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# POSITIVE OPERATORS ON HILBERT SPACE: A DIFFERENTIAL GEOMETRIC APPROACH

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## 1. Introduction.

This paper is devoted to expose several results about the geometrical structure of the set of positive bounded operators in a Hilbert space $\mathbf{H}$.

The geometry of the invertible positive operators is well known. This set has a very rich structure as a homogeneous space of the group of the invertible operators of $\mathbf{L}(\mathbf{H})$, with a canonical connection and a Finsler structure. The geodesics of this connection are short when measured with the Finsler metric. Many other important geometric properties have been studied, see [11], [12], [14], [15].

However, in mathematical physics the study of the geometry of not invertible positive operators appears in a natural way. Among others, Uhlmann [39], [40], [41], Dabrowski and Jadczyk [17], Dabrowski and Grosse [16], Dittman and Rudolph [19], Petz [33] have emphasized the relevance of the search of geometrical structure of parts of $\mathbf{L}(\mathbf{H})^+$. For example, in Uhlmann's explanation of the geometric phase (or Berry phase [7]) a central role is played by the set of generalized density operators, i. e. the positive operators belonging to the trace class.

As a first attempt to understand the geometry of not invertible positive operators we study some subsets of $\mathbf{L}(\mathbf{H})^+$: $\mathbf{L}(\mathbf{H})^+$ is a closed convex cone in $\mathbf{L}(\mathbf{H})_h$, the set of Hermitian elements of $\mathbf{L}(\mathbf{H})$. There are two important metrics defined in this cone, the Hilbert's projective metric and the Thompson part metric [24], [38]. Thompson defines an equivalence relation in $\mathbf{L}(\mathbf{H})^+$ and a complete metric in each class or "component". It turns out that each component is a differential submanifold of an appropriate Banach space and a homogeneous space of a certain group. A natural connection and a Finsler structure can be defined in these components and the geodesics here are short, too. The geodesic metric obtained coincides with the Thompson's part metric in each component.

The contents of the paper are the following. Section 2 contains a survey of the differential geometry of $\mathbf{G}^+$,the invertible positive elements of a $C^*$-algebra $\mathbf{A}$, as studied in [11], [12],[14],[15], with a description of its homogeneous structure, the canonical connection, its geodesics and the Finsler metric. Also a variational characterization of horizontal lifts is included. Section 3 contains a description of the components of $\mathbf{L}(\mathbf{H})^+$ and the results are essentially a consequence of a well known theorem of R. G. Douglas [22]. One of the characterizations of the components is that $A, B \in \mathbf{L}(\mathbf{H})^+$ belong to the same component if and only if $A^{1/2}$ and $B^{1/2}$ have the same range. This description allows a parametrization of the set of all components of $\mathbf{L}(\mathbf{H})^+$ by means of the set of all operator ranges of $\mathbf{H}$. The subspaces which are operator ranges have been characterized by Dixmier ( see [20], [21] and [23]). In section 4, different Hilbert spaces associated to a positive operator $A$ are studied and they are used in Section 5 to describe the geometrical structure of each component $C_A$, as a homogeneous space of the invertible group of one of these spaces. A principal connection is defined and the geodesics of this connection are characterized. We introduce a Finsler metric on each $C_A$ for which the geodesics are short. The geodesic metric coincides with the Thompson metric defined in the component (see [28]).

Finally, in section 6 we present a brief survey of Uhlmanns results and compare them to the construction of section 2.

## 2. Differential geometry of $\mathbf{G}^+$.

This section contains a description of the differential geometry of $\mathbf{G}^+$, the set of positive elements of a $C^*$-algebra. Most of the results we mention are contained in the papers [11], [12], [15] but there is some new material and shorter proofs of some results.

In what follows, $\mathbf{A}$ denotes a unital $C^*$-algebra with invertible group $\mathbf{G}$, $\mathbf{A}_h$ is the (real) subspace of Hermitian or self-adjoint elements of $\mathbf{A}$, $\mathbf{A}_{ah}$ is the set of skew-hermitian elements of $\mathbf{A}$,

$$\begin{aligned}\mathbf{A}_{ah} &= \{X \in \mathbf{A} : X + X^* = 0\} \\ &= \{X \in \mathbf{A} : iX \in \mathbf{A}_h\},\end{aligned}$$

$\mathbf{A}^+$ denotes the set of positive elements of $\mathbf{A}$ and $\mathbf{G}^+ = \mathbf{G} \cap \mathbf{A}^+$. Throughout this paper no deep $C^*$-algebra result is needed, so that in general the reader may suppose that $\mathbf{A}$ is the algebra $\mathbf{L}(\mathbf{H})$ of all bounded linear operators in a Hilbert space $\mathbf{H}$.

The group $\mathbf{G}$ is an open subset of $\mathbf{A}$ and $\mathbf{G}^+$ is an open subset of the (real) Banach space $\mathbf{A}_h$. Thus, $\mathbf{G}^+$ has the natural differentiable structure of an open submanifold of $\mathbf{A}_h$.

There is also a natural action of $\mathbf{G}$ over $\mathbf{G}^+$, defined by $L : \mathbf{G} \times \mathbf{G}^+ \to \mathbf{G}^+$, $L_g a = gag^*$, $g \in \mathbf{G}$, $a \in \mathbf{G}^+$. This action is clearly differentiable, and transitive: if $a, b \in \mathbf{G}^+$ then $L_g a = b$ for $g = b^{1/2}a^{-1/2}$ (in this paper power $\frac{1}{2}$ denotes the positive square root and power $-\frac{1}{2}$ denotes the inverse of the positive square root). Thus, for each $a \in \mathbf{G}^+$, the map

$$p_a : \mathbf{G} \to \mathbf{G}^+, \quad p_a(g) = gag^*$$

is surjective and $s(b) = b^{1/2}a^{-1/2}$ defines a global cross section of $p_a$, i.e., $p_a \circ s = \mathrm{id}_{G^+}$. Because $\mathbf{G}^+$ is open in $\mathbf{A}_h$, for all $a \in \mathbf{G}^+$ the tangent space $(\mathcal{T}\mathbf{G}^+)_a$ naturally identifies with $\mathbf{A}_h$ and the tangent map $(\mathcal{T}p_a)_1$ identifies with

$$\mathcal{T}_a : \mathbf{A} \to \mathbf{A}_h, \quad \mathcal{T}_a(X) = Xa + aX^* \ .$$

