SOME WEIGHTED NORM INEQUALITIES FOR A ONE-SIDED VERSION OF g_λ^*

L. de Rosa and C. Segovia

Abstract.

In this paper, we study the boundedness of the one-sided operator $g^+_{\lambda,\varphi}$ between the weighted spaces $L^p(M^-w)$ and $L^p(w)$ for every weight w. If $\lambda = 2/p$ whenever $1 < p < 2$, and in the case $p = 1$ for $\lambda > 2$, we proved the weak-type of $g^+_{\lambda,\varphi}$. For every $\lambda > 1$ and $p = 2$, or $\lambda > 2/p$ and $1 < p < 2$, the boundedness of this operator is obtained. For the case when $p > 2$ and $\lambda > 1$, we obtained the boundedness of $g^+_{\lambda,\varphi}$ from $L^p((M^-)^{[p/2]+1}w)$ to $L^p(w)$, where $(M^-)^{[p/2]+1}$ denotes the operator M^- iterated $[p/2] + 1$ times.

1. Notations and Definitions

As usual, $\mathcal{S}$ denotes de class of all those $C^\infty-$functions defined on $I\!R$ such that

$$\sup_{x\in I\!R} |x^m(D^n\varphi)(x)| < \infty,$$

for all non-negative integers m and n. We also consider the space C_0^∞ of all $C^\infty-$functions defined on $I\!R$ with compact support.

If $E \subset I\!R$ is a Lebesgue measurable set, we denote its Lebesgue measure by $|E|$, and the characteristic function of E by $\chi_E(x)$.

Let f be a measurable function defined on $I\!R$, the one-sided Hardy-Littlewood maximal functions M^-f and M^+f are given by

$$M^-f(x) = \sup_{h>0} \frac{1}{h}\int_{x-h}^{x} |f(t)|\,dt \quad \text{and} \quad M^+f(x) = \sup_{h>0} \frac{1}{h}\int_{x}^{x+h} |f(t)|\,dt.$$

A weight w is a measurable and non-negative function defined on $I\!R$. If $E \subset I\!R$ is a measurable set, we denote its $w-$measure by $w(E) = \int_E w(t)dt$. Given $p \geq 1$, $L^p(w)$ is the space of all measurable functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{-\infty}^{\infty} |f(x)|^p w(x)dx\right)^{1/p} < \infty.$$

1991 *Mathematics Subject Classification.* Primary 42B25, 26A33.

Key words and phrases. One-sided maximal functions, Littlewood-Paley theory, one-sided

If $w=1$, we simply write L^p and $\|f\|_{L^p}$.

We shall say that a function $B:[0,\infty)\longrightarrow[0,\infty)$ is a Young function, if it is continuous, convex, increasing and satisfies $\lim_{t\to\infty}B(t)=\infty$. The Luxemburg norm of a function f is given by

$$\|f\|_B=\inf\Big\{\lambda>0:\int B\Big(\frac{|f|}{\lambda}\Big)\le 1\Big\},$$

and the average over an interval I is:

$$\|f\|_{B,I}=\inf\Big\{\lambda>0:\frac{1}{|I|}\int_I B\Big(\frac{|f|}{\lambda}\Big)\le 1\Big\}.$$

The one-sided maximal operators associated to B are defined as

$$M_B^+(f)(x)=\sup_{h>0}\|f\|_{B,[x,x+h]}\quad\text{ and }\quad M_B^-(f)(x)=\sup_{h>0}\|f\|_{B,[x-h,x]}.$$

Let φ belong to $\mathcal{S}$, suported on $(-\infty,0]$ with $\int\varphi(x)dx=0$. For every $\lambda>1$, the one-sided operator $g^+_{\lambda,\varphi}$ was defined in [RoSe], as

$$g^+_{\lambda,\varphi}(f)(x)=\left(\int_0^\infty\int_x^\infty\left(\frac{t}{t+y-x}\right)^\lambda|f*\varphi_t(y)|^2\frac{dydt}{t^2}\right)^{1/2}.$$

Throughout this paper the letter C will always mean a positive constant not necessarily the same at each occurrence. If $1<p<\infty$ then p' denotes its conjugate exponent: $p+p'=pp'$.

2. Statement of the results

In [CW], S. Chanillo and R. Wheeden obtained the boundedness of the Area Integral between the spaces $L^p(Mw)$ and $L^p(w)$ when $1<p\le 2$. For $p=2$ and $\lambda>1$, if the support of φ is compact, they showed in Lemma (1.1) that the operator $g^*_{\lambda,\varphi}$ maps $L^2(Mw)$ into $L^2(w)$. We shall give, in Theorem A, a one sided-version of this result without the restriction about support of φ. For $1<p<2$ and $\lambda=2/p$, in order to prove Theorem B, we partially borrow some arguments due to C. Fefferman (see [F]). As a consequence of Theorem A and Theorem B, for $1<p\le 2$ and $\lambda>2/p$, we shall obtain, in Theorem C, the boundedness of $g^+_{\lambda,\varphi}$ between $L^p(M^-w)$ and $L^p(w)$. For $p>2$, known techniques (see [P]), allow us to prove Theorem D.

Next, we state the already mentionated Theorems A, B, C and D.

Theorem A. *Let φ belong to $\mathcal{S}$, supported on $(-\infty, 0]$, and $\int \varphi(x)dx = 0$. Then, for every $\lambda > 1$,*

$$\left(\int_{-\infty}^{\infty} g^{+}_{\lambda,\varphi}(f)(x)^2 w(x)dx\right)^{1/2} \leq C_{\lambda,\varphi}\left(\int_{-\infty}^{\infty} |f(x)|^2 M^{-}w(x)dx\right)^{1/2},$$

with a constant $C_{\lambda,\varphi}$ not depending on f.

