \leq u_t(x_0,\tau)\operatorname{dist}(z_0,\partial_p Q) \leq -m_2 \operatorname{dist}(z_0,\partial_p Q)$. Hence

$$u(x_0,t_0) \leq u(x_0,t_{\text{bdy}}) - m_2 \operatorname{dist}(z_0,\partial_p Q) = 1 - m_2 \operatorname{dist}(z_0,\partial_p Q) \leq 1 - m_2\,\epsilon.$$

□

Lemma 4.6. *Let $z \notin Q_h^*(z_0)$ and T_p a parabolically affine transformation normalizing $Q_h^*(z_0)$. Then there exist structural positive constants C and ν so that*

$$T_p(Q^*_{(1-\epsilon)h}(z_0)) \cap K(T_p(z), C\,\epsilon^\nu) = \emptyset,$$

for $0 < \epsilon < 1$, where $K(z,R)$ is the standard parabolic cylinder given by $K(z,R) = B_R(x) \times (t - R^2, t + R^2]$; $z = (x,t)$.

Proof. With the notation of Lemma (4.3) and similarly to the proof of Lemma (4.2) we have

$$\begin{aligned}
&(u - \ell_{z_0})((x)^{(1-\epsilon)h}_{\min}, t_0 + \delta(1-\epsilon)h) + (1-\epsilon)h \\
&\leq (u - \ell_{z_0})((x)^{h}_{\min}, t_0 + \delta(1-\epsilon)h) + (1-\epsilon)h \\
&\leq (u - \ell_{z_0})((x)^{h}_{\min}, t_0 + \delta h) + \sup|u_t|\,\delta\epsilon h + (1-\epsilon)h \\
&\leq (u - \ell_{z_0})((x)^{h}_{\min}, t_0 + \delta h) + (1 - \frac{\epsilon}{2})h.
\end{aligned}$$

Hence

$$Q^*_{(1-\epsilon)h}(z_0) \subset Q_{(1-\frac{\epsilon}{2})h}((x)^h_{\min}, t_0 + \delta h) \cap \{t \leq t_0 + \delta(1-\epsilon)h\}.$$

Let

$$T_p(x,t) = \left(Tx, \frac{t - (t_0 + \delta h)}{h}\right)$$

normalizing $Q_h(z_h)$ with $z_h = ((x)^h_{\min}, t_0 + \delta h)$. Set

$$T_p(Q_h(z_h)) = \hat{Q}_1(0); \qquad T_p(Q_{1-\frac{\epsilon}{2}}(z_h)) = \hat{Q}_{1-\frac{\epsilon}{2}}(0);$$

and

$$\hat{u}(\hat{x},\hat{t}) = \frac{1}{h}\{u(T_p^{-1}(\hat{x},\hat{t}) - \ell_{z_h}(T^{-1}\hat{x})\}.$$

To prove the lemma it is sufficient to show that

$$\operatorname{dist}(\partial\hat{Q}_1(0), \hat{Q}_{1-\frac{\epsilon}{2}}(0) \cap \{t \leq -\epsilon\,\delta\}) \geq C_\delta\,\epsilon^\nu.$$

This follows from Proposition (4.5), item 1, because

$$\operatorname{dist}(\partial_p\hat{Q}_1(0), \hat{Q}_{1-\frac{\epsilon}{2}}(0)) \geq C\,\epsilon^{n+1},$$

and

$$\operatorname{dist}(\{t = 0\}, \{t \leq -\epsilon\,\delta\}) = \epsilon\,\delta.$$

□

4.4. Size of parabolic sections.

LEMMA 4.7. *Let $Q^*_{h_0}(z_0)$ be a section and T_p a transformation that normalizes it. If $h \le h_0$ and $Q^*_h(z') \cap Q^*_{h_0}(z_0) \neq \emptyset$ then $|T_p(z')| \le K$ and*

$$K\left(T_p(z'), C_1\left(\frac{h}{h_0}\right)^{\epsilon_1}\right) \subset T_p(Q^*_h(z')) \subset K\left(T_p(z'), C_2\left(\frac{h}{h_0}\right)^{\epsilon_2}\right),$$

with K, C_1, C_2, ϵ_1 and ϵ_2 positive constants depending only on the structure.

PROOF. We can assume $h_0 = 1$, $Q^*_1(z_0)$ is already normalized and T_p =identity. Applying lemma (4.3) several times we get $Q^*_1(z') \subset Q^*_{\theta/2}(z_0)$. Since $Q^*_1(z_0)$ is normalized, it follows from lemma (4.4) that $Q^*_\theta(z_0)$ is also normalized. We have $|z'| \le K$ and

$$\begin{aligned} Q^*_h(z') &= Q_h((x')^h_{\min}, t' + \delta h) \\ &\subset S_h((x')^h_{\min}|t' + \delta h) \times (t' - ch, t' + \delta h] \\ &\subset B((x')^h_{\min}, C_{2,\theta}\, h^{\epsilon_2}) \times (t' - ch, t' + \delta h] \qquad \text{by [\textbf{GH00}, Theorem 2.3]} \\ &\subset B(x', 2C\, h^{\epsilon_2}) \times (t' - ch, t' + \delta h]. \end{aligned}$$

On the other hand

$$S_{ch}(\hat{x}|t' - \delta h) \times (t' - \delta h, t' + \delta h) \subset Q^*_h(z'),$$

where $S_{ch}(\hat{x}|t' - \delta h)$ is the projection of $Q^*_h(z') \cap \{t' - \delta h\}$ and is a section of $u(\cdot, t' - \delta h)$ at the minimum point $\hat{x}$ of height ch (assume $\ell_{((x')^h_{\min}, t' + \delta h)} = 0$; $c \ge 3/4$ and δ small). Since (x', t') is minimum point at $\{t'\}$, we have

$$\begin{aligned} u(x', t' - \delta h) &\le u(x', t') + \hat{c}\delta h \\ &\le u(\hat{x}, t') + \hat{c}\delta h \\ &\le u(\hat{x}, t' - \delta h) + \hat{c}\delta h. \end{aligned}$$

Hence $x' \in S_{ch/2}(\hat{x}|t' - \delta h)$. Then $S_{\eta h}(x'|t' - \delta h) \subset S_{ch}(\hat{x}|t' - \delta h)$, and therefore

$$\begin{gathered} B(x', Ch^{\epsilon_1}) \times (t' - \delta h, t' + \delta h] \subset \\ S_{\eta h}(x'|t' - \delta h) \times (t' - \delta h, t' + \delta h] \subset Q^*_h(z'). \end{gathered}$$

□

REMARK 4.8. *More precisely, the first inclusion in lemma (4.7) can be written as*

$$K\left(T_p(z'), C_1\left(\frac{h}{h_0}\right)^{\epsilon_1}\right) \cap \{t \le T\} \subset T_p(Q^*_h(z')).$$

4.5. Second size property of sections.

LEMMA 4.9. *Let $Q_1(z_0)$ be an old section normalized. There exist positive constants C and p such that if $0 < r < s < 1$ and $z' \in Q_r(z_0)$ then $Q_h(z') \subset Q_s(z_0)$ for $h \leq C\,(s-r)^p$.*

PROOF. We have

$$\begin{aligned} Q_h(z') &\subset Q^*_{2h}(z') \qquad \text{by (4.2), part 2} \\ &\subset K(z', C\,h^{\epsilon_1}) \qquad \text{by (4.7).} \end{aligned}$$

By the parabolic Aleksandrov maximum principle [**GH98**, Theorem 2.1] we have $|Du(z')| \leq \dfrac{C}{(1-r)^{n+1}}$. If $z \in Q_h(z')$ then

$$\begin{aligned} u(z) &\leq u(z') + Du(z') \cdot (x - x') + h \\ &\leq r + \frac{C}{(1-r)^{n+1}}\,C\,h^{\epsilon_1} + h \qquad (\text{assume } \ell_{z_0} = 0) \\ &\leq s, \end{aligned}$$

when $h \leq \dfrac{1}{2}(s-r)$ and $h^{\epsilon_1} \leq \eta\,(s-r)^{n+2}$. That is $Q_h(z') \subset Q_s(z_0)$. □

4.6. Besicovitch's type covering lemma.

LEMMA 4.10. *Let Q be a parabolic section and $\mathcal{O} \subset Q$. Suppose that for each $z \in \mathcal{O}$ a section $Q^*_r(z)$ is given so that $r \leq M$. Assume that the engulfing and size properties hold (lemmas (4.3), (4.6) and (4.7)). Then we can choose a countable subfamily $\{Q^*_{r_k}(z_k)\}_{k=1}^\infty$ with the following properties:*

(1) $\mathcal{O} \subset \cup_{k=1}^\infty Q^*_{r_k}(z_k)$;
(2) $z_k \notin \cup_{j<k} Q^*_{r_j}(z_j)$ *for* $k \geq 2$;
(3) $\sum_{k=1}^\infty \chi_{Q^*_{(1-\epsilon)r_k}(z_k)}(z) \leq C_0 \log \dfrac{1}{\epsilon}$;

where χ_E denotes the characteristic function of the set E.

PROOF. It is the same as the one given in [**CG96**] and [**Hua99**]. □

4.7. A Calderón-Zygmund type decomposition.

We now give a proposition needed in the proof of the Calderón-Zygmund decomposition.

PROPOSITION 4.11. *There exists a positive constant C depending only on the structure such that*

$$|Q^*_h(z_0) \setminus Q^*_{(1-\epsilon)h}(z_0)| \leq C\,\sqrt{\epsilon}\,|Q^*_h(z_0)|,$$

with $0 < \epsilon < 1$.

PROOF. The inequality is affine invariant. We may assume $h = 1$ and

t in $Q_1^*(0)$. Also let $m(t) = \min_x u(x,t)$ and $x^t_{\min}$ the point where the minimum of $u(\cdot,t)$ is attained. We write

$$\begin{aligned}|Q_1^*(0)\setminus Q_{1-\epsilon}^*(0)| &\le \int_{\delta(1-\epsilon)}^{\delta} |S_{1-m(t)}(x^t_{\min}|t)|\,dt\\ &+\int_{t_{\min}+\sqrt{\epsilon}}^{\delta(1-\epsilon)} |S_{1-m(t)}(x^t_{\min}|t)\setminus S_{1-\epsilon-m(t)+\alpha}(x^t_{\min}|t)|\,dt\\ &+\int_{t_{\min}}^{t_{\min}+\sqrt{\epsilon}} |S_{1-m(t)}(x^t_{\min}|t)|\,dt\\ &= I+II+III,\end{aligned}$$

where

$$Q_1^*(0)\cap\{t\} = S_{1-m(t)}(x^t_{\min}|t)\times\{t\} \qquad \text{and}$$
$$\alpha = u(x^{1-\epsilon}_{\min},(1-\epsilon)\delta) \le u(x^1_{\min},(1-\epsilon)\delta) \le \sup|u_t|\,\epsilon\delta \le \frac{\epsilon}{2}.$$

We have

$$I \le \delta\,\epsilon|S_1(x^1_{\min}|1)| \le C\,\epsilon \le \epsilon\,|Q_1^*(0)|,$$

since $|Q_1^*(0)| \approx$ constant. Also

$$III \le \sqrt{\epsilon}\,|S_{1-m(t')}(x^{t'}_{\min}|t')| \le C\,\sqrt{\epsilon},$$

with $t' = t_{\min}+\sqrt{\epsilon}$. By the elliptic result $|S_h(x_0)\setminus S_{h(1-\epsilon)}(x_0)| \le n\,\epsilon\,|S_h(x_0)|$, and for $t \ge t_{\min}+\sqrt{\epsilon}$ we have $m(t) \le u(\hat{x},t) \le u(\hat{x},t_{\min}) + u_t\,(t-t_{\min}) \le 1 - C\,\sqrt{\epsilon}$. Hence

$$\begin{aligned}II &\le \int_{t_{\min}+\sqrt{\epsilon}}^{\delta(1-\epsilon)} |S_{1-m(t)}(x^t_{\min}|t)\setminus S_{(1-\frac{\epsilon-\alpha}{1-m(t)})(1-m(t))}(x^t_{\min}|t)|\,dt\\ &\le \int_{t_{\min}+\sqrt{\epsilon}}^{\delta(1-\epsilon)} n\,\frac{\epsilon-\alpha}{1-m(t)}|S_{1-m(t)}(x^t_{\min}|t)|\,dt\\ &\le C\,\frac{\epsilon}{c\sqrt{\epsilon}} = C\,\sqrt{\epsilon} \approx C\,\sqrt{\epsilon}\,|Q_1^*(0)|.\end{aligned}$$

□

We conclude this section with the decomposition needed in the proof of the $W^{2,p}$-estimates.