The *isotropy group* $\mathcal{I}_a$ of $a \in \mathbf{G}^+$ is the group of all $g \in \mathbf{G}$ such that $L_g a = a$, i.e.

$$\mathcal{I}_a = \{g \in \mathbf{G} : gag^* = a\} \ .$$

In particular, $\mathcal{I}_1 = \mathcal{U}$, the unitary group of $\mathbf{A}$. Observe that the Lie algebra of $\mathcal{U}$, i.e. the tangent space $(\mathcal{T}\mathcal{U})_1$ coincides with $\mathbf{A}_{ah}$. In general, $(\mathcal{T}\mathcal{I}_a)_1 = \{X \in \mathbf{A} : Xa + aX^* = 0\}$.

From now on, we shall consider the case $a = 1$, so we get the projection

$$p : \mathbf{G} \to \mathbf{G}^+, \quad p(g) = gg^*$$

with tangent map at $1 \in \mathbf{G}$

$$\mathcal{T} : \mathbf{A} \to \mathbf{A}_h, \quad \mathcal{T}(X) = X + X^* \ .$$

The projection $p$ is related to the polar decomposition of elements of $\mathbf{G}$. Recall that every $g \in \mathbf{G}$ admits a unique decomposition $g = \lambda u$ with $\lambda \in \mathbf{G}^+$ and $u \in \mathcal{U}$. It is easy to show that $\lambda^2 = gg^* = p(g)$, (see [35]).

Observe that $\mathbf{A}$ splits as a direct sum $\ker \mathcal{T} \oplus R(\mathcal{T}) = \mathbf{A}_{ah} \oplus A_h$ and that $\frac{1}{2}\mathcal{T}$ is the projection onto $\mathbf{A}_h$ with kernel $\mathbf{A}_{ah}$.

Since $\mathcal{U}$ acts over $\mathbf{G}$ by right translation without fixed points, the triple $\{\mathbf{G}^+, \mathbf{G}, \mathcal{U}\}$ defines a trivial fibre bundle with structural group $\mathcal{U}$. Moreover, $\mathbf{G}^+$ is a homogeneous space of $\mathbf{G}$. The reader is referred to [26] for all differential geometric notions used here. There is a natural connection on $\mathbf{G}^+$, i.e. a smooth distribution of subspaces of $\mathbf{A}$, $g \mapsto \mathcal{H}_g$ such that

(i) $\mathbf{A} = \mathcal{H}_g \oplus \mathcal{V}_g$, if $g \in \mathbf{G}$ and

$$\begin{aligned} \mathcal{V}_g &= \{X \in \mathbf{A} : (\mathcal{T}p)_g X = 0\}, \\ &= \{X \in \mathbf{A} : Xg^* + gX^* = 0\}; \end{aligned}$$

(ii) $u\mathcal{H}_1 u^* = \mathcal{H}_1$ if $u \in \mathcal{U}$;

(iii) $\mathcal{H}_g u = \mathcal{H}_{gu}$ if $g \in \mathbf{G}$, $u \in \mathcal{U}$.

In fact, $\mathcal{H}_g = g\mathbf{A}_h$ satisfies these properties; *smooth* means that the map which assigns to each $g \in \mathbf{G}$ the projection on $\mathbf{A}$ which has $\mathcal{V}_g$ as kernel and $\mathcal{H}_g$ as range, is a differentiable map from $\mathbf{G}$ to $\mathbf{L}(\mathbf{A})$, the algebra of bounded linear operator on $\mathbf{A}$. The projection $\phi_g : \mathbf{A} \to \mathcal{H}_g$ can be given explicitely as follows: if $X \in \mathbf{A}$ then $\phi_g(X) = \frac{1}{2}(X + gX^*g^{*-1})$. The spaces $\mathcal{H}_g$ are called *horizontal.* In every locally trivial fibre bundle $p : S \to B$ a (continuous) curve $\gamma : [0,1] \to B$ admits a lift, i.e. there is a (continuous) curve $\Gamma : [0,1] \to S$ such that $p \circ \Gamma = \gamma$.

If the fibre bundle has a connection, a smooth curve admits a unique horizontal lift, i.e. a lift $\Gamma$ such that each tangent vector is horizontal:

$$\dot{\Gamma}(t) = \frac{d}{dt}\Gamma(t) \in H_{\Gamma(t)}, \quad t \in [0,1] \ .$$

If $\gamma : [0,1] \to \mathbf{G}^+$ is a $C^\infty$ curve and $p(g) = \gamma(0)$ then the horizontal lift $\Gamma : [0,1] \to \mathbf{G}$ such that $\Gamma(0) = g$ is the (unique) solution of the linear differential equation

$$\begin{cases} \dot{\Gamma} & = \frac{1}{2}\dot{\gamma}\gamma^{-1}\Gamma \\ \Gamma(0) & = g \end{cases}$$

Or, also, a $C^\infty$ lift $\Gamma$ of $\gamma : [0,1] \to \mathbf{G}^+$ is horizontal if and only if $\dot{\Gamma}\Gamma^* = \Gamma\dot{\Gamma}^*$, i.e. if $\dot{\Gamma}\Gamma^*$ is selfadjoint.

The horizontal lift $\Gamma$ can also be characterized by means of its unitary part: a $C^\infty$ lift $\Gamma$ of $\gamma$ is horizontal if and only if its unitary part $u : [0,1] \to \mathcal{U}$ satisfies

$$\dot{u}u^{-1} = \frac{1}{2}\left(\left(\gamma^{1/2}\right)^{\bullet}\gamma^{-1/2} - \gamma^{-1/2}\left(\gamma^{1/2}\right)^{\bullet}\right) \ .$$

See [12] for a proof of these results.