Theorem B. *Let φ belong to $\mathcal{S}$, supported on $(-\infty, 0]$, and $\int \varphi(x)dx = 0$. Let $\lambda > 2$ if $p = 1$, and $\lambda = \frac{2}{p}$ whenever $1 < p < 2$. Then, there exists a constant $C_{p,\lambda,w,\varphi}$ such that*

$$w\left(\{x : g^{+}_{\lambda,\varphi}(f)(x) > \mu\}\right) \leq \frac{C_{p,\lambda,w,\varphi}}{\mu^p}\int_{-\infty}^{\infty} |f(x)|^p M^{-}w(x)dx,$$

holds for every function f, and $\mu > 0$.

Theorem C. *Let φ belong to $\mathcal{S}$, supported on $(-\infty, 0]$, and $\int \varphi(x)dx = 0$. Let $1 < p \leq 2$. If $\lambda > 2/p$, then there exists a constant $C_{p,\lambda,w,\varphi}$ such that*

$$\int_{-\infty}^{\infty} g^{+}_{\lambda,\varphi}(f)(x)^p w(x)dx \leq C_{p,\lambda,w,\varphi}\int_{-\infty}^{\infty} |f(x)|^p M^{-}w(x)dx,$$

for every function f.

Theorem D. *Let φ belong to $\mathcal{S}$, supported on $(-\infty, 0]$, and $\int \varphi(x)dx = 0$. Let $\lambda > 1$ and $p > 2$. Then, there exists a constant $C_{p,\lambda,w,\varphi}$ such that*

$$\int_{-\infty}^{\infty} g^{+}_{\lambda,\varphi}(f)(x)^p w(x)dx \leq C_{p,\lambda,w,\varphi}\int_{-\infty}^{\infty} |f(x)|^p (M^{-})^{[p/2]+1}(w)(x)dx.$$

3. Proof of the results

The following lemma and remark will be used in the proof of Theorem A.

Lemma 1. *Let φ belong to C_0^{∞}, supported on the interval $[-2^s, 0]$, $s \geq 0$, and $\int \varphi(x)dx = 0$. Then,* *with a constant C_λ depending neither on f nor φ.*

Proof. By Fubini's Theorem, we have that

$$\begin{aligned}\int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^2 w(x)dx &= \int_{-\infty}^{\infty}\int_0^\infty\int_x^\infty \left(\frac{t}{t+y-x}\right)^\lambda |f*\varphi_t(y)|^2\frac{dydt}{t^2}w(x)dx\\ &= \int_0^\infty\int_{-\infty}^{\infty}|f*\varphi_t(y)|^2\left(\frac{1}{t}\int_{-\infty}^{y}\left(\frac{t}{t+y-x}\right)^\lambda w(x)dx\right)\frac{dydt}{t}.\end{aligned}$$

For each integer k, we consider the set

$$A_k = \left\{(y,t) : 2^{k-1} < \frac{1}{t}\int_{-\infty}^{y}\left(\frac{t}{t+y-x}\right)^\lambda w(x)dx \le 2^k\right\}.$$

Then,

$$\int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^2 w(x)dx \le \sum_{k\in\mathbb{Z}} 2^k\int_0^\infty\int_{-\infty}^{\infty}|f*\varphi_t(y)|^2\chi_{A_k}(y,t)\frac{dydt}{t}. \tag{2}$$

For every (y,t) belonging to A_k and $y\le z\le y+2^s t$, we have

$$\frac{1}{t}\int_{-\infty}^{z}\left(\frac{t}{t+z-x}\right)^\lambda w(x)dx \ge \frac{1}{2^{(s+1)\lambda}}\frac{1}{t}\int_{-\infty}^{y}\left(\frac{t}{t+y-x}\right)^\lambda w(x)dx > \frac{2^{k-1}}{2^{(s+1)\lambda}}.$$

On the other hand, since $\lambda>1$, there exists a constant C_λ such that for every z,

$$\frac{1}{t}\int_{-\infty}^{z}\left(\frac{t}{t+z-x}\right)^\lambda w(x)dx \le C_\lambda M^- w(z).$$

Therefore, if $(y,t)\in A_k$ and $y\le z\le y+2^s t$ then z belongs to $E_k=\{z : M^-w(z)\ge \frac{C_\lambda}{2^{(s+1)\lambda}}2^{k-1}\}$. Taking into account that support $(\varphi)\subset[-2^s,0]$, we get

$$f*\varphi_t(y) = \int f(z)\chi_{E_k}(z)\varphi_t(y-z)dz = (f\chi_{E_k}*\varphi_t)(y).$$

Then, by Plancherel's and Fubini's Theorems, we have that (2) is majorized by

$$\sum_{k\in\mathbb{Z}} 2^k\int_0^\infty\int_{-\infty}^{\infty}|f\chi_{E_k}*\varphi_t(y)|^2\frac{dydt}{t} = \sum_{k\in\mathbb{Z}} 2^k\int_{-\infty}^{\infty}|\widehat{f\chi_{E_k}}(y)|^2\int_0^\infty|\widehat{\varphi}(ty)|^2\frac{dt}{t}dy.$$

The inner integral is bounded by $C_\varphi=\int_{-\infty}^{\infty}|\widehat{\varphi}(t)|^2\frac{dt}{|t|}$. Thus, applying Plancherel's Theorem again, we get

$$\int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^2 w(x)dx \le C_\varphi\int_{-\infty}^{\infty}|f(y)|^2\sum_{k\in\mathbb{Z}}2^k\chi_{E_k}(y)dy.$$