Theorem 4.12. *Let $\mathcal{O}$ be an open subset of a section Q, $0<\delta<1$ small, and $\gamma>0$. Suppose that for each $z\in\mathcal{O}$ a section $Q^*_{h_z}(z)$ is given with $h_z\le\gamma$, and*

$$\frac{|Q^*_{h_z}(z)\cap\mathcal{O}|}{|Q^*_{h_z}(z)|} = \delta.$$

Then there exists a family of parabolic sections $\{Q^*_{h_k}(z_k)\}_{k=1}^{\infty}$ with the following properties

(3) $\dfrac{|Q^*_{h_k}(z_k)\cap \mathcal{O}|}{|Q^*_{h_k}(z_k)|} = \delta.$

(4) $|\mathcal{O}| \leq \sqrt{\delta}\,|\cup_{k=1}^{\infty} Q^*_{h_k}(z_k)|.$

PROOF. It follows combining lemma (4.10) with proposition (4.11) and using the technique in [**CG96**]. □

5. Approximation Theorem

Let $(x_0,t_0)\in Q$, $\sigma>0$, and

$$P_\sigma(x,t)=\sigma(|x-x_0|^2-(t-t_0))+p\cdot(x-x_0)+u(x_0,t_0).$$

If $(x_0,t_0)\in S\subset Q$ then we say that u is touched from below by P_σ in S if $u(x,t)\geq P_\sigma(x,t)$ for all $(x,t)\in S$. Notice that if $S=Q\cap\{t=t_0\}$ then $p=D_xu(x_0,t_0)$.

Let us define the following sets:

$$A_\sigma(u)=\{(x_0,t_0): u \text{ is touched from below by } P_\sigma \text{ in } Q\cap\{t\leq t_0\}\},$$

and

$$A^*_\sigma(u)=\{(x_0,t_0): u(x,t_0) \text{ is touched from below by } P_\sigma(x,t_0) \text{ in } Q\cap\{t=t_0\}\}.$$

Notice that $A_\sigma\subset A^*_\sigma$, and if $u_t\leq-\sigma$ in Q then we also have $A^*_\sigma\subset A_\sigma$.

THEOREM 5.1 (Approximation Theorem). *Let Q be a bowl-shaped domain in $\mathbb{R}^{n+1}$ such that*

$$B_\delta(0)\times(-\delta^2,0]\subset Q\subset B_1(0)\times(-1,0],$$

with $Q=\{(x,t):\Phi(x,t)<0,t\leq 0\}$ with Φ parabolically convex.[1] *Suppose that $0<\epsilon<1/2$, and u a parabolically convex function in Q classical solution of*

$$(1-\epsilon)^{n+1}\leq\mathcal{M}u\leq(1+\epsilon)^{n+1},\qquad \text{in } Q \tag{5.1}$$

$$u=0,\qquad \text{on } \partial_pQ, \tag{5.2}$$

and

$$-m_1\leq u_t\leq -m_2,\qquad \text{in } Q, \tag{5.3}$$

where m_1 and m_2 are positive constants.

Let $0<\alpha<1$ and set

$$Q_\alpha=\{(x,t)\in Q: u(x,t)<(1-\alpha)\min_Q u\}.$$

Then there exist positive constants $\sigma>0$ and C_n depending only on the dimension n and α, and both independent of Q (depending only on the time derivatives of the function Φ defining Q), u and ϵ such that

$$|Q_\alpha\setminus A_\sigma|<C_n\,\epsilon\,|Q_\alpha|.$$

PROOF. By [**GH98**, Lemma 2.1] have that

$$-\min_Q u \approx C.$$

Let w be a parabolically convex solution of

$$(5.4)\qquad \begin{aligned} -w_t \det D^2 w &= 1 && \text{in } Q \\ w &= 0 && \text{on } \partial_p Q. \end{aligned}$$

We have that $w \in C(\overline{Q}) \cap C^\infty(Q)$. We use the following:

Comparison Principle: If $\mathcal{M}v \geq \mathcal{M}u$ in Q then

$$u(x,t) - v(x,t) \geq \min_{\partial_p Q}\{u(x,t) - v(x,t)\}, \qquad \text{for all } (x,t) \in Q,$$

see [**WW93**, Proposition 2.3].

In our case we have

$$\begin{aligned} \mathcal{M}((1+\epsilon)w) &= (1+\epsilon)^{n+1}\mathcal{M}w = (1+\epsilon)^{n+1} \geq \mathcal{M}u \\ &\geq (1-\epsilon)^{n+1} = (1-\epsilon)^{n+1}\mathcal{M}w = \mathcal{M}((1-\epsilon)w)). \end{aligned}$$

So we get the estimate

$$(5.5)\qquad (1+\epsilon)w \leq u \leq (1-\epsilon)w \qquad \text{in } Q.$$

Thus

$$(5.6)\qquad \left(\frac{1}{2}+\epsilon\right) w \leq u - \frac{w}{2} \leq \left(\frac{1}{2}-\epsilon\right) w \qquad \text{in } Q.$$

We have $w < 0$ in Q, so

$$\left(\frac{1}{2}+\epsilon\right)(-w) \geq |u - \frac{w}{2}| \geq \left(\frac{1}{2}-\epsilon\right)(-w) \qquad \text{in } Q.$$

Since Q is normalized, it follows again by [**GH98**, Lemma 2.1] that $|\min_Q w| \approx 1$ and consequently

$$\max_Q |u - \frac{w}{2}| \approx 1.$$

Let $\Gamma(x,t)$ be the parabolic convex envelope of $u - \dfrac{w}{2}$ in Q, see definition (7.1).

Claim 1. For all $(x,t) \in Q$ we have the inequality

$$|\frac{w(x,t)}{2} - \Gamma(x,t)| \leq C_n \epsilon.$$

Indeed, by (5.6) and the fact that w is parabolically convex we have

$$\left(\frac{1}{2}+\epsilon\right) w(x,t) \leq \Gamma(x,t) \leq \left(\frac{1}{2}-\epsilon\right) w(x,t) \qquad \text{in } Q,$$

which yields

$$\epsilon w(x,t) \leq \Gamma(x,t) - \frac{w(x,t)}{2} \leq -\epsilon w(x,t),$$

Claim 2. We need the following. Let D be a bowl-shaped open and bounded domain, $u, v \in C(\bar{D})$ parabolically convex, $u = v$ on $\partial_p D$, and $v \le u$ in D. Then

$$\mathcal{P}_u(D) \subset \mathcal{P}_v(D), \qquad a.e,$$

that is $\mathcal{P}_u(D) \setminus E \subset \mathcal{P}_v(D)$ for some $|E| = 0$. Indeed, let $(p, h) \in \mathcal{P}_u(D)$. Then $\ell(x) = u(x_0, t_0) + p \cdot (x - x_0) \le u(x, t)$ for all $t \le t_0$ and $x \in D(t)$, with $(x_0, t_0) \in D$. Slide ℓ in a parallel fashion in the direction of t negative until it touches for the last time the graph of v. Say ℓ touches v at (x_1, t_1), $t_1 \le t_0$. Then $\ell(x_1) = v(x_1, t_1) = u(x_0, t_0) + p \cdot (x_1 - x_0)$ and so $p \cdot x_1 - v(x_1, t_1) = p \cdot x_0 - u(x_0, t_0) = h$. If $(x_1, t_1) \in \partial_p D$ then $v(x_1, t_1) = u(x_1, t_1) = p \cdot x_1 - h$. Since $u(x, t) \ge u(x_1, t_1) + p \cdot (x - x_1)$ for all $t \le t_0$ and $x \in D(t)$, it follows that $(p, h) \in P_u(x_1, t_1)$. That is $(p, h) \in P_u(x_1, t_1) \cap P_u(x_0, t_0)$, but if $(x_0, t_0) \ne (x_1, t_1)$ then this set of (p, h) has measure zero and the claim follows.

Since $w = 0$ on $\partial_p Q$, it follows that

$$\mathcal{P}_{(\frac{1}{2}-\epsilon)w}(Q) \subset \mathcal{P}_\Gamma(Q) \subset \mathcal{P}_{(\frac{1}{2}+\epsilon)w}(Q), \qquad a.e.$$

By the results in section (7), Corollary (7.7),

$$\mathcal{M}\Gamma \le \mathcal{M}(u - \frac{w}{2})\chi_{\mathcal{C}}, \tag{5.7}$$

where

$$\mathcal{C} = \{(x, t) \in Q : \Gamma(x, t) = u(x, t) - \frac{w(x, t)}{2}\}.$$

If $(x_0, t_0) \in \mathcal{C}$ then $u - \dfrac{w}{2} - \Gamma$ attains its minimum 0 at the point (x_0, t_0) and hence

$$D_x^2 \left(u - \frac{w}{2}\right)(x_0, t_0) \ge 0, \qquad \text{and} \qquad \left(u - \frac{w}{2}\right)_t (x_0, t_0) \le 0. \tag{5.8}$$

On the other hand, if $a \ge 0$ and $A \ge 0$ is an $n \times n$ symmetric matrix then

$$\begin{aligned}&(a \det A)^{1/(n+1)} \\ &= \frac{1}{n+1} \inf \{\mathrm{trace}(B\, A) + b\, a : b > 0,\ B > 0 \text{ symmetric with } b \det B = 1\}.\end{aligned}$$

Thus

$$((a_1 + a_2) \det(A_1 + A_2))^{1/(n+1)} \ge (a_1 \det A_1)^{1/(n+1)} + (a_2 \det A_2)^{1/(n+1)},$$

for $a_i \ge 0$ and $A_i \ge 0$ symmetric, $i = 1, 2$. Since $\dfrac{w}{2}$ is parabolically convex, it follows using the previous inequality and (5.8) that

$$\{\mathcal{M}u(x, t)\}^{1/(n+1)} \ge \left\{\mathcal{M}\left(u - \frac{w}{2}\right)(x, t)\right\}^{1/(n+1)} + \left\{\mathcal{M}\left(\frac{w}{2}\right)(x, t)\right\}^{1/(n+1)},$$

for $(x, t) \in \mathcal{C}$. Consequently

$$\left\{\mathcal{M}\left(u - \frac{w}{2}\right)(x_0, t_0)\right\}^{1/(n+1)} \le \{\mathcal{M}u(x_0, t_0)\}^{1/(n+1)} - \left\{\mathcal{M}\left(\frac{w}{2}\right)(x_0, t_0)\right\}^{1/(n+1)}$$

an inequality valid for a.e. point in $\mathcal{C}$. By claim 2

$$|\mathcal{P}_{(\frac{1}{2}-\epsilon)w}(Q)| \leq |\mathcal{P}_\Gamma(Q)|.$$

Therefore

$$\left(\frac{1}{2}-\epsilon\right)^{n+1} \int_Q \mathcal{M}w(x,t)\,dxdt \leq \int_{\mathcal{C}} \mathcal{M}\Gamma(x,t)\,dxdt \leq \left(\frac{1}{2}+\epsilon\right)^{n+1} |\mathcal{C}|,$$

by (5.7) and (5.9). This yields the estimate

$$|\mathcal{C}| \geq \left(\frac{\frac{1}{2}-\epsilon}{\frac{1}{2}+\epsilon}\right)^n |Q| \geq (1-C_n\epsilon)|Q|,$$

which implies

$$|Q \setminus \mathcal{C}| \leq C_n\,\epsilon\,|Q|. \tag{5.10}$$

We now prove that there exists a universal constant $\sigma > 0$ so that

$$Q_\alpha \cap \mathcal{C} \subset A_\sigma.$$

We recall the following variant of a theorem due to Pogorelov, see [**GH98**, Theorem 2.2].