The differential equation

$$\dot{\Gamma} = \frac{1}{2}\dot{\gamma}\gamma^{-1}\Gamma \ ,$$

which is called the *transport equation*, induces a covariant derivative of a tangent field $X$ along $\gamma$, namely

$$\begin{aligned} \frac{DX}{dt} &= \dot{X} - \frac{1}{2}\left(X\gamma^{-1}\dot{\gamma} + \dot{\gamma}\gamma^{-1}X\right) \\ &= \Gamma(t)\frac{d}{dt}\left(\left(TL_{\Gamma(t)^{-1}}\right)_{\gamma(t)} X(t)\right)\Gamma(t)^* \ . \end{aligned}$$

The field $X$ is *parallel* if $\frac{DX}{dt} = 0$. A curve $\gamma$ is a *geodesic* if $\dot{\gamma}$ is parallel, i.e. $\ddot{\gamma} = \dot{\gamma}\gamma^{-1}\dot{\gamma}$.

It is easy to prove that the connection is $\mathbf{G}$-invariant in the sense that if $\gamma$ is a geodesic then $g\gamma g^*$ is a geodesic for all $g \in \mathbf{G}$. The unique geodesic $\gamma$ such that $\gamma(0) = a$ and $\dot{\gamma}(0) = X \in (\mathcal{T}G^+)_a$ is

$$\text{(2.1)} \qquad \gamma(t) = a^{1/2}e^{ta^{-1/2}Xa^{-1/2}}a^{1/2}, \qquad t \in [0,1] \ .$$

The exponential map at $a \in \mathbf{G}^+$ $\exp_a : (\mathcal{T}\mathbf{G}^+)_a \to \mathbf{G}^+$ is given by $\exp_a X = a^{1/2} e^{a^{-1/2} X a^{-1/2}} a^{1/2}$. Notice that $\exp_a$ is a diffeomorphism and its inverse map is $\log_a : \mathbf{G}^+ \to (\mathcal{T}\mathbf{G}^+)_a$, $\log_a b = a^{1/2} \log\left(a^{-1/2} b a^{-1/2}\right) a^{1/2}$.

On the other hand, for every $a, b \in \mathbf{G}^+$ there is one and only one geodesic $\gamma_{a,b}$ such that $\gamma_{a,b}(0) = a$ and $\gamma_{a,b}(1) = b$, namely

$$\gamma_{a,b}(t) = a^{1/2} \left(a^{-1/2} b a^{-1/2}\right)^t a^{1/2}, \qquad t \in [0,1] \,. \tag{2.2}$$

In order to introduce a Finsler structure on $\mathbf{G}^+$ it is convenient to suppose that $\mathbf{A}$ consists of bounded linear operators on a Hilbert space $\mathbf{H}$. For the basic definitions and results on Finsler manifolds the reader is referred to [32].

For each $a \in \mathbf{G}^+$ the formula

$$B_a(x, y) = < ax, y >, \qquad x, y \in \mathbf{H}$$

defines a scalar product on $\mathbf{H}$. Denote by $\mathbf{H}_a$ the Hilbert space $(\mathbf{H}, B_a)$ and $\|x\|_a = B_a(x,x)^{1/2} = \|a^{1/2} x\|$. For each $a \in \mathbf{G}^+$ and $g \in \mathbf{G}$, $g^* : \mathbf{H}_{gag^*} \to \mathbf{H}_a$ is an isometric isomorphism.

A *Finsler structure* on $\mathbf{G}^+$ consists of a smooth assignation of a complete norm $\| \ \|_a$ on the tangent space $(\mathcal{T}\mathbf{G}^+)_a$.

Observe that $(\mathcal{T}\mathbf{G}^+)_a$ can be naturally identified with $\mathbf{A}_h$. Now, for $a \in \mathbf{G}^+$ and $X \in (\mathcal{T}\mathbf{G}^+)_a$ let $B_X : \mathbf{H}_a \times \mathbf{H}_a \to \mathbb{C}$ be the sesquilinear form defined by

$$B_X(x, y) = < Xx, y >, \quad x, y \in \mathbf{H}_a \,.$$

Define $\|X\|_a = \|B_X\| = \sup\{| < Xx, y > | : \|x\|_a \leq 1, \ \|y\|_a \leq 1\}$. An explicit formula for $\|X\|_a$ is given by

$$\|X\|_a = \|a^{-1/2} X a^{-1/2}\|, \qquad X \in (\mathcal{T}\mathbf{G}^+)_a,$$

and the norm verifies $\|X\|_a = \|gXg^*\|_{gag^*}$, $X \in (\mathcal{T}\mathbf{G}^+)_a$, $g \in \mathbf{G}$.

For a $C^\infty$ curve $\gamma : [0,1] \to \mathbf{G}^+$ its *length* $L(\gamma)$ is

$$\int_0^1 \|\dot\gamma(t)\|_{\gamma(t)} dt \,.$$

The length of the unique geodesic $\gamma_{a,b}$ joining $a \in \mathbf{G}^+$ to $b \in \mathbf{G}^+$ is $\| \log\left(a^{-1/2} b a^{-1/2}\right) \|$.

The next step consists in showing that geodesics are short curves, i.e., if $\delta$ is another curve joining $a$ to $b$ then $L(\delta) \geq L\left(\gamma_{a,b}\right)$.