Finally, we observe that by definition of E_k,

$$\sum_{k\in\mathbb{Z}}2^k\chi_{E_k}(y)\le C_\lambda 2^{s\lambda}M^-w(y),$$

for almost every y, ending the proof of the lemma. $\square$

Remark. *We observe that if* φ *belongs to* $\mathcal{S}$ *and* $\int \varphi(x)dx = 0$, *then*

$$\int_{-\infty}^{\infty} |\widehat{\varphi}(s)|^2 \frac{ds}{|s|} \leq 4\pi^2 \left(\int_{-\infty}^{\infty} |s||\varphi(s)|ds \right)^2 + \int_{-\infty}^{\infty} |\varphi(s)|^2 ds. \tag{3}$$

In fact, since $\int \varphi(x)dx = 0$, we have,

$$|\widehat{\varphi}(s)| = \left| \int_{-\infty}^{\infty} \varphi(t)(e^{-2\pi ist} - 1)dt \right| \leq 2\pi|s| \int_{-\infty}^{\infty} |t||\varphi(t)|dt.$$

In consequence,

$$\int_{|s|\leq 1} |\widehat{\varphi}(s)|^2 \frac{ds}{|s|} \leq 4\pi^2 \left(\int_{-\infty}^{\infty} |s||\varphi(s)|ds \right)^2.$$

On the other hand, in virtue of Plancherel's Theorem

$$\int_{|s|\geq 1} |\widehat{\varphi}(s)|^2 \frac{ds}{|s|} \leq \int_{-\infty}^{\infty} |\widehat{\varphi}(s)|^2 ds \leq \int_{-\infty}^{\infty} |\varphi(s)|^2 ds,$$

which shows that (3) holds.

Let η be a non-negative and $C_0^\infty-$ function with support contained in $[-2,-1]$ and $\int \eta(x)dx = 1$. For every non-negative integer k, let $\eta_k(x) = 2^{-k}\eta(2^{-k}x)$. We define

$$\theta(x) = \int_{\frac{|x|}{2} \leq |t| \leq |x|} \eta(t)dt.$$

The function θ belongs to C_0^∞ and its support is contained in $[-4,-1] \cup [1,4]$. For every positive integer k, let

$$\theta_k(x) = \theta(2^{-k+1}x),$$

and for $k = 0$, let

$$\theta_0(x) = 1 - \int_{|y|\leq|x|} \eta(y)dy.$$

Then, $\sum_{k=0}^{\infty} \theta_k(x) = 1$ for every x. Given a function φ belonging to $\mathcal{S}$, supported on $(-\infty, 0]$ and $\int \varphi(x)dx = 0$, we define

$$a_k = \int \sum_{h=0}^{k} \theta_h(y)\varphi(y)dy, \quad k \geq 0$$

and $a_{-1} = 0$. For every non-negative integer k, let ρ_k be given by

It is easy to check that support$(\rho_k) \subset [-2^{k+1}, -2^{k-1}]$ for $k \geq 1$, and support$(\rho_0) \subset [-2, 0]$. Besides, $\int \rho_k(x)dx = 0$ for every $k \geq 0$, and $\sum_{k=0}^{\infty} = \varphi$. We shall show that for every $N > 2$,

$$C_{\rho_k} = \int_{-\infty}^{\infty} |\widehat{\varphi_k}(s)|^2 \frac{ds}{|s|} \leq C_{N,\varphi} 2^{-2k(N-2)}, \tag{5}$$

holds. By definition of ρ_k,

$$\begin{aligned} \left(\int_{-\infty}^{\infty} |\rho_k(x)|^2 dx \right)^{1/2} \leq & \left(\int_{-\infty}^{\infty} |\theta_k(x)\varphi(x)|^2 dx \right)^{1/2} \\ & + |a_{k-1}| \left(\int_{-\infty}^{\infty} |\eta_{k-1}(x)|^2 dx \right)^{1/2} + |a_k| \left(\int_{-\infty}^{\infty} |\eta_k(x)|^2 dx \right)^{1/2}. \end{aligned} \tag{6}$$

Since $0 \leq \theta_k(x) \leq 1$, support$(\theta_k \varphi) \subset [-2^{k+1}, -2^{k-1}]$ for $k \geq 1$, and support$(\theta_0 \varphi) \subset [-2, 0]$, we have that

$$\begin{aligned} \left(\int_{-\infty}^{\infty} |\theta_k(x)\varphi(x)|^2 dx \right)^{1/2} \leq & \left(\int_{\mathrm{supp}(\theta_k \varphi)} \frac{C_{N,\varphi}}{(1+|x|)^{2N}} dx \right)^{1/2} \\ \leq & C_{N,\varphi} 2^{-k(N-1/2)}. \end{aligned} \tag{7}$$

By definition of a_k, and taking into account that $\int \varphi(x)dx = 0$, we get

$$\begin{aligned} |a_k| = & \left| - \int \sum_{h=k+1}^{\infty} \theta_h(y)\varphi(y)dy \right| \leq \int_{|y|\geq 2^k} |\varphi(y)|dy \\ \leq & C_{N,\varphi} \int_{|y|\geq 2^k} \frac{dy}{(1+|y|)^N} \leq C_{N,\varphi} 2^{-k(N-1)}. \end{aligned}$$

Thus,

$$|a_k| \left(\int_{-\infty}^{\infty} |\eta_k(x)|^2 dx \right)^{1/2} = \frac{|a_k|}{2^{k/2}} \left(\int_{-\infty}^{\infty} |\eta(x)|^2 dx \right)^{1/2} \leq C_{N,\varphi} 2^{-k(N-1/2)}. \tag{8}$$

Then, by (6), (7) and (8),

$$\int_{-\infty}^{\infty} |\rho_k(x)|^2 dx \leq C_{N,\varphi} 2^{-2k(N-1/2)}.$$

Simple calculations shows that

$$\int^{\infty}$$

Now, using (3) we obtain (5).