Theorem 5.2. *Let $D \subset \mathbb{R}^{n+1}$ be a bounded open bowl-shaped domain and $v \in C(\overline{D})$ such that v is parabolically convex in D. Suppose that v is a smooth solution of*

$$-v_t \det D^2 v = 1, \qquad \text{in } \overline{D} \setminus \partial_p D \tag{5.11}$$

$$v(x,t) = 0, \qquad \text{for } (x,t) \in \partial_p D. \tag{5.12}$$

Let $\alpha \in \mathbb{R}^n$, $|\alpha| = 1$,

$$w(x,t) = |v(x,t)| D_{\alpha\alpha} v(x,t) e^{\frac{1}{2}(D_\alpha v(x,t))^2},$$

and $M = \max_{\overline{D}} w(x,t)$. Then there exists $P \in \overline{D} \setminus \partial_p D$ where the maximum M is attained and the following inequality holds

$$M \leq C_n\ (1 + |D_\alpha v(P)|)\, e^{\frac{1}{2}(D_\alpha v(P))^2},$$

with C_n a positive constant depending only on the dimension n.

We apply this theorem to w. Let $\delta > 0$ and

$$Q_\delta(w) = \{(x,t) \in Q : w(x,t) < -\delta\}.$$

Notice that since $0 < \epsilon < 1/2$, it follows from (5.5) that $\frac{3}{2}w \leq u \leq \frac{1}{2}w$ and consequently

$$Q_\delta(u) \subset Q_{2\delta/3}(w) \qquad \text{and} \qquad Q_\delta(w) \subset Q_{\delta/2}(u). \tag{5.13}$$

Arguing as in the proof of (3-7) and (3-8) of [**GH98**], we obtain

$$|D_x w(x,t)| \leq C(\delta), \qquad \text{for } (x,t) \in Q_\delta(w),$$

which implies that

on the same set and for all $|\alpha| = 1$. Thus

$$D_x^2 w(x,t) \le M_\delta Id, \qquad \forall (x,t) \in Q_\delta(w). \tag{5.14}$$

This estimate used together with the equation yields the following upper bound for the time derivative of w

$$w_t(x,t) \le -C(\delta) \tag{5.15}$$

for $(x,t) \in Q_\delta(w)$, with $C(\delta) > 0$. To obtain the lower estimate of w_t we invoke [**WW92**, Lemma 3.3] (notice that this estimate depends on the time derivative of the defining function Φ). Thus from (5.14) and the equation (5.4) we obtain

$$D_x^2 w(x,t) \ge M'_\delta Id, \qquad \forall (x,t) \in Q_\delta(w). \tag{5.16}$$

Consequently, if $(x_0,t_0) \in Q_\delta(w)$ by the convexity of w we then obtain the estimate

$$w(x,t_0) \ge w(x_0,t_0) + D_x w(x_0,t_0)\cdot(x-x_0) + m\,|x-x_0|^2,$$

for all $(x,t_0) \in Q$ with m a positive constant depending only on n and δ (here we use the Taylor polynomial of second order of $w(\cdot,t_0)$ with the remainder written in integral form and the convexity of $w(.,t_0)$ together with (5.16) to obtain the inequality valid in all $Q(t_0)$).

Recall that $\Gamma(x,t) \le u(x,t) - \dfrac{w(x,t)}{2}$ for all $(x,t) \in Q$. Since $\Gamma(x,t_0)$ is convex, let ℓ_{x_0} be a supporting hyperplane to $\Gamma(x,t_0)$ at $x = x_0$. Then

$$\begin{aligned} u(x,t_0) &\ge \ell_{x_0}(x) + \frac{w(x,t_0)}{2} \\ &\ge \ell_{x_0}(x) + \frac{1}{2}\left(w(x_0,t_0) + D_x w(x_0,t_0)\cdot(x-x_0) + m\,|x-x_0|^2\right) \\ &\ge \ell(x) + \frac{1}{2} m\,|x-x_0|^2 \\ &= P_{m/2}(x,t_0), \end{aligned}$$

for all $x \in Q(t_0)$. If $(x_0,t_0) \in \mathcal{C}$ then $\Gamma(x_0,t_0) = u(x_0,t_0) - \dfrac{w(x_0,t_0)}{2}$. Hence $u(x_0,t_0) = P_{m/2}(x_0,t_0)$, that is $P_{m/2}(x,t_0)$ touches $u(x,t_0)$ from below in $Q(t_0)$. Since $u_t \le -m_2$ in Q, if we take $\sigma = \min\{m_2, \frac{1}{2}m\}$ then $u_t \le -\sigma$. Hence $(x_0,t_0) \in A^*_\sigma \subset A_\sigma$, that is, $Q_\delta(w) \cap \mathcal{C} \subset A_\sigma$.

Now taking into account (5.13) and choosing δ so that $\dfrac{3\delta}{2} \approx -(1-\alpha)\min_Q u$ we obtain

$$Q_\alpha \setminus A_\sigma \subset Q \setminus \mathcal{C}.$$

Then by (5.10) and since $|Q_\alpha| \approx |Q|$ we obtain the theorem. □

6. $W^{2,p}$ estimates

We prove here L^p estimates of the second derivatives in x of solutions to $\mathcal{M}u = f$. This done in several steps. We first establish a strict convexity result, lemma (6.1). Second, we prove a density result, proposition (6.3), for which the approximation theorem (5.1) is used. This density result combined with the decomposition theorem (4.12) and once more the approximation theorem (5.1), yields the power decay, proposition (6.4). Once this is done, we obtain $W^{2,p}$ estimates on parabolic sections, theorem (6.5), that is with zero boundary data. Next, we use an strict convexity result due to Caffarelli, theorem (6.6), which coupled with theorem (3.2) yields a the strict convexity result in the parabolic case, theorem (6.7). This last result and theorem (6.5) yield by means of a covering argument the main result in the paper, theorem (2.1).

Let $u(x,t)$ be a parabolically convex function in the bowl-shaped domain Q with $u = 0$ on $\partial_p Q$. Given $0 < \alpha \leq 1$ consider the set

$$Q_\alpha = \{(x,t) \in Q : u(x,t) < (1-\alpha) \min_Q u\}.$$

We have $Q_\alpha \subset Q_\beta$ for $0 < \alpha \leq \beta \leq 1$. Given $z_0 = (x_0, t_0) \in Q$, let us keep in mind definitions (2.1), (2.2), and (4.1).

We recall that, see section (5),

$$A^*_\sigma(u) = \{(x_0,t_0) \in Q :$$
$$u(x,t_0) \text{ is touched from below by } \sigma|x-x_0|^2 + p\cdot(x-x_0) + u(x_0,t_0) \text{ in } Q \cap \{t_0\} \ \}.$$

We have the following strict convexity result which states that sections with base points in Q_α are contained in Q for sufficiently small values of the parameter h and independently of the base point.

LEMMA 6.1. *Let Q be a normalized bowl-shaped domain and u solution to $\lambda \leq \mathcal{M}u \leq \Lambda$ in Q with $u = 0$ on $\partial_p Q$ and $0 < m_2 \leq -u_t \leq m_1$ in Q. Given $0 < \alpha \leq \alpha_0 < 1$ there exists $\eta_\alpha > 0$ such that if $h \leq \eta_\alpha$ and $(x_0,t_0) \in Q_\alpha$ then*

$$Q_h(x_0,t_0) \subset Q_{(\alpha_0+1)/2}.$$

PROOF. It is by contradiction. Suppose there exist $z_j = (x_j, t_j)$, u_j, Q^j such that $z_j \in Q^j_\alpha$, $Q_{1/j}(z_j) \not\subset Q^j_{\alpha_0}$, Q^j is normalized, $\lambda \leq \mathcal{M}u_j \leq \Lambda$ in Q^j, $0 < m_2 \leq -(u_j)_t \leq m_1$ in Q^j, and $u_j = 0$ on $\partial_p Q^j$. Let $(y_j, s_j) \in Q_{1/j}(z_j) \setminus Q^j_{\alpha_0}$. Then

$$(1-\alpha) \min_{Q^j} u_j \leq u_j(y_j, s_j) \leq \ell_j(y_j) + \frac{1}{j},$$

where ℓ_j is a supporting hyperplane to u_j at z_j. Letting $j \to \infty$ by compactness we obtain a function u_∞ and a normalized convex domain Q^∞ such that $\lambda \leq \mathcal{M}u_\infty \leq \Lambda$ in Q^∞, $0 < m_2 \leq -(u_\infty)_t \leq m_1$ in Q^∞, and $u_\infty = 0$ on $\partial_p Q^\infty$. Moreover,

$$(1-\alpha) \min_{Q^\infty} u_\infty \leq u_\infty(y_\infty, s_\infty) = \ell_\infty(y_\infty),$$

where ℓ_∞ is a supporting hyperplane to u^∞ and $z_\infty = (x_\infty, s_\infty) \in Q^\infty_\alpha$. Therefore

$-u_t \approx$ constant. Since $\det D_x^2 u_\infty(\cdot, t_\infty) \approx$ constant in $Q^\infty \cap \{t = t_\infty\}$, it follows from Caffarelli's result, [**Caf90b**, Theorem 1], that all extremal points of $\{u_\infty = \ell_\infty\}$ cannot be inside the interior of the domain and must be outside $Q^\infty_{\alpha_0} \cap \{t = t_\infty\}$. The point x_∞ must be convex combination of extremal points, i.e, $x_\infty = \sum_{i=1}^k \lambda_i P_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$, and $u_\infty(P_i) \geq (1-\alpha_0) \min_{Q^\infty} u_\infty$. Thus

$$\begin{aligned}(1-\alpha) \min_{Q^\infty} u_\infty \geq u_\infty(x_\infty) = \ell_\infty(x_\infty) \\ &= \sum_{i=1}^k \lambda_i \, \ell_\infty(P_i) = \sum_{i=1}^k \lambda_i \, u_\infty(P_i) \\ &\geq (1-\alpha_0) \min_{Q^\infty} u_\infty,\end{aligned}$$

a contradiction. □

For $\lambda > 0$ and $0 < \alpha \leq \alpha_0 < 1$, we define the set

$$D_\lambda^\alpha = \{(x_0, t_0) \in Q_\alpha : S_h(x_0|t_0) \subset B_{\lambda\sqrt{h}}(x_0), \quad \text{for all } h \leq \eta_0\},$$

where $\eta_0 = \eta_{\alpha_0}$ is the number in lemma (6.1) corresponding to $\alpha = \alpha_0$.

Lemma 6.2. *Let u be a parabolically convex function on a bounded bowl-shaped domain Q and $0 < \alpha_0 < 1$. There exists a constant $C_1 > 0$ depending only on α_0 and $\max_t\{diam(Q \cap \{t\})\}$ such that*

$$D_\lambda^\alpha = Q_\alpha \cap A^*_{1/\lambda^2}(u),$$

for all $\lambda \geq C_1$ and $0 < \alpha \leq \alpha_0 < 1$.

Proof. Suppose $z_0 = (x_0, t_0) \in D_\lambda^\alpha$. Let $\ell_{z_0}(x)$ be a supporting hyperplane to $u(\cdot, t_0)$ at $x = x_0$, i.e., $u(x, t_0) \geq \ell_{z_0}(x)$ for all $x \in Q \cap \{t_0\}$. Let $x \in Q \cap \{t_0\}$ and $\mu = u(x, t_0) - \ell_{z_0}(x)$, so $x \in S_\mu(x_0|t_0)$. If $\mu < \eta_0$ then $S_\mu(x_0|t_0) \subset B_{\lambda\sqrt{\mu}}(x_0)$ and $\dfrac{1}{\lambda^2}|x - x_0|^2 + \ell_{z_0}(x) \leq u(x, t_0)$. On the other hand, if $\mu \geq \eta_0$, then

$$u(x, t_0) - \ell_{z_0}(x) = \mu \geq \frac{\eta_0}{\operatorname{diam}(Q \cap \{t_0\})^2} \operatorname{diam}(Q \cap \{t_0\})^2 \geq \frac{\eta_0}{\operatorname{diam}(Q \cap \{t_0\})^2} |x - x_0|^2.$$

If $\dfrac{1}{\lambda^2} \leq \dfrac{\eta_0}{\max_t\{\operatorname{diam}(Q \cap \{t\})^2\}}$ then $u(x, t_0) - \ell_{z_0}(x) \geq \dfrac{1}{\lambda^2} |x - x_0|^2$ and the inclusion follows taking $C_1 = \dfrac{\max_t\{\operatorname{diam}(Q \cap \{t\})\}}{\sqrt{\eta_0}}$.