**Theorem 2.3.** *For every $a, b \in \mathbf{G}^+$ and every $C^\infty$ curve $\delta : [0,1] \to \mathbf{G}^+$ such that $\delta(0) = a$, $\delta(1) = b$ it holds $L(\delta) \geq L(\gamma_{a,b}) = \| \log \left(a^{-1/2} b a^{-1/2}\right) \|$.*

*The geodesic metric of $\mathbf{G}^+$ is defined by*

$$d(a,b) = \inf L(\gamma)$$

*where the infimum is taken over all smooth curves joining $a$ to $b$.*

*The result above says the unique geodesic joining $a, b \in \mathbf{G}^+$ is a shortest curve. However, there are infinite many curves (not geodesics) which are shortest (see [28],page 1659). As a consequence of the above theorem we have that for all $a, b \in \mathbf{G}^+$*

$$d(a,b) = \| \log(a^{-1/2} b a^{-1/2}) \| = L(\gamma_{a,b}) \; . \tag{2.4}$$

**Remark** *The Finsler structure defined in $\mathbf{G}^+$ is a special case of the Finsler structure for Thompson metric which is described in [27]. Also the geodesic metric on $\mathbf{G}^+$ coincides with the Thompson metric. We will see most of these results in section 3.*

*It can be shown (see the next section for a proof, using Thompson's results) that the metric space $(\mathbf{G}^+, d)$ is complete. However, a direct proof can be given as follows. Recall that, as a part of his work on quantum mechanics, I. Segal [36] proved that, for all selfadjoint operators $X, Y \in \mathbf{L}(\mathbf{H})$ the following inequality holds:*

$$\|e^{X+Y}\| \leq \|e^{X/2} e^Y e^{X/2}\| \; .$$

*From Segal's result it follows quite easily (see [14]) that $d$ satisfies*

$$d(a,b) \geq \| \log a - \log b \|, \quad a, b \in \mathbf{G}^+ \; . \tag{2.5}$$

*Indeed, both inequalities are equivalent. Now, if $\{a_n\}$ is a Cauchy sequence in $(\mathbf{G}^+, d)$ then, by (2.5) it follows that $\{\log a_n\}$ is Cauchy in $(\mathbf{A}_h, \| \; \|)$. Then there exists $X \in \mathbf{A}_h$ such that $\| \log a_n - X \| \to 0$ and then $a_n = e^{\log a_n} \to a = e^X \in \mathbf{G}^+$. Now, it is obvious that $d(a_n, a) = \| \log \left( a_n^{-1/2} a a_n^{-1/2} \right) \| \to 0$.*

*We close this section with a remark due to E. Vesentini [42]. Concerning the particular form of the geodesic distance $d$, he observes that it comes from the fact that the Haar measure on $\mathbf{G}^+_{\mathbb{C}} = \mathbb{R}^+ - \{0\}$ of an open interval $(\alpha, \beta)$ is*

$$\begin{aligned} m(\alpha, \beta) &= |\log \beta - \log \alpha| \\ &= \left| \log \frac{\beta}{\alpha} \right| \\ &= \left| \log \alpha^{-1/2} \beta \alpha^{-1/2} \right| \; . \end{aligned}$$

*We present now a variational characterization of the horizontal lift of a smooth curve* $\gamma : [0,1] \to \mathbf{G}^+$ .

**Theorem 2.6.** *The horizontal lift* $\Gamma_0$ *is a solution of the variational problem*

$$\inf\Big\{ \int_0^1 \|\Gamma^{-1}\dot\Gamma\| dt : \Gamma \text{ is a } C^\infty \text{ curve in } \mathbf{G}, \quad \gamma(t) = \Gamma(t)\Gamma^*(t) \ \ \forall\, t \in [0,1] \Big\} .$$

*Moreover*

$$\inf_\Gamma \int_0^1 \|\Gamma^{-1}\dot\Gamma\| dt = \int_0^1 \|\Gamma_0^{-1}\dot\Gamma_0\| dt = \frac{1}{2}\int_0^1 \|\dot\gamma(t)\|_{\gamma(t)} dt \ .$$

**Remark.** *Every horizontal lift* $\Gamma_0$ *of* $\gamma$ *verifies* $\dot\Gamma_0\Gamma_0^* = (\dot\Gamma_0\Gamma_0^*)^*$ . *Then,* $a = \Gamma_0^{-1}\dot\Gamma_0$ *is selfadjoint and*

$$\frac{1}{2}L(\gamma) = \int_0^1 \|a\| dt \ .$$

*In general, for every lift* $\Gamma$ *of* $\gamma$ *there is a decomposition*

$$\Gamma^{-1}\dot\Gamma = X_\Gamma + Y_\Gamma \quad , \quad X_\Gamma \in \mathcal{H}_1, \ \ Y_\Gamma \in \mathcal{V}_1$$

*where* $\Gamma = \gamma^{1/2}u$ *and*

$$X_\Gamma = \frac{1}{2}u^{-1}\left(\gamma^{-1/2}\left(\gamma^{1/2}\right)^{\bullet} + \left(\gamma^{1/2}\right)^{\bullet}\gamma^{-1/2}\right)u \ .$$

*Then*

$$\begin{aligned}\|X_\Gamma\| &= \frac{1}{2}\|\gamma^{-1/2}\left(\gamma^{1/2}\right)^{\bullet} + \left(\gamma^{1/2}\right)^{\bullet}\gamma^{-1/2}\| \\ &= \frac{1}{2}\|\gamma^{-1/2}\left(\left(\gamma^{1/2}\right)^{\bullet}\gamma^{1/2} + \gamma^{1/2}\left(\gamma^{1/2}\right)^{\bullet}\right)\gamma^{-1/2}\| \\ &= \frac{1}{2}\|\gamma^{-1/2}\dot\gamma\gamma^{-1/2}\| \\ &= \frac{1}{2}\|\dot\gamma\|_\gamma \ .\end{aligned}$$

*This shows that*

$$L(\gamma) = 2\int_0^1 \|X_\Gamma\| dt \ ,$$

*where* $X_\Gamma$ *is the selfadjoint component of* $\Gamma^{-1}\dot\Gamma$ , *i.e.* $X_\Gamma = Re\,(\Gamma^{-1}\dot\Gamma)$ .