Proof of Theorem A. We consider the sequence of functions $\{\rho_k, k \geq 0\}$ defined in (4). Since $\sum_{k=0}^{\infty} \rho_k = \varphi$ and $\sum_{k=0}^{\infty} \chi_{supp(\rho_k)}(x) \leq 3$, we have that

$$f * \varphi_t(y) = \sum_{k=0}^{\infty} f * (\varphi_k)_t(y),$$

for every y. Then,

$$\text{(9)} \qquad \left(\int_{-\infty}^{\infty} g_{\lambda,\varphi}^{+}(f)(x)^2 w(x) dx \right)^{1/2}$$

$$\leq \sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{x}^{\infty} \left(\frac{t}{t+y-x} \right)^{\lambda} |f * (\rho_k)_t(y)|^2 \frac{dydt}{t^2} w(x) dx \right)^{1/2}$$

$$= \sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} g_{\lambda,\rho_k}^{+}(f)(x)^2 w(x) dx \right)^{1/2}.$$

Keeping in mind that support$(\rho_k) \subset [-2^{k+1}, 0]$ and $\int \rho_k(x) dx = 0$, we can apply Lemma 1. Then, by the estimation (5) with $N > \lambda + 2$, we get that (9) is bounded by a constant times

$$\sum_{k=0}^{\infty} 2^{(k+1)\lambda/2} \left(\int_{-\infty}^{\infty} |\widehat{\rho_k}(t)|^2 \frac{dt}{|t|} \right)^{1/2} \left(\int_{-\infty}^{\infty} |f(x)|^2 M^{-} w(x) dx \right)^{1/2}$$

$$\leq C_{\lambda,\varphi} \left(\int_{-\infty}^{\infty} |f(x)|^2 M^{-} w(x) dx \right)^{1/2}. \quad \square$$

In order to prove Theorem B, we shall need the following one-sided version of Fefferman-Stein type inequality and Lemma 11.

Lemma 10. *There exists a positive constant C, such that*

$$w\left(\{x : M^{+}(f)(x) > \mu\} \right) \leq \frac{C}{\mu} \int_{-\infty}^{\infty} |f(x)| M^{-} w(x) dx,$$

for every function f, and $\mu > 0$.

Proof. The proof of this lemma is similar to the proof of Theorem 1 in [M] page 693, and shall not be given. $\square$

Lemma 11. *Let $I=(\alpha,\beta)$ a bounded interval, $1<\lambda<2$, and $k\geq 4$. Then, there exists a constant $C_{\lambda,k}$ such that for every $x<\alpha-2|I|$, we have*

$$\int_0^\infty\int_x^{\alpha-2|I|}\left(\frac{t}{t+y-x}\right)^\lambda\left(\frac{t}{t+\alpha-y}\right)^k\frac{dydt}{t^4}\leq C_{\lambda,k}\frac{|I|^{\lambda-2}}{(\alpha-x)^\lambda}.$$

Proof. Changing the variables (y,t) by $z=\frac{\alpha-y}{t}$ and $u=\frac{\alpha-x}{t}$, we obtain that

$$\int_0^\infty\int_{\alpha-x\geq\alpha-y\geq 2|I|}\left(\frac{1}{1+\frac{y-x}{t}}\right)^\lambda\left(\frac{1}{1+\frac{\alpha-y}{t}}\right)^k\frac{dydt}{t^4}$$

$$=\frac{1}{(\alpha-x)^2}\int_0^\infty\int_{u\geq z\geq\frac{2|I|u}{\alpha-x}}\frac{1}{(1+u-z)^\lambda}\frac{1}{(1+z)^k}ududz.$$

We denote $A=\frac{2|I|}{\alpha-x}$. Applying Fubini's Theorem, it is enough to show that

$$\int_0^\infty\frac{1}{(1+z)^k}\int_{z\leq u\leq\frac{1}{A}z}\frac{u}{(1+u-z)^\lambda}dudz\leq C_{\lambda,k}A^{\lambda-2}.$$

Recalling that $1<\lambda<2$, we have

$$\int_0^\infty\frac{1}{(1+z)^k}\int_{z\leq u\leq\frac{1}{A}z,u-z>u/2}\frac{u}{(1+u-z)^\lambda}dudz$$

$$\leq\int_0^\infty\frac{1}{(1+z)^k}\int_0^{\frac{1}{A}z}\left(\frac{2}{u}\right)^\lambda ududz$$

$$=C_\lambda\int_0^\infty\frac{1}{(1+z)^k}\left(\frac{z}{A}\right)^{2-\lambda}dz=A^{\lambda-2}.$$

Since $k\geq 4$, $A<1$ and $\lambda<2$, it follows that

$$\int_0^\infty\frac{1}{(1+z)^k}\int_{z\leq u\leq\frac{1}{A}z,u-z\leq u/2}\frac{u}{(1+u-z)^\lambda}dudz\leq\int_0^\infty\frac{1}{(1+z)^k}\int_0^{2z}ududz$$

$$=2\int_0^\infty\frac{z^2}{(1+z)^k}dz\leq C_kA^{\lambda-2},$$

which ends the proof of this lemma. $\square$

Proof of Theorem B. By an argument of density it is enough to consider f belonging to both

$f \in L^p$, each I_j is a bounded interval, and it is well known (see [HSt], pp. 421-424), that

$$\frac{1}{|I_j|}\int_{I_j} |f(x)|^p dx = \mu^p. \tag{12}$$

Given $I_j = (\alpha_j, \beta_j)$, we denote $I_j^- = (\alpha_j - 4|I_j|, \alpha_j)$. By (12), we have that

$$w(I_j^-) = \frac{1}{\mu^p}\int_{I_j} |f(x)|^p \frac{w(I_j^-)}{|I_j|} dx \le \frac{5}{\mu^p}\int_{I_j} |f(x)|^p M^- w(x) dx.$$