If $(x_0, t_0) \in Q_\alpha \cap A^*_{1/\lambda^2}(u)$ then $u(x, t_0) \geq \dfrac{1}{\lambda^2} |x - x_0|^2 + \ell_{z_0}(x)$ for all $x \in Q \cap \{t_0\}$. Let $x \in S_h(x_0|t_0)$ and $h < \eta_0$ then $S_h(x_0|t_0) \subset Q \cap \{t_0\}$ and $\dfrac{1}{\lambda^2} |x - x_0|^2 + \ell_{z_0}(x) \leq u(x, t_0) < \ell_{z_0}(x) + h$ and so $x \in B_{\lambda\sqrt{h}}(x_0)$. □

only on n and σ in theorem (5.1) such that if $z_0 = (x_0, t_0) \in Q_{\alpha_0}$ and $h \le \eta_0/2$ then we have

$$\frac{|Q_h(z_0) \setminus A^*_{c_0\,h}(u)|}{|Q_h(z_0)|} \le C_n\, \epsilon.$$

Moreover, if $\lambda \ge \dfrac{2}{c_0\,\eta_0}$ then

$$\frac{|Q_h(z_0) \setminus A^*_{1/\lambda}(u)|}{|Q_h(z_0)|} \le C_n\, \epsilon,$$

for $h \ge \dfrac{1}{c_0\,\lambda}$; ($C_n$ is the constant in the approximation theorem (5.1)).

Proof. The idea of the proof is to normalize u and then apply theorem (5.1). Consider the elliptic section $S_h(x_0|t_0)$, $h < \eta_0$, and let T be the affine transformation that normalizes $S_h(x_0|t_0)$. That is,

$$B_{\alpha_n}(0) \subset T\left(S_h(x_0|t_0)\right) \subset B_1(0).$$

We define

$$T_p(x,t) = \left(Tx, \frac{t-t_0}{h}\right) = (y,s).$$

Then from the estimates for u_t, see [**GH98**, Lemma 3.1], we have that

$$(-\epsilon_1, 0] \times B_{\alpha_n}(0) \subset T_p\left(Q_h(z_0)\right) \subset (-\epsilon_2, 0] \times B_1(0),$$

where ϵ_1 and ϵ_2 are constants. Set

$$Q^*_h(z_0) = T_p\left(Q_h(z_0)\right).$$

We have

$$T_p^{-1}(y,s) = \left(T^{-1}y, t_0 + h\,s\right).$$

Let $\ell_{z_0}(x)$ be the affine function defining $Q_h(z_0)$. We define

$$v(x,t) = \frac{C}{h}\left(u(x,t) - \ell_{z_0}(x) - h\right),$$

where C is a constant that will be determined in a moment.

Let

$$u^*(y,s) = v(T_p^{-1}(y,s)) = v(T^{-1}y, t_0 + h\,s).$$

We have $u^*_s(y,s) = \dfrac{C}{h}\, h\, u_t(T_p^{-1}(y,s))$, and

$$D^2u^*(y,s) = \frac{C}{h}\,(T^{-1})^t (D^2u)(T_p^{-1}(y,s))(T^{-1}).$$

Hence

$$\det D^2u^*(y,s) = \left(\frac{C}{h}\right)^n |\det T^{-1}|^2 \det D^2u(T_p^{-1}(y,s)).$$

Consequently,

We now pick C such that

$$\frac{C^{n+1}}{h^n}\,|\det T^{-1}|^2 = 1. \tag{6.1}$$

Since u satisfies the equation (5.1), it follows that u^* satisfies

$$(1-\epsilon)^{n+1} \leq -u_t^* \det D^2 u^* \leq (1+\epsilon)^{n+1} \qquad \text{in } Q_h^*(z_0) \tag{6.2}$$

$$u^* = 0 \qquad \text{on } \partial Q_h^*(z_0). \tag{6.3}$$

By definition of u^* we have that

$$\min_{Q_h^*(z_0)} u^* = -C.$$

By properties of the elliptic sections, see [**GH00**, Proposition 1.1], we have that $|S_h(x_0|t_0)| \approx h^{n/2}$, and since $|T(S_h(x_0|t_0))| \approx 1$, it follows that $|\det T|\,|S_h| \approx 1$ and consequently $|\det T| \approx h^{-n/2}$. Therefore $C \approx C_n$ by (6.1). Applying theorem (5.1) with $Q \to Q_h^*(z_0)$, $\alpha \to \beta$, and $u \to u^*$ we obtain

$$\frac{|Q^*_{\beta h}(z_0) \setminus A^*_\sigma|}{|Q^*_{\beta h}(z_0)|} \leq C_n\,\epsilon, \qquad \text{with } T_p(Q_{\beta h}(z_0)) = Q^*_{\beta h}(z_0),$$

where $A^*_\sigma = A^*_\sigma(u^*)$ (notice that $(Q_h^*(z_0))_\beta = T_p(Q_{\beta h}(z_0)) = Q^*_{\beta h}(z_0)$ and $A_\sigma(u^*) \subset A^*_\sigma(u^*)$). We now show that there exist universal constants $0 < \beta < 1$ and $c_0 > 0$ such that

$$T_p^{-1}\left(Q^*_{\beta h}(z_0) \cap A^*_\sigma\right) \subset Q_{\beta h}(z_0) \cap A^*_{c_0 h}(u). \tag{6.4}$$

Let $z_1^* = (x_1^*, t_1^*) \in Q^*_{\beta h}(z_0) \cap A^*_\sigma$ and $z_1 = (x_1, t_1) = T_p^{-1} z_1^* \in Q_{\beta h}(z_0)$. Since $z_1^* \in A^*_\sigma$, we have that

$$u^*(x^*, t_1^*) - \ell^*(x^*) \geq \sigma\,|x^* - x_1^*|^2,$$

for all $x^* \in Q_h^* \cap \{t_1^*\}$ with $\ell^*(x^*) = \ell_{z_1}(T^{-1}x^*)$ where ℓ_{z_1} is a supporting hyperplane to $u(\cdot, t_1)$ at $x = x_1$. Hence

$$\frac{C(u(x,t_1) - \ell_{z_0}(x) - h)}{h} - \frac{C(\ell_{z_1}(x) - \ell_{z_0}(x) - h)}{h} \geq \sigma\,|Tx - Tx_1|^2,$$

for $x \in Q_h(z_0) \cap \{t_1\}$. Therefore

$$u(x,t_1) - \ell_{z_1}(x) \geq \frac{1}{C}\,\sigma\,h\,|Tx - Tx_1|^2, \qquad \text{in } Q_h(z_0) \cap \{t_1\}.$$

By rotating the coordinates, we may assume that the ellipsoid of minimum volume containing $S_h(x_0|t_0)$ with center at x_h, the center of mass of $S_h(x_0|t_0)$, has axes on the coordinate axes. That is, $Tx = (\dfrac{x_1 - x_h^1}{\mu_1}, \cdots, \dfrac{x_n - x_h^n}{\mu_n})$ where μ_i are the axes the ellipsoid. Since Q is bounded, we have that $\mu_i \leq \textit{const}$, and so $\mu_i^{-1} \geq \textit{const}$. Therefore $|Tx - Tx_1| \geq C'\,|x - x_1|$. Consequently,

$$u(x,t_1) - \ell_{z_1}(x) \geq C''\,\sigma\,h\,|x - x_1|^2, \qquad \text{in } Q_h(z_0) \cap \{t_1\}. \tag{6.5}$$

We now want to show that a similar inequality holds in $Q \cap \{t_1\}$. Since $z_1 \in Q_{\beta h}(z_0)$, by the engulfing property lemma (4.3), we have $Q_{\beta h}(z_0) \subset Q_{\theta \beta h}(z_1)$. Again by the then $u(x,t_1) - \ell_{z_1}(x) \geq h/\theta$, and since Q is normalized, $h \geq h\,C''' \sigma\,|x - x_1|^2$. Therefore $(x_1,t_1) \in A^*_{\bar{C}\sigma h}(u)$ and letting $c_0 = \bar{C}\sigma$ we obtain (6.4) with $\beta = 1/\theta^2$.

Therefore (6.4) implies that

$$Q_{\beta\,h}(z_0) \setminus A^*_{c_0 h}(u) \subset T_p^{-1}(Q^*_{\beta\,h}(z_0) \setminus A^*_\sigma),$$

and consequently

$$\frac{|Q_{\beta\,h}(z_0) \setminus A^*_{c_0 h}(u)|}{|Q_{\beta\,h}(z_0)|} \leq \frac{|T_p^{-1}(Q^*_{\beta\,h}(z_0) \setminus A^*_\sigma)|}{|T_p^{-1}(Q^*_{\beta\,h}(z_0))|} = \frac{|Q^*_{\beta\,h}(z_0) \setminus A^*_\sigma|}{|Q^*_{\beta\,h}(z_0)|} \leq C_n\,\epsilon, \qquad h < \eta_0,$$

which yields the first conclusion of the proposition.

To prove the second conclusion, notice that if $\sigma \geq \mu$ then $A^*_\sigma \subset A^*_\mu$. Hence $A^*_{c_0\,h}(u) \subset A^*_{1/\lambda}(u)$ for $1/\lambda \leq c_0\,h$.

□

We recall the definition of D^α_λ,

$$D^\alpha_\lambda = \{(x_0,t_0) \in Q_\alpha : S_h(x_0|t_0) \subset B_{\lambda\sqrt{h}}(x_0) \quad \text{for all } h \leq \eta_0\};$$

here $0 < \alpha \leq \alpha_0 < 1$, $\lambda > 0$, and $\eta_0 > 0$ is from lemma (6.1) so that $Q_h(x_0,t_0) \subset Q_{(\alpha_0+1)/2}$ for $h \leq \eta_0$.

The following proposition gives the power decay needed for the proof of the $W^{2,p}$-estimates.

Proposition 6.4 (Power decay). *Let $0 < \epsilon < 1/2$ and u a solution satisfying the hypotheses of theorem* (5.1)*. We set*

$$(D^\alpha_\lambda)^c = Q_\alpha \setminus D^\alpha_\lambda; \qquad (D^\tau_{M\lambda})^c = Q_\tau \setminus D^\tau_{M\lambda}, \quad 0 < \tau < \alpha \leq \alpha_0.$$

There exist positive constants M, p_0 and C_2 so that

$$|(D^\tau_{M\lambda})^c| \leq \sqrt{C_n\,\epsilon}\,|(D^\alpha_\lambda)^c|, \tag{6.6}$$

for all $\lambda \geq C_2$ and $\alpha - \tau \approx (M\lambda)^{-p_0}$.

Proof. By lemma (6.2) we have

$$D^\tau_{M\lambda} = Q_\tau \cap A^*_{1/(M\lambda)^2}(u), \qquad \text{for } M\,\lambda \geq C_1 \text{ and } \tau < \alpha_0.$$

Since $D^\tau_{M\lambda}$ is closed, $\mathcal{O} = (D^\tau_{M\lambda})^c$ is open and we obtain

$$\mathcal{O} = Q_\tau \setminus D^\tau_{M\lambda} = Q_\tau \cap (A^*_{1/(M\lambda)^2}(u))^c$$

for $M\,\lambda \geq C_1$ and $\tau < \alpha_0$. Consequently

$$Q_h(z_0) \cap \mathcal{O} \subset Q_h(z_0) \cap Q_\tau \cap (A^*_{1/(M\lambda)^2}(u))^c \subset Q_h(z_0) \setminus A^*_{1/(M\lambda)^2}(u).$$

Therefore by Proposition (6.3) we obtain

for
(6.7)

$$M\,\lambda \geq \max\left\{C_1, \sqrt{\frac{2}{c_0\eta_0}}\right\}, \qquad \tau < \alpha_0, \qquad \frac{1}{(\lambda M)^2} \leq h \leq \eta_0/2, \qquad z_0 \in Q_{\alpha_0}.$$

Let us now consider the sections $Q_h^*(x_0,t_0)$ defined by (4.1), and keep in mind (4.2). Since the set $\mathcal{O}$ is open we have that

$$\lim_{h\to 0} \frac{|Q_h^*(z_0)\cap \mathcal{O}|}{|Q_h^*(z_0)|} = 1, \qquad z_0 \in \mathcal{O}. \tag{6.8}$$

By Proposition (6.3) we have that

$$\frac{|Q_h^*(x_0,t_0)\cap \mathcal{O}|}{|Q_h^*(x_0,t_0)|} = \frac{|Q_h((x_0)^h_{\min}, t_0+\delta\, h)\cap \mathcal{O}|}{|Q_h((x_0)^h_{\min}, t_0+\delta\, h)|} \leq C_n\,\epsilon,$$

with h satisfying (6.7), since $(x_0,t_0)\in Q_\alpha$ implies that $((x_0)^h_{\min}, t_0+\delta\,h)\in Q_{(\alpha+1)/2}$, see remark (4.1) $m_1\,\delta < \eta_0/2$.