*The geometric study of* $\mathbf{G}^+$ *seems to be relevant for its applications to several areas like statistics,( Ohara [29], [30]), interpolation methods, (Amari [1], Semmes [37], Andruchow, Corach, Stojanoff [4]), quantum statistics (Uhlmann [39], [40], [41]), Petz [31], [32]) and others.*

## 3. Thompson's part metric on $A^+$.

In an early paper on the fundations of geometry, David Hilbert [24] introduced a metric on projective space, now called the *Hilbert projective metric*. In 1957 Garrett Birkhoff [8] discovered that Hilbert's metric is a useful tool to find solutions of some linear integral equations. A slight modification of Hilbert's projective metric discovered by A.C. Thompson [38] widens the range of problems for which these methods are applicable. Thompson's metric coincides with the so-called *part metric*, a tool discovered by H.S. Bear ([5],[6]) to study Gleason parts in uniform algebras. We shall use the name *Thompson's part metric*. The reader will find an excellent treatement of Hilbert's and Thompson's metrics in R. Nussbaum's papers [27], [28] (see also Bushell [9], Wojtkowski [43]).

If $K$ is a non empty closed convex cone in a Banach space $E$ and $\leq$ is the partial order on $E$ induced by $K$, $K$ is said to be *normal* if there is a constant $r > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq r\|y\|$.

Consider the following equivalence relation on $K : x \sim y$ if there exist $r > 0$, $s > 0$ such that $x \leq ry$, $y \leq sx$. Each equivalence class is called a *component*. Thompson proved that, if $K$ is normal,

$$d_T(x,y) = \log\max\{\inf\{r>0 : x \leq ry\}, \inf\{s>0 : y \leq sx\}\}$$

defines a complete metric on each component of $K$. It is easy to show that

$$d_T(x,y) = \inf\{\log m : m > 0,\ \frac{1}{m}x \leq y \leq mx\}\ .$$

Observe that if $\mathbf{H}$ is a Hilbert space then $\mathbf{L}(\mathbf{H})^+$ is a closed normal convex cone. In this case, the infimum is attained.

This section is devoted to characterize the components of $\mathbf{L}(\mathbf{H})$ and is a set of variations of the following results of R. G. Douglas [22].

**Theorem 3.1.** *If* $A, B \in \mathbf{L}(\mathbf{H})$ *the following conditions are equivalent*

*(1)* $\mathbf{R}(A) \subset \mathbf{R}(B)$;

*(2) there exists* $r > 0$ *such that* $AA^* \leq rBB^*$;

*(3) there exists* $C \in \mathbf{L}(\mathbf{H})$ *such that* $A = BC$.

*If one of these conditions holds then there exists a unique* $C \in \mathbf{L}(\mathbf{H})$ *such that* *(a)* $\|C\|^2 = \inf\{r > 0 : AA^* \leq rBB^*\}$

*(b)* $\ker C = \ker A$

*(c)* $\mathbf{R}(C) \subset \overline{\mathbf{R}(B^*)}$

**Corollary 3.2.** *If $A, B \in \mathbf{L}(\mathbf{H})$ then $\mathbf{R}(A) = \mathbf{R}(B)$ and $\dim\ker A = \dim\ker B$ if and only if there exists $C \in \mathbf{GL}(\mathbf{H})$ such that $A = BC$. In particular, positive operators $A, B$ have the same range if and only if there exists $C \in \mathbf{GL}(\mathbf{H})$ such that $A = BC$. The reader is referred to Fillmore and Williams [23] for a survey on these subjects and related matters.*

*We proceed now to characterize Thompson's components of $\mathbf{L}(\mathbf{H})^+$.*

**Theorem 3.3.** *For $A, B \in \mathbf{L}(\mathbf{H})^+$ the following conditions are equivalent:*

*(i)* $B \in \mathbf{C}_A$;

*(ii)* $\mathbf{R}(A^{1/2}) = \mathbf{R}(B^{1/2})$;

*(iii) there exists* $V \in \mathbf{GL}(\mathbf{H})$ *such that* $B^{1/2} = A^{1/2}V$;

*(iv) there exists* $P \in \mathbf{GL}(\mathbf{H})^+$ *such that* $B = A^{1/2}PA^{1/2}$;

*(v)* $\mathbf{R}(B) \subset \mathbf{R}(A^{1/2}) \subset \overline{\mathbf{R}(B)}$ *and*

$$\left(A^{1/2}\Big|_{\overline{\mathbf{R}(A^{1/2})}}\right)^{-1} B \left(A^{1/2}\Big|_{\overline{\mathbf{R}(A^{1/2})}}\right)^{-1} : \mathbf{R}\left(A^{1/2}\right) \to \mathbf{R}\left(A^{1/2}\right)$$

*extends to a positive invertible operator* $\mathbf{H} \to \mathbf{H}$.

*Proof.* First consider the case $A \in \mathbf{L}(\mathbf{H})^+$ injective.

The equivalence between (i) and (ii) follows from Douglas' theorem. Its corollary shows the equivalence between (ii) and (iii). The uniqueness of $V$ follows from the injectivity of $A$.

(iii) $\Rightarrow$ (iv) If $B^{1/2} = A^{1/2}V$ for some $V \in \mathbf{GL}(\mathbf{H})$ then $B = A^{1/2}VV^*A^{1/2}$ and $P = VV^* \in \mathbf{GL}(\mathbf{H})^+$ is uniquely determined because $A$ is injective.

(iv) $\Rightarrow$ (v) If $B = A^{1/2}PA^{1/2}$ for some $P \in \mathbf{GL}(\mathbf{H})^+$ then $A^{-1/2}BA^{-1/2}$ is a bounded linear operator on $\mathbf{R}(A^{1/2})$, so that $A^{-1/2}BA^{-1/2}$ is a bounded linear operator on $\mathbf{R}(A^{1/2})$ which admits a unique extension to $\mathbf{H}$, namely $P$, which is invertible.