Therefore, if we denote $\tilde{\Omega} = \cup_{j\ge 1} I_j \cup I_j^-$, applying Lemma 10 we obtain that

$$\begin{aligned} w(\tilde{\Omega}) &\le w(\Omega) + \sum_{j\ge 1} w(I_j^-) \\ &\le \frac{C}{\mu^p}\int_{-\infty}^{\infty} |f(x)|^p M^- w(x) dx + \frac{5}{\mu^p}\sum_{j\ge 1}\int_{I_j} |f(x)|^p M^- w(x) dx \\ &\le \frac{C}{\mu^p}\int_{-\infty}^{\infty} |f(x)|^p M^- w(x) dx. \end{aligned}$$

In consequence, it is enough to prove that

$$w\left(\{x \notin \tilde{\Omega} : g^+_{\lambda,\varphi}(f)(x) > \mu\}\right) \le \frac{C}{\mu^p}\int_{-\infty}^{\infty} |f(x)|^p M^- w(x) dx. \tag{13}$$

We define $g(x) = f(x)\chi_{\Omega^c}(x) + \sum_{j\ge 1}\left(\frac{1}{|I_j|}\int_{I_j} f\right)\chi_{I_j}(x)$, and

$$b_j(x) = \left(f(x) - \frac{1}{|I_j|}\int_{I_j} f\right)\chi_{I_j}(x)$$

for every $j \ge 1$. Then, $f = g + b$ where $b = \sum_{j\ge 1} b_j$.

By Chebyshev's inequality and applying Theorem A, we get

$$\begin{aligned} w\left(\{x \notin \tilde{\Omega} : g^+_{\lambda,\varphi}(f)(x) > \mu\}\right) &\le \frac{1}{\mu^2}\int_{\tilde{\Omega}^c} g^+_{\lambda,\varphi}(g)(x)^2 w(x) dx \\ &\le \frac{C}{\mu^2}\int_{-\infty}^{\infty} |g(x)|^2 M^-(w\chi_{\tilde{\Omega}^c})(x) dx \\ &= \frac{C}{\mu^2}\int_{-\infty}^{\infty} |g(x)|^{2-p}|g(x)|^p M^-(w\chi_{\tilde{\Omega}^c})(x) dx. \end{aligned} \tag{14}$$

We observe that $|g(x)| \le \mu$ almost everywhere. Then, by the definition of g and Hölder's inequality, (14) is bounded by,

$$C\left[\int \ldots |f(x)|^p M^-(\ldots)(x)dx + \sum \int \left(\frac{1}{\ldots}\int \ldots |f(x)|^p dx\right) M^-(\ldots)(x)dx\right]$$

It is easy to see that $M^-(w\chi_{\tilde{\Omega}^c})(x) \leq CM^-(w)(z)$ for every $x, z \in I_j$. Thus,

$$w\left(\{x \notin \tilde{\Omega} : g^+_{\lambda,\varphi}(g)(x) > \mu\}\right) \leq \frac{C}{\mu^p} \int_{-\infty}^{\infty} |f(x)|^p M^- w(x) dx. \tag{15}$$

We denote $I_j^* = (\alpha_j - 2|I_j|, \beta_j)$ for every $j \geq 1$. We can write

$$g^+_{\lambda,\varphi}(b)(x) \leq g^1(x) + g^2(x), \tag{16}$$

where

$$g^1(x) = \left(\int_0^\infty \int_x^\infty \left(\frac{t}{t+y-x}\right)^\lambda \left|\sum_{i:y\notin I_i^*} b_i * \varphi_t(y)\right|^2 \frac{dydt}{t^2}\right)^{1/2},$$

and

$$g^2(x) = \left(\int_0^\infty \int_x^\infty \left(\frac{t}{t+y-x}\right)^\lambda \left|\sum_{i:y\in I_i^*} b_i * \varphi_t(y)\right|^2 \frac{dydt}{t^2}\right)^{1/2}.$$

Let us consider $g^1(x)$. Taking into account that $b_i * \varphi_t(y) = 0$ if $y > \beta_i$, and $\int |b_i(z)|dz \leq 2|I_i|\mu$, it follows that

$$\left|\sum_{i:y\notin I_i^*} b_i * \varphi_t(y)\right| \leq \frac{2\mu}{t} \sum_{i:y\notin I_i^*, y<\beta_i} |I_i| \sup_{z\in I_i} \left|\varphi\left(\frac{y-z}{t}\right)\right|.$$

Since $\varphi \in \mathcal{S}$, and support$(\varphi) \subset (-\infty, 0]$, we have that $\left|\varphi\left(\frac{y-z}{t}\right)\right| \leq \frac{C}{\left(1+\frac{w-y}{t}\right)^2}$ for $y \notin I_i^*$ and $z, w \in I_i$. Then,

$$\left|\sum_{i:y\notin I_i^*} b_i * \varphi_t(y)\right| \leq \frac{C\mu}{t} \sum_{i:y\notin I_i^*, y<\beta_i} \int_{I_i} \frac{dw}{\left(1+\frac{w-y}{t}\right)^2} \leq c\mu.$$

Therefore,

$$\begin{aligned} g^1(x)^2 \leq & C\mu \int_0^\infty \int_x^\infty \left(\frac{t}{t+y-x}\right)^\lambda \left|\sum_{i:y\notin I_i^*} b_i * \varphi_t(y)\right| \frac{dydt}{t^2} \\ = & C\mu F(x), \end{aligned}$$

and, by Chebyshev's inequality we get

Since $\int b_i(z)dz = 0$, applying Mean Value Theorem for every $y \leq \alpha_i - 2|I_i|$, we obtain the estimation

$$
\begin{aligned}
|b_i * \varphi_t(y)| &\leq \frac{1}{t}\int |b_i(z)| \left|\varphi\left(\frac{y-z}{t}\right) - \varphi\left(\frac{y-\alpha_i}{t}\right)\right| dz \\
&\leq \frac{C}{t}\int_{I_i} |b_i(z)| \left|\frac{z-\alpha_i}{t}\right| \left(\frac{t}{t+\alpha_i - y}\right)^4 dz \\
&\leq C|I_i| \frac{t^2}{(t+\alpha_i-y)^4}\int_{I_i}|f(z)|dz.
\end{aligned}
$$