If $\mathcal{O} = Q_\tau \setminus D^\tau_{M\lambda}$ then for $z\in\mathcal{O}$ we pick h_z the largest h such that $\dfrac{|Q_h^*(z)\cap\mathcal{O}|}{|Q_h^*(z)|} \geq C_n\,\epsilon$. Then by (6.7) and (6.8) we get $h_z \leq \dfrac{1}{(M\lambda)^2}$. Applying theorem (4.12) to this $\mathcal{O}$ with $\gamma = \dfrac{1}{(M\lambda)^2}$, and $\delta = C_n\,\epsilon$, we obtain a family of sections $\{Q^*_{h_k}(z_k)\}_{k=1}^\infty$, $z_k = (x_k,t_k)$, with $h_k \leq \dfrac{1}{(M\lambda)^2}$.

We shall prove that

$$Q^*_{h_k}(x_k,t_k) \subset (D^\alpha_\lambda)^c = Q_\alpha \setminus D^\alpha_\lambda. \tag{6.9}$$

By remark (4.1) and lemma (4.9) we have that if $(x_k,t_k)\in Q_\tau$ then $((x_k)^h_{\min}, t_k) \in Q_{m_1\,\delta\,h}(x_k,t_k) \subset Q_{\tau+c(\delta h)^{1/p}} \subset Q_{\tau+(M\lambda)^{-p_0}}$. That is, $Q^*_{h_k}(x_k,t_k)\subset Q_{\tau+2(M\lambda)^{-p_0}}$. For $\tau = \alpha-(M\lambda)^{-p_0}$ and since $(x_k, t_k+\delta h_k)\in Q_\tau$, it follows that $Q_{h_k}((x_k)^h_{\min}, t^h_k)\subset Q_\alpha$ where $t^h_k = t_k+\delta h_k$.

To complete the proof of (6.9) we proceed by contradiction. Suppose there exists $z_0 = (x_0,t_0)\in Q_{h_k}((x_k)^h_{\min}, t^h_k)\cap D^\alpha_\lambda$. By the engulfing property of elliptic sections at different times, lemma (4.4), we have that

$$S_{h_k}((x_k)^h_{\min}|t^h_k) \subset S_{\theta h_k}(x_0|t_0) \subset B_{\lambda\sqrt{\theta h_k}}(x_0),$$

with $z_0\in D^\alpha_\lambda$ ($\theta\,h_k \leq \eta_0$ by choosing M large). As in the proof of Proposition (6.3) we normalize the section $S_{2h_k}((x_k)^h_{\min}|t^h_k)$. That is, $B_{\alpha_n}(0)\subset T(S_{2h_k}((x_k)^h_{\min}|t^h_k))\subset B_1(0)$, and let $Q^*_k = T_p\left(Q_{2h_k}((x_k)^h_{\min}, t^h_k)\right)$ normalized. We set $u^* = \dfrac{c}{2h_k}(u-\ell-2h_k)(T_p^{-1}(x^*,t^*))$, and we have $(1-\epsilon)^{n+1}\leq Mu^*\leq(1+\epsilon)^{n+1}$ in Q^*_k. By the approximation theorem (5.1) we then have

$$|(Q^*_k)_{1/2}\setminus A^*_\sigma| < C_n\,\epsilon\,|(Q^*_k)_{1/2}|, \tag{6.10}$$

(notice that $(Q^*_k)_{1/2} = T_p\left(Q_{h_k}((x_k)^h_{\min}, t^h_k)\right)$). We now claim that

Let $z_1^* = (x_1^*, t_1^*) \in (Q_k^*)_{1/2} \cap A_\sigma^*$ and $z_1 = T_p^{-1} z_1^* = (x_1, t_1) \in Q_{h_k}\left((x_k)^h_{\min}, t_k^h)\right)$. Since $(x_1^*, t_1^*) \in A_\sigma^*$, we have that $u^*(x^*, t_1^*) - \ell^*(x^*) \geq \sigma\,|x^* - x_1^*|^2$ and hence $S_h^*(x_1^*|t_1^*) \subset B(x_1^*, \sqrt{h/\sigma})$. Therefore $T^{-1}\left(S_h^*(x_1^*|t_1^*)\right) \subset T^{-1}\left(B(x_1^*, \sqrt{h/\sigma})\right)$ for $h \leq const$, and consequently

$$S_{c\,h_k\,h}(x_1|t_1) \subset T^{-1}(B(x_1^*, \sqrt{h/\sigma})) \subset B(x_1, \lambda\sqrt{\theta 2 h_k}\,\sqrt{h/\sigma}),$$

because T dilates at least $(\lambda\sqrt{\theta 2 h_k})^{-1}$ and T^{-1} contracts at least $\lambda\sqrt{\theta 2 h_k}$. Then $S_h(x_1|t_1) \subset B(x_1, \lambda\sqrt{ch/\sigma})$ for $h \leq const\, h_k$. If $h_k \leq h \leq \eta_0$ then $(x_0, t_0), (x_1, t_1) \in Q_h((x_k)^h_{\min}, t_k^h)$. By the engulfing property at different times $S_h(x_1|t_1) \subset S_{\theta h}(x_k|t_k^h) \subset S_{\theta^2 h}(x_0|t_0) \subset B_{\lambda\sqrt{\theta^2 h}}(x_0)$, since $z_0 \in D_\lambda^\alpha$. Therefore $(x_1, t_1) \in Q_\alpha \cap D_{M\lambda}^\alpha$ for some M large, and the proof of (6.11) is complete.

Therefore by Lemma (6.2),

$$\begin{aligned} Q_{h_k}((x_k)^h_{\min}, t_k^h) \cap \mathcal{O} &= Q_{h_k}((x_k)^h_{\min}, t_k^h) \cap (Q_\tau \setminus D_{M\lambda}^\tau) \\ &= Q_{h_k}((x_k)^h_{\min}, t_k^h) \cap Q_\tau \cap \left(A^*_{(M\lambda)^{-2}}(u)\right)^c \\ &\subset Q_{h_k}((x_k)^h_{\min}, t_k^h) \setminus D_{M\lambda}^\alpha \subset T_p^{-1}((Q_k^*)_{1/2} \setminus A_\sigma^*), \end{aligned}$$

and by (6.10) we obtain

$$\frac{|Q^*_{h_k}(x_k, t_k) \cap \mathcal{O}|}{|Q^*_{h_k}(x_k, t_k)|} \leq \frac{|(Q_k^*)_{1/2} \setminus A_\sigma^*|}{|(Q_k^*)_{1/2}|} < C_n\,\epsilon,$$

which contradicts (3) in Theorem (4.12). This completes the proof of the power decay. □

THEOREM 6.5. *Let Q be a normalized bowl-shaped bounded domain and u satisfying the hypothesis of theorem* (5.1). *Then given $0 < p < \infty$ there exists $\epsilon(p) > 0$ such that*

$$\iint_{Q_\tau} D_{ee} u(x,t)^p\, dxdt \leq C,$$

for all $|e| = 1$ and $0 < \epsilon < \epsilon(p)$ with C a constant depending only on the structure.

PROOF. We iterate the inequality in proposition (6.4). Notice that we can pick M large so that the statement of proposition (6.4) holds for all $\lambda \geq M$. We begin the iteration with $\lambda = M$ and therefore $(\tau =)\alpha_1 = \alpha - (M^2)^{-p_0}$ and we get

$$|Q_{\alpha_1} \setminus D^{\alpha_1}_{M^2}| \leq \sqrt{C_n\,\epsilon}\,|Q_\alpha \setminus D_M^\alpha|.$$

Continuing in this way, we let $\lambda = M^k$ and $\alpha_k = \alpha - \sum_{j=1}^k M^{-p_0(j+1)}$ obtaining

$$|Q_{\alpha_k} \setminus D^{\alpha_k}_{M^{k+1}}| \leq C\left(\sqrt{C_n\,\epsilon}\right)^k, \qquad \text{for } k = 1, 2, \cdots.$$

We fix $\tau < \alpha$ and choose M large so that $\alpha_k \geq \alpha - \sum_{j=1}^\infty M^{-(j+1)p_0} \geq \tau$. We claim that if $(x_0, t_0) \in A_\sigma^*(u)$ then $u(x, t_0) \leq C(n)\,\sigma^{-n+1}\,|x - x_0|^2 + \ell_{z_0}(x)$ for all x sufficiently close to x_0. Indeed, we have $S_h(x_0|t_0) \subset B_{\sqrt{h/\sigma}}(x_0)$ and by properties of the elliptic sections $|S_h(x_0|t_0)| \approx h^{n/2}$. Applying Aleksandrov's max-

$(x_0,t_0) \in A^*_\sigma(u)$ then $D_{ee}u(x_0,t_0) \le 2\,C(n)\,\sigma^{-n+1}$ for any $|e|=1$. By lemma (6.2), if $(x_0,t_0) \in D^{\alpha_i}_{M^{i+1}}$ then $(x_0,t_0) \in Q_\alpha \cap A^*_{1/M^{2(i+1)}}(u)$ and consequently $D_{ee}u(x_0,t_0) \le 2\,C(n)\,M^{2(n-1)(i+1)}$. Therefore

$$D^{\alpha_i}_{M^{i+1}} \subset \{(x,t) \in Q_{\alpha_i} : D_{ee}u(x,t) \le 2\,C(n)\,M^{2(n-1)(i+1)}\},$$

and consequently

$$\{(x,t) \in Q_{\alpha_i} : D_{ee}u(x,t) > 2\,C(n)\,M^{2(n-1)(i+1)}\} \subset Q_{\alpha_i} \setminus D^{\alpha_i}_{M^{i+1}}.$$

Therefore

$$\begin{aligned}
&\|D_{ee}u\|^p_{L^p(Q_\tau)}\\
&\le M^{2(n-1)p}\,|Q_\tau| + \sum_{i=0}^\infty \int_{\{(x,t)\in Q_\tau : M^{2(n-1)(i+1)} < D_{ee}u(x,t) \le M^{2(n-1)(i+2)}\}} D_{ee}u(x,t)^p\,dxdt\\
&\le M^{2(n-1)p}\,|Q_\tau| + \sum_{i=0}^\infty \int_{\{(x,t)\in Q_{\alpha_i} : M^{2(n-1)(i+1)} < D_{ee}u(x,t) \le M^{2(n-1)(i+2)}\}} D_{ee}u(x,t)^p\,dxdt\\
&\le C(M,n,\alpha,\tau,p) + \sum_{i=0}^\infty |Q_{\alpha_i} \setminus D^{\alpha_i}_{M^{i+1}}|\,M^{2(n-1)(i+2)p}\\
&\le C(M,n,\alpha,\tau,p) + C(n)\,\sum_{i=0}^\infty (\sqrt{C_n\epsilon})^{i+1}\,M^{2(n-1)(i+2)p} < \infty,
\end{aligned}$$

for ϵ sufficiently small. □

To complete the proof of the $W^{2,p}$ estimates we need the following result due to Caffarelli, see [**Caf90b**].

THEOREM 6.6. *Let u be a convex solution to*

$$\lambda \le \det D^2u \le \Lambda, \qquad \text{in } \Omega \tag{6.12}$$

$$u = f, \qquad \text{on } \partial\Omega, \tag{6.13}$$

where $\Omega \subset \mathbb{R}^n$ is a $C^{1,\alpha}$ normalized convex domain and $f \in C^{1,\alpha}$, with $\alpha > 1 - \frac{2}{n}$. Then for each $h > 0$ there exists $\delta > 0$ such that for $x_0 \in \Omega_h = \{x \in \Omega : \text{dist}(x,\partial\Omega) > h\}$ we have

$$S(x_0,\delta) = \{x : u(x) < \ell_{x_0}(x) + \delta\} \subset \Omega_{h/2},$$

where δ depends only on $h, \lambda, \Lambda, n, \alpha$ and the $C^{1,\alpha}$ norms of f and Ω.

Now we are ready to prove the following result for the parabolic case.