(v) $\Rightarrow$ (i) The positive operator $A^{-1/2}BA^{-1/2}$ on $\mathbf{R}(A^{1/2})$ satisfies $\alpha I \leq A^{-1/2}BA^{-1/2} \leq \beta I$, where

$$\alpha = \inf\{\|A^{-1/2}BA^{-1/2}y\| : \ y \in \mathbf{R}(A^{1/2}), \|y\| = 1\} \ \text{ and}$$
$$\beta = \sup\{\|A^{-1/2}BA^{-1/2}y\| : \ y \in \mathbf{R}(A^{1/2}), \|y\| = 1\} \ .$$

This shows $\alpha A \leq B \leq \beta A$ and then $B \in \mathbf{C}_A$.

The general case follows from the injective case considering $A^{1/2}\Big|_{\overline{\mathbf{R}(A^{1/2})}}$.

Observe that $B \in \mathbf{C}_A$ if and only if $B^{1/2}A^{-1/2}$ is a bounded operator on $\mathbf{R}(A^{1/2})$ (which can be extended to $\mathbf{H}$).

**Remarks** a) It should be noticed that the invertible operator $V$ of part (iii) is not unique, as it is in the injective case. However $V(\ker A^{1/2}) = \ker A^{1/2}$ and $V(\mathbf{M}) = \mathbf{M}$. Then, condition (iii) is equivalent to

(iii)' $\left(A^{1/2}|_{\mathbf{M}}\right)^{-1} B^{1/2}|_{\mathbf{M}} = V|_{\mathbf{M}} \in \mathbf{GL}(\mathbf{M})$.

b) For every $A \in \mathbf{L}(\mathbf{H})^+$ the component $\mathbf{C}_A$ coincides with the set $A^{1/2}\mathbf{GL}(\mathbf{H})^+A^{1/2}$. In particular $\mathbf{C}_A$ is contained in the trace class ideal if $A$ does.

As a consequence of Theorem 3.3, we obtain a parametrization of the set of components $\{\mathbf{C}_A : A \in \mathbf{L}(\mathbf{H})^+\}$ by means of the set of operator ranges of $\mathbf{H}$, i.e. subspaces $\mathbf{S}$ of $\mathbf{H}$ such that there exists a bounded linear operator $C \in \mathbf{L}(\mathbf{H})$ with $\mathbf{R}(C) = \mathbf{S}$. These subespaces have been studied by Dixmier [20], [21] under the name of "variétés de Julia". The reader will find in [23] a modern treatment including simplified proofs of Dixmier's results. Using the polar decomposition it can be proved that a subspace $\mathbf{S}$ of $\mathbf{H}$ is an operator range if and only if there exists $A \in \mathbf{L}(\mathbf{H})^+$ such that $\mathbf{R}(A) = \mathbf{S}$. Thus, there exists as many Thompson's components of $\mathbf{L}(\mathbf{H})^+$ as operator ranges of $\mathbf{H}$.

The next result shows that, for $A \in \mathbf{L}(\mathbf{H})^+$ the values of the curve $t \mapsto A^t$ lie in the same component if and only if $\mathbf{R}(A)$ is closed. Moreover, if $\mathbf{R}(A)$ is not closed then each $A^t$ lies in a different component.

**Theorem 3.4.** *For a positive operator $A \in \mathbf{L}(\mathbf{H})^+$ the following alternative holds:*

*1) or $\mathbf{R}(A)$ is closed and then $\mathbf{R}(A^t) = \mathbf{R}(A)$ for every $t \in [0,1]$*

*2) or $\mathbf{R}(A)$ is not closed and then $\mathbf{R}(A^t)$ is not closed for every $t \in (0,1]$ and*

$$\mathbf{R}(A) \subset \mathbf{R}(A^t) \subsetneq \mathbf{R}(A^s) \subset \overline{\mathbf{R}(A)}$$

*for $0 \le s < t \le 1$.*

*Proof.* See [13].

**Corollary 3.5.** *If $A \in \mathbf{L}(\mathbf{H})^+$ then $\mathbf{R}(A)$ is closed and then the curve $\gamma(t) = A^t$ lies in the Thompson's component of $A$; or $\mathbf{R}(A)$ is not closed and then each $A^t$ lies in a different component.*

## 4. Hilbert spaces associated to a positive operator.

Each injective positive operator $A$ defines a scalar product on $\mathbf{H}$ by

$$\langle x, y\rangle_A = \langle Ax, y\rangle \qquad (x, y \in \mathbf{H})$$

and a norm $\| \ \|_A$ by

$$\|x\|_A = \langle x, x\rangle_A^{1/2} = \|A^{1/2}x\| \qquad (x \in \mathbf{H}) \ .$$

Thus $A^{1/2} : (\mathbf{H}, \| \ \|_A) \to (\mathbf{H}, \| \ \|)$ is an isometry onto $\mathbf{R}(A^{1/2})$. If $\mathbf{H}_A$ denotes the completion of $(\mathbf{H}, \langle \ , \ \rangle_A)$ then $A^{1/2}$ admits an extension

$$\widetilde{A^{1/2}} : \mathbf{H}_A \to \mathbf{H}$$

which is an isometric isomorphism. Observe that the densely defined operator

$$A^{-1/2} : \left(\mathbf{R}\left(A^{1/2}\right), \| \ \|\right) \to (\mathbf{H}, \| \ \|_A)$$

is an isometry, so that it can be extended to an isometric isomorphism

$$\widehat{A^{-1/2}} : \mathbf{H} \to \mathbf{H}_A$$

which is the inverse map of $\widetilde{A^{1/2}}$. (We shall use different symbols to denote extensions to $\mathbf{H}$ and $\mathbf{H}_A$).

Every $\widetilde{B} \in \mathbf{L}(\mathbf{H}_A, \mathbf{H})$ induces, by restriction, an operator $B = \widetilde{B}|_{\mathbf{H}} : \mathbf{H} \to \mathbf{H}$ which is bounded because for $x \in \mathbf{H}$

$$\begin{aligned} \|Bx\| &= \|\widetilde{B}x\| \le \|\widetilde{B}\| \ \|x\|_A \\ &= \|\widetilde{B}\| \ \|A^{1/2}x\| \le \|\widetilde{B}\| \ \|A^{1/2}\| \ \|x\| \ . \end{aligned}$$

Moreover, by Douglas' theorem and

$$B^*B \le \|\widetilde{B}\|^2 A$$

it follows that $\mathbf{R}(B^*) \subset \mathbf{R}(A^{1/2})$.