Then, by definition of $F(x)$, (17) is majorized by

$$
\frac{C}{\mu}\sum_{i\geq 1}\int_{I_i}|f(z)|dz\int_{\tilde{\Omega}^c}|I_i|\int_0^\infty\int_{x<y<\beta_i, y\notin I_i^*}\left(\frac{t}{t+y-x}\right)^{\lambda'}\frac{1}{(t+\alpha_i-y)^4}dydtw(x)dx, \tag{18}
$$

where $1 < \lambda' < \inf(\lambda, 2)$. Now, applying Lemma 11 with $k = 4$, we have that (18) is bounded by

$$
\frac{C}{\mu}\sum_{i\geq 1}\int_{I_i}|f(z)|dz\int_{-\infty}^{\alpha_i - 4|I_i|}\frac{|I_i|^{\lambda'-1}}{(\alpha_i - x)^{\lambda'}}w(x)\chi_{\tilde{\Omega}^c}(x)dx.
$$

The inner integral is bounded by $CM^-(w\chi_{\tilde{\Omega}^c})(\alpha_i)$. It is easy to verify that, by Hölder's inequality and (12),

$$
\frac{1}{\mu}\int_{I_i}|f| \leq \frac{1}{\mu^p}\int_{I_i}|f|^p.
$$

Thus, we obtain that

$$
\begin{aligned}
w\left(\{x\notin\tilde{\Omega} : g^1(x) > \mu\}\right) &\leq \frac{C}{\mu^p}\sum_i\int_{I_i}|f(z)|^p dz M^-(w\chi_{\tilde{\Omega}^c})(\alpha_i) \\
&\leq \frac{C}{\mu^p}\int_{-\infty}^{\infty}|f(z)|^p M^- w(z)dz.
\end{aligned} \tag{19}
$$

Now, let us consider $g^2(x)$. By (12), there exists an integer k_0 such that $|I_j| \leq \|f\|_p^p\mu^{-p} \leq 2^{k_0}$ for every $j \geq 1$. Let $A_k = \{j : 2^{k-1} < |I_j| \leq 2^k\}, k \leq k_0$. We can write

$$
\cup_{j\geq 1} I_j^* = \cup_{k\leq k_0}\cup_{j\in A_k} E_j^*,
$$

where $E_j^* = I_j^* \setminus \cup_{l>k}\cup_{s\in A_l} I_s^*$, for each $j \in A_k$. We observe that if $I_i^* \cap E_j^*$ is not empty then, $I_i^* \subset I_j^{'}$, where $I_j^{'}$ is the interval with the same center of I_j and its measure is equal to $20|I_j|$. For each $x \notin \tilde{\Omega}$, we have that

$$
g^2(x)^2 \ldots \sum\sum\int^{\infty}\int \left(\frac{t}{\ldots}\right)^{\lambda}\left|\sum b \ldots\right|^2 \frac{dydt}{\ldots}
$$

We observe that if $x \notin \tilde{\Omega}^c, x < y$ and $y \in E_j^*$ then $x < \alpha_j - 4|I_j|$ and $t + y - x \geq (\alpha_j - x) - (\alpha_j - y) \geq \frac{\alpha_j - x}{2}$. Then,

$$g^2(x)^2 \leq C \sum_{k \leq k_0} \sum_{j \in A_k, x < \alpha_j} \frac{1}{(\alpha_j - x)^\lambda} \int_0^\infty \int_{x<y, y \in E_j^*} t^{\lambda-2} \left| \sum_{i : y \in I_i^*} b_i * \varphi_t(y) \right|^2 dydt. \tag{20}$$

If we denote $D_j = \cup_{i : E_j^* \cap I_i^* \neq \emptyset} I_i$, and $b^j(x) = |b(x)| \chi_{D_j}(x)$ then, for every $y \in E_j^*$, we obtain

$$\begin{aligned}\left| \sum_{i : y \in I_i^*} b_i * \varphi_t(y) \right| &\leq \sum_{i : y \in I_i^*} \int_{I_i} |b(z)||\varphi_t(y-z)|dz \\ &\leq \int_{\cup_{i : E_j^* \cap I_i^* \neq \emptyset} I_i} |b(z)||\varphi_t(y-z)|dz \\ &\leq \int_{D_j} |b(z)||\varphi_t(y-z)|dz = (b^j * |\varphi|_t)(y).\end{aligned}$$

In consequence, by (20), we have

$$g^2(x)^2 \leq C \sum_{k \leq k_0} \sum_{j \in A_k, x < \alpha_j} \frac{1}{(\alpha_j - x)^\lambda} \int_0^\infty \int_{x<y, y \in E_j^*} t^{\lambda-2} |(b^j * |\varphi|_t)(y)|^2 dydt. \tag{21}$$

We claim that

$$\int_0^\infty \int_{E_j^*} t^{\lambda-2} |(b^j * |\varphi|_t)(y)|^2 dydt \leq C |E_j^*|^{\lambda - \frac{2}{p}} \|b^j\|_p^2. \tag{22}$$

In fact, by Fubini's Theorem, we have

$$\int_0^\infty t^{\lambda-2} |(b^j * |\varphi|_t)(y)|^2 dt$$

$$= \int_y^\infty b^j(z) \int_y^\infty b^j(w) \int_0^\infty t^{\lambda-4} |\varphi| \left(\frac{y-z}{t} \right) |\varphi| \left(\frac{y-w}{t} \right) dtdwdz.$$