THEOREM 6.7. *Let u be a solution to $\mathcal{M}u = f$ in the cylinder $Q = \Omega \times (0,T]$ with $u = \phi$ on $\partial_p Q$. Suppose that*

(1) $B_{\alpha_n}(0) \subset \Omega \subset B_1(0)$, $\partial\Omega \in C^{1,\alpha}$ *with* $\alpha > 1 - \frac{2}{n}$.
(2) $0 < \lambda \le f \le \Lambda$, $f \in C(\bar Q)$, $f_t \in L^{n+1}(Q)$ *and* $\exp(A(-f_t)^+) \in L^1(Q)$ *for*

Then for each $h > 0$ there exists $\delta > 0$ such that for $(x_0, t_0) \in \Omega_h \times (h, T]$, $\Omega_h = \{x \in \Omega : dist(x, \partial\Omega) > h\}$, we have that

$$Q_\delta(x_0, t_0) = \{(x, t) \in Q : u(x, t) < \ell_{x_0}(x) + \delta, \quad t \le t_0\} \subset \Omega_{h/2} \times (h/2, T],$$

where δ depends only on h and the parameters.

PROOF. By theorem (3.2) we get that $-m_1 \le u_t \le -m_2$ in Q. Therefore $u_0(\cdot) = u(\cdot, t_0)$ satisfies (6.12) and by theorem (6.6) there exists δ such that if $x_0 \in \Omega_h$ then $S_\delta(x_0|t_0) = \{x : u(x, t_0) < \ell_{x_0}(x) + \delta\} \subset \Omega_{h/2}$. Since $-m_1 \le u_t \le -m_2$, it follows that $Q_\delta(x_0, t_0) \subset S_\delta(x_0|t_0) \times (t_0 - c\,\delta, t_0] \subset \Omega_{h/2} \times (h/2, T]$. □

We are now in a position to complete the proof of the main result in the paper.

PROOF OF THEOREM (2.1) (B). The proof will follow combining theorems (6.5) and (6.7). Let $z_0 = (x_0, t_0) \in \Omega_h \times (h, T]$ and suppose that we have a section $Q^\delta = Q_u(z_0, \delta) \subset \Omega_{h/2} \times (h/2, T]$ such that $|f(z_0) - f(z)| \le \epsilon$, for each $z = (x, t) \in Q_u(z_0, \delta)$. Taking δ sufficiently small, by theorem (6.7) we may assume that $Q_u(z_0, \delta) \subset \Omega_{h/2} \times (h/2, T]$. Notice that since Q is normalized we have from the property of size of sections, lemma (4.7), that

$$K(z_0, K_1\,\delta^{\epsilon_1}) \subset Q_u(z_0, \delta) \subset K(z_0, K_2\,\delta^{\epsilon_2}), \tag{6.14}$$

with K_i, ϵ_i, positive constants depending only on λ, Λ and n, and $K(z, R)$ is the standard parabolic cylinder defined in the statement of (4.6). Let T be an affine transformation normalizing $S_\delta(x_0|t_0)$, $T_p(x, t) = \left(Tx, \dfrac{t - t_0}{\delta}\right)$ as in the comment following remark (4.1), and consider the function

$$v(x, t) = \frac{C}{\delta}\,\left(u(T_p^{-1}(x, t)) - \ell_y(T_p^{-1}(x, t)) - \delta\right),$$

where ℓ_{x_0} is the supporting hyperplane to $u(\cdot, t_0)$ at x_0, and C is a constant that will be determined in a moment. We look at v on the set $T_p(Q_u(z_0, \delta))$, and we have $v = 0$ on $\partial T_p(Q_u(z_0, \delta))$,

$$D_x^2 v(x, t) = \frac{C}{\delta}\,\left\{(T^{-1})^t\,(D_x^2 u)(T_p^{-1}(x, t))\,T^{-1}\right\}, \qquad \text{and}$$
$$v_t(x, t) = C\,u_t(T_p^{-1}(x, t)).$$

Hence

$$\mathcal{M}v(x, t) = \frac{C^{n+1}}{\delta^n}\,|\det T|^{-2}\,f(T_p^{-1}(x, t)) = \frac{f(T_p^{-1}(x, t))}{f(z_0)},$$

for $C = \dfrac{\delta^{n/(n+1)}\,|\det T|^{2/(n+1)}}{f(z_0)^{1/(n+1)}}$. Now $f(z_0) - \epsilon \le f(z) \le f(z_0) + \epsilon$ for $z \in Q^\delta$, and so

$$1 - \frac{\epsilon}{f(z_0)} \le \frac{f(T_p^{-1}z)}{f(z_0)} \le 1 + \frac{\epsilon}{f(z_0)},$$

for $z \in T_p(Q^\delta)$. Since $f(z_0) \ge \lambda$, it follows that

Then applying our result on the set $T_p(Q^\delta)$ to the function v we get that

$$\int_{(T_p(Q^\delta))_h} D_{ee}v(x,t)^p\,dxdt \le C(n,h,p),$$

for each unit vector e and $\epsilon \le \epsilon_p$.

By definition of v we have that

$$D_x^2 u(x,t) = \frac{\delta}{C}\,T^t\,(D_x^2 v)(T_p(x,t))\,T,$$

and consequently

$$\begin{aligned}
D_{ee}u(x,t) &= \langle D_x^2 u(x,t)\,e,\ e\rangle \\
&= \frac{\delta}{C}\,\langle (D_x^2 v)(T_p(x,t))\,Te, Te\rangle \\
&= \frac{\delta}{C}\,|Te|^2\,\langle (D_x^2 v)(T_p(x,t))\,e', e'\rangle \qquad e' = \frac{Te}{|Te|} \\
&= \frac{\delta}{C}\,|Te|^2\,(D_{e'e'}v)(T_p(x,t)).
\end{aligned}$$

We have $(T_p(Q^\delta))_h = T_p((Q^\delta)_h)$. Therefore

$$\begin{aligned}
\int_{(Q^\delta)_h} D_{ee}u(x,t)^p\,dxdt &= \left(\frac{\delta}{C}\right)^p |Te|^{2p} \int_{(Q^\delta)_h} (D_{e'e'}v)(T_p(x,t))^p\,dxdt \\
&= \left(\frac{\delta}{C}\right)^p |Te|^{2p} \int_{(T(Q^\delta))_h} (D_{e'e'}v)(z)^p\,|\det T|^{-1}\,\delta\,dz \\
&\le f(z_0)^{p/(n+1)} \left(\frac{\delta^{\frac{1}{p}+\frac{1}{n+1}}\,|Te|^2}{|\det T|^{\frac{2}{n+1}+\frac{1}{p}}}\right)^p C(h,n,p).
\end{aligned}$$

To estimate the term between parenthesis, let E be the ellipsoid of minimum volume containing $S_\delta(x_0|t_0)$, and let $\mu_1,\cdots,\mu_n$ be the axes of E. If δ is small, then by regularity theory we have that $|S_\delta(x_0|t_0)| \approx \delta^{n/2}$. The affine transformation that normalizes $S_\delta(x_0|t_0)$ has the form

$$Tx = \left(\frac{x_1 - x_1^0}{\mu_1},\ldots,\frac{x_n - x_n^0}{\mu_n}\right),$$

where $(x_1^0,\cdots,x_n^0)$ is the center of the ellipsoid E (the center of mass of $S_\delta(x_0|t_0)$). We have $|\det T| \approx \delta^{-n/2}$, and from (6.14) it follows that $\mu_i \ge K_1\,\delta^{\epsilon_1}$. Hence

$$\frac{\delta^{\frac{1}{p}+\frac{1}{n+1}}\,|Te|^2}{|\det T|^{\frac{2}{n+1}+\frac{1}{p}}} \approx |Te|^2\,\delta^{1+\frac{1}{p}+\frac{n}{2p}} \le C\,\delta^{1+\frac{1}{p}+\frac{n}{2p}-2\epsilon_1},$$

and consequently

$$\int_{(Q^\delta)_h} D_{ee}u(x,t)^p\,dxdt \le C(\lambda,\Lambda,n,h,p)\,\delta^{p+1+\frac{n}{2}-2p\epsilon_1}. \tag{6.15}$$

We now pick δ small depending only on the parameters λ,Λ,h and the modulus

$\{K(z_j, K_1\,\delta^{\epsilon_1})\}_{j=1}^N$ with $z_j \in \Omega_h \times (h, T]$. The desired inequality then follows by adding (6.15) over $(Q(z_j, \delta))_h$. □

7. APPENDIX: The parabolic convex envelope on a bowl-shaped domain

Let Q be a bowl-shaped domain in $\mathbb{R}^{n+1}$, and $u \in C(\overline{Q})$. We define *the parabolic convex envelopes* Γ_u and Γ_u^p as follows. Given $(x_0, t_0) \in Q$ we let

$$\Gamma_u(x_0, t_0) = \sup\{v(x_0, t_0) : v \le u \text{ in } Q \text{ with } v \in C(Q) \text{ and } p\text{-convex in } Q\}; \tag{7.1}$$

$$\begin{aligned}\Gamma_u^p(x_0, t_0) = \sup\{&v(x_0, t_0) : v \le u,\\ &\text{in } Q \cap \{t \le t_0\} \text{ with } v \text{ continuous and } p\text{-convex in } Q \cap \{t \le t_0\}\}.\end{aligned}$$

The set $\mathcal{C}$ *of contact points, or contact set,* is given by

$$\mathcal{C} = \{(x, t) \in \overline{Q} : u(x, t) = \Gamma_u(x, t)\}.$$

LEMMA 7.1. *We have*

$$\Gamma_u = \Gamma_u^p \qquad \text{in } Q. \tag{7.2}$$

PROOF. We obviously have $\Gamma_u \le \Gamma_u^p$ in Q. Given $(x_0, t_0) \in Q$ and $\epsilon > 0$, let v be continuous and p-convex in $Q \cap \{t \le t_0\}$ such that $v \le u$ in $Q \cap \{t \le t_0\}$ and

$$v(x_0, t_0) \ge \Gamma_u^p(x_0, t_0) - \epsilon.$$

Since v is p-convex there exists a supporting hyperplane $\ell_{x_0}(x)$ such that

$$\begin{aligned}\ell_{x_0}(x) &\le v(x, t) \qquad \text{in } Q \cap \{t \le t_0\} \text{ and}\\ \ell_{x_0}(x_0) &= v(x_0, t_0).\end{aligned}$$

By continuity of ℓ_{x_0} and u, there exists $\delta > 0$ so that

$$\ell_{x_0}(x) - \epsilon \le u(x, t) \qquad \text{in } Q \cap \{t \le t_0 + \delta\}.$$

Let $0 \le \alpha(t) \le 1$ be a continuous and nonincreasing function on $(0, t_0 + \delta)$ with $\alpha(t) = 1$ on $(0, t_0)$ and $\alpha(t_0 + \delta) = 0$. Set

$$w(x, t) = \alpha(t)(\ell_{x_0}(x) - \epsilon) + (1 - \alpha(t))K,$$

where $K = \min\{\min_Q(\ell_{x_0} - \epsilon), \min_Q u\}$. It is easy to see that w is continuous and p-convex in Q, and satisfies

$$\begin{aligned}w &\le \ell_{x_0} - \epsilon \le u \qquad \text{in } Q \cap \{t \le t_0 + \delta\}\\ w &= K \le u \qquad \text{in } Q \cap \{t > t_0 + \delta\}.\end{aligned}$$

Hence $w \le u$ in Q. Therefore

$$\Gamma_u(x_0, t_0) \ge w(x_0, t_0) = \ell_{x_0}(x_0) - \epsilon = v(x_0, t_0) - \epsilon \ge \Gamma_u^p(x_0, t_0) - 2\epsilon,$$

LEMMA 7.2. *Let $u \in C^{2,1}(\bar{Q})$. If $(x_0,t_0) \in \mathcal{C} \cap Q$ then there exist $\epsilon_0 > 0$, $M > 0$, and $p = D_x u(x_0,t_0)$, depending only on u (bounded by the $C^{2,1}$-norm of u in $\bar{Q}$) such that*

$$\Gamma_u(x,t) \le \Gamma_u(x_0,t_0) + p\cdot(x-x_0) + M\left(|x-x_0|^2 + t_0 - t\right), \tag{7.3}$$

for all $(x,t) \in B_{\sqrt{\epsilon_0}}(x_0) \times (t_0-\epsilon_0, t_0] \cap Q$.