Thus, the restriction defines a transformer

$$\begin{aligned} \mathbf{L}(\mathbf{H}_A, \mathbf{H}) &\to \mathbf{L}(\mathbf{H}) \\ \widetilde{B} &\mapsto B \end{aligned}$$

with image $\{B \in \mathbf{L}(\mathbf{H}) : \mathbf{R}(B^*) \subset \mathbf{R}(A^{1/2})\} = \mathbf{L}(\mathbf{H})A^{1/2}$ (the equality is, again, a consequence of Douglas' result). The same argument shows that $\widetilde{B} \in \mathbf{L}(\mathbf{H}_A, \mathbf{H})$ is invertible (i.e., there exists $C \in \mathbf{L}(\mathbf{H}, \mathbf{H}_A)$ such that $\widetilde{B} \circ C = 1_{\mathbf{H}}$, $C \circ \widetilde{B} = 1_{\mathbf{H}_A}$) if and only if $B = VA^{1/2}$ for some $V \in \mathbf{GL}(\mathbf{H})$.

Regarding the norm of $\widetilde{B}$, observe that

$$\|\widetilde{B}\|_{\mathbf{L}(\mathbf{H}_A, \mathbf{H})} = \sup_{\widetilde{x} \ne 0} \frac{\|\widetilde{B}\widetilde{x}\|}{\|\widetilde{x}\|_A} \ .$$

Due to the fact that $\mathbf{H}$ is dense in $\mathbf{H}_A$ it follows that

$$\begin{aligned}\|\widetilde{B}\|_{\mathbf{L}(\mathbf{H}_A,\mathbf{H})} &= \sup_{x\neq 0}\frac{\|Bx\|}{\|A^{1/2}x\|} \qquad\qquad x\in\mathbf{H}\\ &= \sup\left\{\frac{\|BA^{-1/2}z\|}{\|z\|} : z\in\mathbf{R}(A^{1/2}), z\neq 0\right\}\ .\end{aligned}$$

Thus, $\widetilde{B}$ is a bounded operator if and only if $BA^{-1/2}$ is a bounded operator on $\mathbf{R}(A^{1/2})$ and, in this case,

$$\|\widetilde{B}\|_{\mathbf{L}(\mathbf{H}_A,\mathbf{H})} = \|BA^{-1/2}\|\ .$$

If $(BA^{-1/2})^\wedge$ denotes the unique extension of $BA^{-1/2}$ to $(\mathbf{H},\|\ \|)$ we obtain

$$(BA^{-1/2})^\wedge = \widetilde{B}\circ\widehat{A^{-1/2}} = \widetilde{B}\circ\widetilde{A^{1/2}}^{-1} = \widetilde{B}\circ\widehat{A^{-1/2}}\ .$$

because $\widetilde{A^{1/2}}^{-1} = \widetilde{A^{1/2}}^{*}\in\mathbf{L}(\mathbf{H},\mathbf{H}_A)$.

Another useful remark is that $\mathbf{R}(A^{1/2})$ is complete with the norm $\|\ \|_{A^{-1}}$ induced by the inner product

$$\langle x,w\rangle_{A^{-1}} = \left\langle A^{-1/2}z, A^{-1/2}w\right\rangle \qquad\qquad z,w\in\mathbf{R}(A^{1/2})$$

Moreover, $A^{1/2} : (\mathbf{H},\|\ \|)\to\left(\mathbf{R}(A^{1/2},\|\ \|_{A^{-1}}\right)$ is an isometric isomorphism. If $\mathbf{H}'$ denotes the Hilbert space $\left(\mathbf{R}(A^{1/2}),\langle\ ,\ \rangle_{A^{-1}}\right)$ then $\mathbf{L}(\mathbf{H},\mathbf{H}') = \{B\in\mathbf{L}(\mathbf{H})$: there is $X\in\mathbf{L}(\mathbf{H})$ such that $B = A^{1/2}X\} = A^{1/2}\mathbf{L}(\mathbf{H})$ and $\|B\|_{\mathbf{L}(\mathbf{H},\mathbf{H}')} = \|A^{-1/2}B\|_{\mathbf{L}(\mathbf{H})}$.

It may be useful to characterize the elements of $\mathbf{L}(\mathbf{H}_A)$, i.e. linear operators $\mathbf{H}_A\to\mathbf{H}_A$ which are bounded with respect to $\|\ \|_A$. It is easy to see that the map $\phi : B\mapsto\widehat{A^{-1/2}}B\widetilde{A^{1/2}}$ is a isometric isomorphism from $\mathbf{L}(\mathbf{H})$ onto $\mathbf{L}(\mathbf{H}_A)$ which preserves the involution.

In a similar way, $\mathbf{L}(\mathbf{H}') = \{B : \mathbf{R}(A^{1/2})\to\mathbf{R}(A^{1/2})$: there exists $X\in\mathbf{L}(\mathbf{H})$ such that $B = A^{1/2}XA^{-1/2}\} = A^{1/2}\mathbf{L}(\mathbf{H})A^{-1/2}$.

Finally $\mathbf{L}(\mathbf{H}_A,\mathbf{H}') = A^{1/2}\mathbf{L}(\mathbf{H})\widetilde{A^{1/2}}$ and there is a natural isomorphism $\psi$ : $\mathbf{L}(\mathbf{H}')\to\mathbf{L}(\mathbf{H}_A)$,

$$\psi\left(A^{1/2}XA^{-1/2}\right) = \widehat{A^{-1/2}}X^*\widetilde{A^{1/2}} = \widetilde{A^{1/2}}^{-1}X^*\widetilde{A^{1/2}}\ .$$

Using this notions we obtain another characterization of $\mathbf{C}_A$ which will be useful in the next sections.