Since $\varphi \in \mathcal{S}$, and $\lambda < 3$,

$$\int_0^\infty t^{\lambda-4} |\varphi| \left(\frac{y-z}{t} \right) |\varphi| \left(\frac{y-w}{t} \right) dt \leq C \int_0^\infty t^{\lambda-4} \frac{1}{\left(1 + \frac{z-y}{t}\right)^2} \frac{1}{\left(1 + \frac{w-y}{t}\right)^2} dt$$

Then, the left hand side of (22) is bounded by

$$\begin{aligned}
&C\int_{E_j^*}\int_y^\infty b^j(z)\int_y^\infty b^j(w)\frac{1}{(z+w-2y)^{3-\lambda}}dwdzdy\\
&\leq C'\int_{E_j^*}\int_y^\infty \frac{b^j(z)}{(z-y)^{\frac{3-\lambda}{2}}}dz\int_y^\infty\frac{b^j(w)}{(w-y)^{\frac{3-\lambda}{2}}}dwdy\\
&\leq C'\int_{E_j^*}|I^+_{\frac{\lambda-1}{2}}(b^j)(y)|^2dy,
\end{aligned}$$

where $I^+_{\frac{\lambda-1}{2}}$ denotes the one-sided fractional integral operator of order $\frac{\lambda-1}{2}$. In the case $1<p<2$ and $\lambda=\frac{2}{p}$ since, as it is well known, $I^+_{\frac{\lambda-1}{2}}$ is a bounded operator from L^p to L^2 then it follows that (22) holds. For $2<\lambda<3$, the operator $I^+_{\frac{\lambda-1}{2}}$ maps L^1 into weak-$L^{\frac{2}{3-\lambda}}$. Then, by Kolmogorov's condition (see [GRu], page 485), we obtain (22).

On the other hand, since $\int|b_i(y)|^pdy\leq(2\mu)^p|I_i|$, we have that

$$\|b^j\|_p\leq\left(\sum_{i:E_j^*\cap I_i^*\neq\emptyset}(2\mu)^p|I_i|\right)^{1/p}\leq 2\mu|I_j^{'}|^{1/p}=C\mu|I_j|^{1/p}.$$

Therefore, by (21) and (22) we get

$$g^2(x)^2\leq C'\mu^2\sum_{k\leq k_0}\sum_{j\in A_k, x<\alpha_j}\frac{|I_j|^\lambda}{(\alpha_j-x)^\lambda}.$$

In consequence,

$$\begin{aligned}
w\left(\{x\notin\tilde\Omega: g^2(x)>\mu\}\right)&\leq C\sum_j|I_j|^\lambda\int_{-\infty}^{\alpha_j-4|I_j|}\frac{w(x)\chi_{\tilde\Omega^c}(x)}{(\alpha_j-x)^\lambda}dx\\
&\leq\frac{C}{\mu^p}\sum_j\int_{I_j}|f(z)|^pdzM^-(w\chi_{\tilde\Omega^c})(\alpha_j) \qquad (23)\\
&\leq\frac{C}{\mu^p}\int_{-\infty}^\infty|f(z)|^pM^-w(z)dz.
\end{aligned}$$

From (15), (16), (19) and (23) we obtain that (13) holds for $\lambda=2/p$ if $1<p<2$ and $2<\lambda<3$ if $p=1$. Taking into account that if $\lambda_1\leq\lambda_2$ then, $g^+_{\lambda_2,\varphi}(f)(x)\leq g^+_{\lambda_1,\varphi}(f)(x)$, the proof of the theorem is complete. $\square$

We shall give a proof of Theorem C, applying the results obtained in Theorem A

Proof of Theorem C. The case $p = 2$ and $\lambda > 1$ was considered in Theorem A. Let $1 < p < 2$ and $2/p < \lambda < 2$. We have that $\lambda = 2/q$ with $1 < q < p$. Then, by Theorem B, $g^{+}_{\lambda,\varphi}$ maps $L^q(M^-w)$ into weak-$L^q(w)$. Since $g^{+}_{\lambda,\varphi}$ from $L^2(M^-w)$ to $L^2(w)$ is bounded, then by interpolation, we get this corollary for $\lambda < 2$. Now, by simple arguments we obtain the case $\lambda \geq 2$. $\square$

The following remark shall show that for $\lambda = 2$ and $p = 1$, a weak type inequality as in Theorem B, can not be valid.

Remark. *Let $\varphi \neq 0$ belong to $\mathcal{S}$, supported on the interval $[-1, 0]$ and $\int \varphi(x)dx = 0$. There exists $f \in L^1$ such that $g^{+}_{2,\varphi}(f)(x) = \infty$ for every x belonging to an unbounded set.*

In fact, we consider $f(t) = \left(\frac{1}{|t|(\ln 1/|t|)^{3/2}} - c\right)\chi_{[-1/2,0]}(t)$, where c is the unique constant that satisfies $\int f(t)dt = 0$. For every $x < -4$, we have that

$$g^{+}_{2,\varphi}(f)(x)^2 \geq \frac{1}{(1-x)^2}\int_0^1\int_{-2}^0 |f * \varphi_t(y)|^2 dydt. \tag{24}$$

The support of $f * \varphi_t$ is contained in $(-\infty, 0]$ and the fractional integral $I_{1/2}(f) \notin L^2$ (see [Z], page 232). Then, applying Plancherel's Theorem, it follows that

$$\begin{aligned} A = \int_0^\infty\int_{-\infty}^0 |f * \varphi_t(y)|^2 dydt &= \int_0^\infty\int_{-\infty}^\infty |\widehat{\varphi}(ty)|^2|\widehat{f}(y)|^2 dydt \\ &\geq C_\varphi \int_{-\infty}^\infty \frac{|\widehat{f}(y)|^2}{|y|}dy \\ &= C_\varphi \int_{-\infty}^\infty |I_{1/2}(f)(y)|^2 dy = \infty. \end{aligned}$$