PROOF. By the Taylor expansion

$$\begin{aligned} u(x,t) &= u(x_0,t_0) + u_t(x_0,t_0)(t-t_0) + Du(x_0,t_0)\cdot(x-x_0) \\ &+ \frac{1}{2}\langle D_x^2 u(x_0,t_0)(x-x_0), x-x_0\rangle + o(|x-x_0|^2 + t_0 - t), \end{aligned}$$

as $x \to x_0$ and $t \to t_0^-$. Hence

$$\begin{aligned} u(x,t) &\le u(x_0,t_0) + u_t(x_0,t_0)(t-t_0) + Du(x_0,t_0)\cdot(x-x_0) \\ &+ \frac{1}{2}\langle D_x^2 u(x_0,t_0)(x-x_0), x-x_0\rangle + \epsilon(|x-x_0|^2 + t_0 - t), \end{aligned}$$

for ϵ small. Since $\Gamma_u(x,t) \le u(x,t)$ and $(x_0,t_0) \in \mathcal{C}\cap Q$, the lemma follows. □

LEMMA 7.3. *Assume $u \in C^{2,1}(\bar{Q})$. Let $(x_0,t_0) \in Q \setminus \mathcal{C}$ and let $L(x) = \alpha + p\cdot x$ be a supporting hyperplane to $\Gamma_u(\cdot,t_0)$ at $x = x_0$. Then there exist at most $n+1$ points $(x_i,t_i) \in \mathcal{C}$ such that*

$$x_0 = \sum_{i=1}^{n+1} \lambda_i\, x_i, \tag{7.4}$$

where $\lambda_i \ge 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$, $t_i \le t_0$, $L(x_i) = \Gamma_u(x_i,t_i) = u(x_i,t_i)$ and $p = D_x u(x_i,t_i)$, $i = 1,\cdots,n+1$.

PROOF. We have $\Gamma_u(x_0,t_0) < u(x_0,t_0)$. Since $L(x)$ is a supporting hyperplane to $\Gamma_u(x,t_0)$ at x_0, then $\Gamma_u(x,t_0) \ge L(x)$ for all $x \in Q\cap\{t=t_0\}$ and $\Gamma_u(x_0,t_0) = L(x_0)$. We have $\Gamma_u(x,t) \ge \Gamma_u(x,t_0)$ for all $(x,t) \in Q\cap\{t\le t_0\}$. Since $u(x,t) \ge \Gamma_u(x,t)$, it follows that

$$u(x,t) \ge L(x), \qquad \text{for all } (x,t) \in Q\cap\{t\le t_0\}. \tag{7.5}$$

Let

$$H = \{x : \text{there exists } t \text{ such that } (x,t) \in \overline{Q}\cap\{t\le t_0\} \text{ and } u(x,t) = L(x)\}.$$

We have $H \ne \emptyset$. Otherwise, by (7.5), $u(x,t) > L(x)$ in $\overline{Q}\cap\{t\le t_0\}$ and by compactness $u(x,t) - L(x) \ge \delta > 0$ on the same set and for some $\delta > 0$. Hence $\Gamma_u^p(x,t_0) \ge L(x) + \delta$. Using (7.2) and letting $x = x_0$ we get a contradiction.

It is clear that the set H is closed.

Let $z \in H$ and $s \le t_0$ such that $u(z,s) = L(z)$. Then $(z,s) \in \mathcal{C}$. Indeed,

$$u(x,t) \ge \Gamma_u(x,t) \ge \Gamma_u(x,t_0) \ge L(x), \qquad \text{for all } (x,t) \in Q\cap\{t\le t_0\},$$

Let $Con(H)$ be the convex hull of H. We claim that $x_0 \in Con(H)$. Assume by contradiction that $x_0 \notin Con(H)$ and let N be a neighborhood of $Con(H)$ and $\ell(x)$ an affine function such that $\ell(x_0) > 0$ and $\ell(x) < 0$ in N. We have

$$\min\{u(x,t) - L(x) : (x,t) \in Q \cap \{t \le t_0\} \setminus N \times [a, t_0]\} \ge \delta > 0,$$

with a lower bound for t when $(x,t) \in Q$. Hence, there exists $\epsilon > 0$ such that $u(x,t) - L(x) \ge \epsilon \ell(x)$ for all $x \notin N$ and $t \le t_0$. Therefore, by (7.5), $u(x,t) \ge L(x) + \epsilon \ell(x)$ for all $(x,t) \in Q \cap \{t \le t_0\}$ and consequently $\Gamma_u(x,t) \ge L(x) + \epsilon \ell(x)$ on the same set. Since $\Gamma_u(x_0, t_0) = L(x_0)$, we obtain a contradiction.

Therefore by Carathéodory's theorem, see [**Sch93**, Theorem 1.1.3, p.3]

$$x_0 = \sum_{i=1}^{n+1} \lambda_i x_i, \tag{7.6}$$

where $\lambda_i \ge 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$, and $x_i \in H$. Let $t_i \le t_0$ be the t's corresponding to x_i's such that $(x_i, t_i) \in Q \cap \{t \le t_0\}$ and $u(x_i, t_i) = L(x_i)$. We have

$$u(x_i, t_i) \ge \Gamma_u(x_i, t_i) \ge \Gamma_u(x_i, t_0) \ge L(x_i) = u(x_i, t_i),$$

and so $u(x_i, t_i) = \Gamma_u(x_i, t_i) = L(x_i)$.

We have that L is a supporting hyperplane to $u(\cdot, t_i)$ at $x = x_i$ for $i = 1, \cdots, n+1$. Since u is regular

$$\begin{aligned} L(x) &\le u(x, t_i) \\ &= u(x_i, t_i) + Du(x_i, t_i) \cdot (x - x_i) + o(|x - x_i|^2) \end{aligned}$$

as $x \to x_i$. Since $L(x_i) = \alpha + p \cdot x_i = u(x_i, t_i)$, we get $L(x) = u(x_i, t_i) + p \cdot (x - x_i)$ and so

$$p \cdot (x - x_i) \le Du(x_i, t_i) \cdot (x - x_i) + o(|x - x_i|^2), \tag{7.7}$$

and the lemma follows. □

LEMMA 7.4. *If $u \in C(\overline{Q})$ then $\Gamma_u \in C(Q)$.*

PROOF. We have that Γ_u is p-convex in Q. We claim that

$$\lim_{t \downarrow t_0} \Gamma_u(x_0, t) = \Gamma_u(x_0, t_0) \qquad (x_0, t_0) \in Q. \tag{7.8}$$

By monotonicity $\Gamma_u(x_0, t) \le \Gamma_u(x_0, t_0)$ for $t \ge t_0$. Hence

$$\lim_{t \downarrow t_0} \Gamma_u(x_0, t) \le \Gamma_u(x_0, t_0).$$

To show the opposite inequality, given $\epsilon > 0$ there exists $v \in C(Q)$, p-convex, so that $v \le u$ in Q and $v(x_0, t_0) + \epsilon \ge \Gamma_u(x_0, t_0)$. Since $v(x_0, t)$ is continuous and nonincreasing in t, there exists $\delta > 0$ so that $0 \le v(x_0, t_0) - v(x_0, t) < \epsilon$ for $t_0 \le t \le t_0 + \delta$. Hence $\Gamma_u(x_0, t_0) \le v(x_0, t) + 2\epsilon$, for $t_0 \le t \le t_0 + \delta$, and taking limit as $t \downarrow t_0$ yields

$$\Gamma_u(x_0, t_0) \le \liminf_{t \downarrow t_0} \Gamma_u(x_0, t) + 2\epsilon.$$

Let $(x_0, t_0) \in \{z \in Q : u(z) = \Gamma_u(z)\}$. We claim that Γ_u is continuous at (x_0, t_0). Notice that by monotonicity if $t \leq t_0$ then $\Gamma_u(x_0, t) \geq \Gamma_u(x_0, t_0)$ and since $\Gamma_u(x_0, t)$ is nonincreasing we get

$$\lim_{t \uparrow t_0} \Gamma_u(x_0, t) \geq \Gamma_u(x_0, t_0). \tag{7.9}$$

Since $u \in C(Q)$ and $(x_0, t_0) \in \mathcal{C}$, it follows that

$$\lim_{t \uparrow t_0} \Gamma_u(x_0, t) \leq \liminf_{t \uparrow t_0} u(x_0, t) = u(x_0, t_0) = \Gamma_u(x_0, t_0).$$

By (7.9) we then have

$$\lim_{t \uparrow t_0} \Gamma_u(x_0, t) = \Gamma_u(x_0, t_0). \tag{7.10}$$

This combined with (7.8) yields

$$\lim_{t \to t_0} \Gamma_u(x_0, t) = \Gamma_u(x_0, t_0). \tag{7.11}$$

On the other hand, since Γ_u is bounded in Q and convex in x, it follows by [**GH00**, Lemma 1.1] that

$$|\Gamma_u(x_1, t) - \Gamma_u(x_2, t)| \leq C\,|x_1 - x_2|,$$

for $(x_1, t), (x_2, t)$ in a neighborhood of (x_0, t_0). Therefore

$$\begin{aligned} |\Gamma_u(x, t) - \Gamma_u(x_0, t_0)| &\leq |\Gamma_u(x, t) - \Gamma_u(x_0, t)| + |\Gamma_u(x_0, t) - \Gamma_u(x_0, t_0)| \\ &\leq C\,|x - x_0| + |\Gamma_u(x_0, t) - \Gamma_u(x_0, t_0)| \to 0, \end{aligned}$$

as $(x, t) \to (x_0, t_0)$ by (7.11).

It remains to show that Γ_u is continuous when $(x_0, t_0) \notin \mathcal{C}$. By substracting L from u we may assume that $u(x_i, t_i) = 0$. and therefore $u(x_i, t_i) = \Gamma_u(x_i, t_i) = 0$ (notice that this implies that $\Gamma_u(x_0, t_0) = 0$). Since (7.8) holds by reviewing the previous argument, we notice that to prove the continuity of Γ_u at (x_0, t_0) it is enough to establish (7.10), actually it is enough to show that

$$\lim_{t \uparrow t_0} \Gamma_u(x_0, t) \leq \Gamma_u(x_0, t_0).$$

Case 1. Suppose $(x_i, t_i) \in Q, t_i < t_0$. Then

$$\begin{aligned} \lim_{\Delta t \to 0^+} \Gamma_u(x_i, t_0 - \Delta t) \leq \lim_{\Delta t \to 0^+} \Gamma_u(x_i, t_i - \Delta t) &\leq \lim_{\Delta t \to 0^+} u(x_i, t_i - \Delta t) \\ &= u(x_i, t_i) = 0. \end{aligned}$$

Case 2. $(x_i, t_i) \in \partial_p Q, t_i < t_0$. For each $\epsilon > 0$ there exist Δx and h so that $|\Delta x| < \epsilon$, $|h| < \epsilon$ and such that $(x_i + \Delta x, t_i + h) \in Q$ and $u(x_i + \Delta x, t_i + h) < u(x_i, t_i) + \epsilon = \epsilon$. Therefore

$$\lim_{\Delta t \to 0^+} \Gamma_u(x_i + \Delta x, t_0 - \Delta t) \leq \Gamma_u(x_i + \Delta x, t_i + h) \leq u(x_i + \Delta x, t_i + h) \leq \epsilon.$$

Case 3. $(x_i, t_i) \in \partial_p Q, t_i = t_0$. For any ϵ there exists $|\Delta x_i| < \epsilon$ such that $(x_i + \Delta x_i, t_0) \in Q$ and $u(x_i + \Delta x_i, t_0) < u(x_i, t_i) + \epsilon$. Hence

$$\lim_{\Delta t \to 0^+} \Gamma_u(x_i + \Delta x_i, t_0 - \Delta t) \leq \lim_{\Delta t \to 0^+} u(x_i + \Delta x_i, t_0 - \Delta t) = u(x_i + \Delta x_i, t_0)$$

Wrapping up, if $x_0 = \sum_{i=1}^{n+1} \lambda_i x_i$ then for each (x_i, t_i) we have

$$\lim_{\Delta t \to 0^+} \Gamma_u(x_i + \Delta x_i, t_0 - \Delta t) \le \epsilon,$$

with some Δx_i possibly equal zero. Therefore

$$\lim_{\Delta t \to 0^+} \Gamma_u(\sum \lambda_i (x_i + \Delta x_i), t_0 - \Delta t) \le \sum \lambda_i \lim_{\Delta t \to 0^+} \Gamma_u(x_i + \Delta x_i, t_0 - \Delta t) \le \epsilon.$$

Since $(x_0, t_0) \in Q$, and all Δx_i, Δt are small, by convexity of Γ_u it follows from [**GH00**, Lemma 1.1] that Γ_u is locally Lipschitz in x with a Lipschitz constant uniform in t. Hence

$$\begin{aligned} &\lim_{\Delta t \to 0^+} \Gamma_u(x_0, t_0 - \Delta t) \\ &\le K |\sum \lambda_i \Delta x_i| + \lim_{\Delta t \to 0^+} \Gamma_u(x_0 + \sum \lambda_i \Delta x_i, t_0 - \Delta t) \le (K+1)\, \epsilon. \end{aligned}$$

That is, $\lim_{\Delta t \to 0^+} \Gamma_u(x_0, t_0 - \Delta t) = 0$, and hence Γ_u is continuous at (x_0, t_0).

□

PROPOSITION 7.5 (Regularity of Γ_u). *Let $u \in C^{2,1}\left(\overline{Q}\right)$, where Q is a bowl-shaped domain, $u = 0$ on $\partial_p Q$, and $u < 0$ in Q. Assume in addition that Q is defined by $Q = \{(x,t) : \Phi(x,t) < 0, t < T\}$ where Φ is p-convex, and if $\Phi(x_0, t_0) = 0$ then there exist $c > 0$ so that $\Phi(x_0 + \Delta x, t_0 - \Delta t) \le 0$, for $|\Delta x| \le c \Delta t^2$. Then Γ_u is locally in $W^{2,1}_\infty(Q)$ and $\mathcal{M}\Gamma_u \le \chi_{\mathcal{C}}\, \mathcal{M}u$, where $\chi_{\mathcal{C}}$ denotes the characteristic function of the contact set $\mathcal{C}$.*

PROOF. If $(x_0, t_0) \in \mathcal{C} \cap Q$ then the proposition follows from lemma (7.2).

Suppose that $(x_0, t_0) \notin \mathcal{C}$. Let $K \Subset Q$ be compact such that $(x_0, t_0) \in K$, and L a supporting hyperplane as in lemma (7.3) and (x_i, t_i) the corresponding points.

Step 1. There exist a compact $K_0 \Subset Q$ and a constant $C > 0$, both depending only on K and u, and at least one (x_i, t_i), say (x_1, t_1), such that $(x_1, t_1) \in K_0$ with $\lambda_1 \ge C$.

Let $-\delta_0 = \max_K u < 0$, and take $K_0 \Subset Q$ such that $u > -\dfrac{\delta_0}{n+1}$ in $\overline{Q} \setminus K_0$. Since $L(x_0) = L\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = \sum_{i=1}^{n+1} \lambda_i\, L(x_i)$, we get $-\delta_0 \ge u(x_0, t_0) \ge L(x_0) = \sum_{i=1}^{n+1} \lambda_i\, u(x_i, t_i)$. Hence $\delta_0 \le (n+1) \max_i \lambda_i |u(x_i, t_i)|$, and assuming the maximum is attained when $i = 1$, we get $\delta_0 \le (n+1)\, \lambda_1 |u(x_1, t_1)|$. If $\lambda_1 \le 1$ then $u(x_1, t_1) \le -\dfrac{\delta_0}{n+1}$, that is $(x_1, t_1) \in K_0$ and consequently $\lambda_1 \ge \dfrac{\delta_0}{(n+1) \max_Q |u|}$.

Step 2. $\Gamma_u(x, t_0)$ is $C^{1,1}$ in x.

Let $\Delta x < \operatorname{dist}(K, \partial_p Q)$. By (7.4), we write

$$\begin{aligned}
\Gamma_u(x_0 + \Delta x, t_0) &= \Gamma_u\left(\sum_{i>1} \lambda_i\, x_i + \lambda_1\left(x_1 + \frac{\Delta x}{\lambda_1}\right), t_0\right)\\
&\leq \sum_{i>1} \lambda_i\, \Gamma_u(x_i, t_0) + \lambda_1\, \Gamma_u\left(\left(x_1 + \frac{\Delta x}{\lambda_1}\right), t_0\right)\\
&\leq \sum_{i>1} \lambda_i\, \Gamma_u(x_i, t_0) + \lambda_1\, \Gamma_u\left(\left(x_1 + \frac{\Delta x}{\lambda_1}\right), t_1\right)\\
&\leq \sum_{i>1} \lambda_i\, L(x_i) + \lambda_1\left(L\left(x_1 + \frac{\Delta x}{\lambda_1}\right) + M\left|\frac{\Delta x}{\lambda_1}\right|^2\right), \qquad \text{by lemma (7.2)}\\
&= L\left(\sum_{i=1}^{n+1} \lambda_i x_i + \Delta x\right) + \frac{M}{\lambda_1}|\Delta x|^2 = L(x_0 + \Delta x) + \frac{M}{\lambda_1}|\Delta x|^2.
\end{aligned}$$

Step 3. $\Gamma_u(x_0, t)$ is Lipschitz in t, $t \leq t_0$.

By assumption $(x_i + \Delta x_i, t_0 - \Delta t) \in Q$ with $|\Delta x| < C\Delta t$. From (7.4), we have

$$\begin{aligned}
&\Gamma_u\left(x_0 + \sum_i \lambda_i \Delta x_i, t_0 - \Delta t\right)\\
&= \Gamma_u\left(\sum_i \lambda_i(x_i + \Delta x_i), t_0 - \Delta t\right) \leq \sum_i \lambda_i\, \Gamma_u((x_i + \Delta x_i), t_0 - \Delta t)\\
&\leq \sum_i \lambda_i\, \Gamma_u((x_i + \Delta x_i), t_i - \Delta t) \leq \sum_i \lambda_i\, \left(L(x_i + \Delta x_i) + M(|\Delta x_i|^2 + \Delta t)\right)\\
&\leq L\left(\sum_i \lambda_i(x_i + \Delta x_i)\right) + C\, M\, \Delta t = L\left(x_0 + \sum_i \lambda_i \Delta x_i\right) + C\, M\, \Delta t.
\end{aligned}$$

On the other hand, since Γ_u is bounded in Q and convex in x by [**GH00**, Lemma 1.1] we have that

$$|\Gamma_u(x_1, t) - \Gamma_u(x_2, t)| \leq C\, |x_1 - x_2|,$$

for $(x_1, t), (x_2, t)$ in a neighborhood of (x_0, t_0). Therefore,

$$\begin{aligned}
&\Gamma_u(x_0, t_0 - \Delta t)\\
&= \Gamma_u(x_0, t_0 - \Delta t) - \Gamma_u(x_0 + \sum_i \lambda_i \Delta x_i, t_0 - \Delta t) + \Gamma_u(x_0 + \sum_i \lambda_i \Delta x_i, t_0 - \Delta t)\\
&\leq C\, |\sum_i \lambda_i \Delta x_i| + L(x_0 + \sum_i \lambda_i \Delta x_i) + CM\Delta t\\
&\leq L(x_0 + \sum_i \lambda_i \Delta x_i) + 2\, C\, M\, \Delta t\\
&\leq L(x_0) + C'\, M\, \Delta t,
\end{aligned}$$

In fact, let $x = \sum \mu_i x_i$ with $\mu_i \geq 0$ and $\sum \mu_i = 1$. Since $\Gamma_u(x_i, t_i) = L(x_i)$ and $\Gamma_u(x,t) \geq \Gamma_u(x, t_0) \geq L(x)$ for all x and $t \leq t_0$, we get

$$\begin{aligned} L(x) &\leq \Gamma_u\left(\sum \mu_i x_i, t_0\right) \leq \sum \mu_i \Gamma_u(x_i, t_0) \\ &\leq \sum \mu_i \Gamma_u(x_i, t_i) = \sum \mu_i L(x_i) = L(x), \end{aligned}$$

and so $\Gamma_u(\sum \mu_i x_i, t_0) = L(\sum \mu_i x_i)$ which proves step 4.

Consequently, $\det D_x^2 \Gamma_u(x, t_0) = 0$ for x in the simplex generated by $\{x_i\}$ and in particular for $x = x_0$. This completes the proof of the proposition. □

Remark 7.6. Let Q be as in proposition (7.5), so $\partial_p Q = \{(x,t) : \Phi(x,t) = 0\}$. Then Γ_u is continuous up to the boundary of Q and $\Gamma_u = 0$ on $\partial_p Q$. Let $(x_0, t_0) \in \partial_p Q$. Let $\Delta t > 0$ be small and $\ell(x) = D_x \Phi(x_{\Delta t}, t_0 + \Delta t) \cdot (x - x_{\Delta t})$ a supporting hyperplane to $Q \cap \{t = t_0 + \Delta t\}$ with $(x_{\Delta x}, t_0 + \Delta t) \in \partial_p Q$. Choose K very negative and $\epsilon > 0$ small so that $K\ell(x) - \epsilon \leq u(x,t)$ in $Q \cap \{t \leq t_0 + \Delta t\}$. Hence $K\ell(x) - \epsilon \leq \Gamma_u(x,t) \leq u(x,t)$ in $Q \cap \{t \leq t_0 + \Delta t\}$. Fixing for a moment Δt and $x_{\Delta t}$, since Φ is Lipschitz we get

$$-K\,C\,|x - x_{\Delta t}| - \epsilon \leq \Gamma_u(x,t) \leq u(x,t),$$

and now letting $(x,t) \to (x_0, t_0)$ yields

$$-K\,C\,|x_0 - x_{\Delta t}| - \epsilon \leq \liminf_{(x,t)\to(x_0,t_0)} \Gamma_u(x,t) \leq 0.$$

Letting $\Delta t \to 0$ we get $x_{\Delta t} \to x_0$ and consequently

$$-\epsilon \leq \liminf_{(x,t)\to(x_0,t_0)} \Gamma_u(x,t) \leq 0,$$

and so $\Gamma_u(x_0, t_0) = 0$.

Corollary 7.7. *Let $u \in C(\bar{Q}) \cap C^{2,1}(Q)$ with $u = 0$ on $\partial_p Q$, $u < 0$ in Q bowl-shaped domain whose defining function is Lipschitz in x. Then $\Gamma_u \in C(\bar{Q})$ and $\Gamma_u = 0$ on $\partial_p Q$ and*

$$\mathcal{M}\Gamma_u \leq \chi_{\mathcal{C}}\, \mathcal{M}u,$$

where $\chi_{\mathcal{C}}$ denotes the characteristic function of the contact set $\mathcal{C}$.

Proof. The first part follows from the previous remark.

Let ϕ be a mollifier in $\mathbb{R}$ and

$$f_\epsilon(x) = \int_{|y|\leq 1} \phi(y)\, g_\epsilon(x - \frac{\epsilon}{3} y)\, dy,$$

where

$$g_\epsilon(x) = \begin{cases} 0, & \text{for } x > -4\epsilon/3 \\ 5\left(x + \dfrac{4\epsilon}{3}\right), & \text{for } -5\epsilon/3 < x < -4\epsilon/3 \end{cases}$$

Then $f_\epsilon \in C^\infty$ and

$$f_\epsilon(x) = \begin{cases} 0, & \text{for } x > -\epsilon \\ \uparrow, & \text{for } -2\epsilon \le x \le -\epsilon \\ x, & \text{for } x < -2\epsilon. \end{cases}$$

Let $u_\epsilon = f_\epsilon(u) \to u$ in $C(\bar{Q})$. Take $Q_\epsilon \uparrow Q$, where Q_ϵ is a smooth bowl-shaped domain such that $u_\epsilon \le 0$ in a small neighborhood of $\partial_p Q_\epsilon$. Then $u_\epsilon \in C^{2,1}(\bar{Q})$ and applying proposition (7.5) to u_ϵ yields $\mathcal{M}\Gamma_{u_\epsilon,Q_\epsilon} \le \mathcal{M}u_\epsilon\ \chi_{\{u_\epsilon=\Gamma_{u_\epsilon,Q_\epsilon}\}}$. Since $\Gamma_{u_\epsilon,Q_\epsilon} \to \Gamma_{u,Q}$ and $\mathcal{M}u_\epsilon = \mathcal{M}u$ for $K \Subset Q$ compact, we obtain $\mathcal{M}\Gamma_{u,Q} \le \mathcal{M}u\ \chi_{\{u=\Gamma_{u,Q}\}}$.

□

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