Applying Mean Value Theorem, we obtain for every $y \leq -2$ the estimation

$$\begin{aligned} |f * \varphi_t(y)| &\leq \frac{1}{t}\int_{-1/2}^0 |f(z)|\left|\varphi\left(\frac{y-z}{t}\right) - \varphi\left(\frac{y}{t}\right)\right| dz \\ &\leq \frac{1}{t}\int_{-1/2}^0 |f(z)|\frac{|z|}{t}C_\varphi\left(\frac{t}{t+|y|}\right)^2 dz \\ &\leq C\frac{1}{(t+|y|)^2}. \end{aligned}$$

Using these inequalities we get

Since $|f * \varphi_t(y)| \leq \frac{1}{t}\|\varphi\|_\infty \|f\|_1$, we have that

$$A_2 = \int_1^\infty \int_{-2}^0 |f * \varphi_t(y)|^2 dydt \leq C \int_1^\infty \int_{-2}^0 \frac{1}{t^2} dydt < \infty.$$

By (24) and the estimations obtained for A, A_1, and A_2 it follows that $g^+_{2,\varphi}(f)(x) = \infty$ for every $x < -4$.

We shall give a proof of Theorem D, proceeding as in Theorem 1.10 in [P], page 150.

Proof of Theorem D. More generally, we shall prove that

$$\int_{-\infty}^\infty g^+_{\lambda,\varphi}(f)(x)^p w(x)dx \leq C \int_{-\infty}^\infty |f(x)|^p M^-_B(w^{2/p})(x)^{p/2} dx,$$

where B is a Young function that satisfies

$$\int_c^\infty \left(\frac{t^{p/2}}{B(t)}\right)^{(p/2)'-1} \frac{dt}{t} < \infty. \tag{25}$$

In the case $B(t) \approx t^{p/2}(1 + \ln^+ t)^{[p/2]}$, we get Theorem D.

Let $r = p/2$. We have that

$$\begin{aligned} I = \|g^+_{\lambda,\varphi}(f)\|^2_{L^p(w)} &= \|g^+_{\lambda,\varphi}(f)^2 w^{1/r}\|_{L^r} \\ &= \int_{-\infty}^\infty g^+_{\lambda,\varphi}(f)(x)^2 w(x)^{1/r} g(x) dx, \end{aligned}$$

for some $g \in L^{r'}$ with unit norm. We recall that

$$M^-(g_1 g_2)(x) \leq M^-_B(g_1)(x) M^-_{\overline{B}}(g_2)(x),$$

where $\overline{B}$ is the complementary function associated to B. Then, applying Theorem A, and Holder's inequality

$$\begin{aligned} I &\leq C \int_{-\infty}^\infty |f(x)|^2 M^-(w^{1/r} g)(x) dx \\ &\leq C \int_{-\infty}^\infty |f(x)|^2 M^-_B(w^{1/r})(x) M^-_{\overline{B}}(g)(x) dx \\ &\leq C \left(\int_{-\infty}^\infty |f(x)|^p M^-_B(w^{1/r})(x)^{p/2} dx\right)^{2/p} \left(\int_{-\infty}^\infty M^-_{\overline{B}}(g)(x)^{r'} dx\right)^{1/r'} \end{aligned}$$

where $v = M_B^-(w^{1/r})(x)^r$. By Theorem 2.6 in [RiRoT], if B satisfies (25), then

$$I \le C\|f\|_{L^p(v)}^2 \|g\|_{L^{r'}} \le C\|f\|_{L^p(v)}^2.$$

It is easy to check that $M_B^-(w^{1/r})(x)^r = M_{\tilde{B}}^-(w)(x)$, where $\tilde{B}(t) = B(t^{1/r})$. If $\tilde{B}(t) = t(1 + \ln^+ t)^{[r]}$ then, B satisfies (25), and by Proposition 2.15 in [RiRoT] there exist two constants C_1 and C_2 such that

$$C_1 M_{\tilde{B}}^-(w)(x) \le (M^-)^{[r]+1} w(x) \le C_2 M_{\tilde{B}}^-(w)(x),$$

which completes the proof. $\square$

References

[CW] S. Chanillo and R. Wheeden, *Some weighted norm inequalities for the Area integral*, Indiana Univ. Math. J. **36** (1987), 277-294.

[F] C. L. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9-36.

[GRu] J. García Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North Holland, Amsterdam 1985.

[HSt] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York, Heidelberg, and Berlin 1965.

[M] F. J. Martín-Reyes, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. **117** (1993), 691-698.

[P] C. Pérez, *Banach function spaces and the two-weight problem for maximal functions*, Proceedings of the Conference on Function Spaces, Differential Operators and Nonlinear Analysis, Paseky (1995), 141-158.

[RiRoT] M. S. Riveros, L. de Rosa and A. de la Torre, *Sufficient conditions for one-sided operators*, to appear in J. Fourier Anal. Appl. (2000).

[RoSe] L. de Rosa and C. Segovia, *One-sided Littlwood-Paley theory*, J. Fourier Anal. Appl. **3** (1997), 933-957.

[Z] A. Zygmund, *Trigonometrical Series*, Monografje Matematyczne, Warszawa - Lwów 1935.

Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina, and Instituto Argentino de Matemática, CONICET, Saavedra 15 3er Piso, 1083 Buenos Aires, Argentina.

E-mail address: lderosa@dm.uba.ar

Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires. 1428 Buenos Aires, Argentina, and Instituto Argentino de Matemática, CONICET, Saavedra 15 3er Piso, 1083 Buenos Aires, Argentina.

E-mail address: segovia@iamba.edu.ar