**Proposition 4.1.** *If $B \in \mathbf{L}(\mathbf{H})^+$ then $B \in \mathbf{C}_A$ if and only if there exists $\widetilde{C} \in \mathbf{L}(\mathbf{H}_A, \mathbf{H})$, $\widetilde{C}$ invertible, such that $B = C^*C$.*

*Now for $A \in \mathbf{L}(\mathbf{H})^+$ not necessarily injective, denote $\mathbf{M} = \overline{\mathbf{R}(A)}$.*
*Then*

$$A|_{\mathbf{M}} : \mathbf{M} \to \mathbf{R}(A)$$

*is an injective operator in $\mathbf{L}(\mathbf{M})^+$. In the same way as before $A|_{\mathbf{M}}$ defines a scalar product in $\mathbf{M}$ and the sets $\mathbf{L}(\mathbf{M}_A, M)$, $\mathbf{L}(\mathbf{M}, \mathbf{M}_A)$, $\mathbf{L}(\mathbf{M}_A)$, $\mathbf{L}(\mathbf{M}_A, \mathbf{M}')$ and $\mathbf{L}(\mathbf{M}')$ can be studied in the same way as in the injective case. (Here $\mathbf{M}'$ denotes the Hilbert space $\left(\mathbf{R}(A^{1/2}), \langle\ ,\ \rangle_{(A|_{\mathbf{M}})^{-1}}\right)$).*

## 5. $\mathbf{C}_A$ as an homogeneous space.

In this section we define an action on $\mathbf{C}_A$ and study the induced homogeneous structure.

Consider $A \in \mathbf{L}(\mathbf{H})^+$ injective and $\mathbf{C}_A$ the component of $A$. If $B \in \mathbf{C}_A$ then $B = A^{1/2}PA^{1/2}$, with $P \in \mathbf{GL}(\mathbf{H})^+$, uniquely determined, as seen in Section 3. If $\widetilde{B} = A^{1/2}P\widetilde{A^{1/2}}$, with $\widetilde{A^{1/2}}$ the extension of $A^{1/2}$ to $\langle \mathbf{H}_A, \|\ \|_A \rangle$, then $\mathbf{C}_A$ identifies with a subset of $\mathbf{L}(\mathbf{H}_A, \mathbf{H}')$.

As in the preceding section

$$\begin{aligned}\mathbf{GL}(\mathbf{H}') &= \{W \in \mathbf{L}(\mathbf{H}') : W = A^{1/2}VA^{-1/2}, V \in \mathbf{GL}(\mathbf{H})\} \\ &= A^{1/2}\mathbf{GL}(\mathbf{H})A^{-1/2}\end{aligned}$$

and

$$\mathbf{GL}(\mathbf{H}_A) = \psi(\mathbf{GL}(\mathbf{H}'))$$

with $\psi : \mathbf{L}(\mathbf{H}') \to \mathbf{L}(\mathbf{H}_A)$ the map defined in Section 4 by

$$\psi(A^{1/2}ZA^{-1/2}) = \left(\widetilde{A^{1/2}}\right)^{-1} Z^* \widetilde{A^{1/2}}\ .$$

Consider the following action on $\mathbf{C}_A$

$$\begin{aligned}\mathbf{L} : \mathbf{GL}(\mathbf{H}') \times \mathbf{C}_A &\to \mathbf{C}_A \\ (W, \widetilde{B}) &\to \mathbf{L}_W \widetilde{B} = W\widetilde{B}\psi(W)\ .\end{aligned}$$

Then

$$\mathbf{L}_W \widetilde{B} = A^{1/2}VPV^*\widetilde{A^{1/2}}$$

where $W = A^{1/2}VA^{-1/2}$ and $\widetilde{B} = A^{1/2}P\widetilde{A^{1/2}}$, $V \in \mathbf{GL}(\mathbf{H})$, $P \in \mathbf{GL}(\mathbf{H})^+$, uniquely determined.

Also, as $P = \left(A^{-1/2}BA^{-1/2}\right)^{\wedge}$, (see Section 4)

$$\mathbf{L}_W\widetilde{B} = A^{1/2}V\left(A^{-1/2}BA^{-1/2}\right)^{\wedge}V^*\widetilde{A^{1/2}} .$$

It is easy to see that $\mathbf{L}$ satisfies

i) $\mathbf{L}_W L_T = \mathbf{L}_{WT}$ because $\psi(WT) = \psi(T)\psi(W), \quad W, T \in \mathbf{GL}(\mathbf{H}')$.

ii) $\mathbf{L}_1 = id$ because $\psi(1_{\mathbf{H}'}) = 1_{\mathbf{H}_A}$.

iii) $\mathbf{L}$ is transitive: if $W = A^{1/2}VA^{-1/2}, V \in \mathbf{GL}(\mathbf{H})$, then $\mathbf{L}_W\widetilde{A} = A^{1/2}VV^*\widetilde{A^{1/2}}$.

$\mathbf{C}_A$ can be considered as a differential submanifold of $\mathbf{L}(\mathbf{H}_A, \mathbf{H}')$ because $\mathbf{C}_A = g\left(\mathbf{GL}(\mathbf{H})^+\right)$, where

$$\begin{aligned} g : \mathbf{L}(\mathbf{H}) &\to \mathbf{L}(\mathbf{H}_A, \mathbf{H}') \\ x &\to A^{1/2}X\widetilde{A^{1/2}} \end{aligned}$$

is an isometric isomorphism.

Consider $p : \mathbf{GL}(\mathbf{H}') \to \mathbf{C}_A$ the map defined by $p(W) = \mathbf{L}_W\widetilde{A} = W\widetilde{A}\psi(W) = A^{1/2}VV^*\widetilde{A^{1/2}}$, with $W = A^{1/2}VA^{-1/2}$.

The isotropy group $\mathcal{I}_A$ of $A$ is

$$\mathcal{I}_A = \{W \in \mathbf{GL}(\mathbf{H}') : p(W) = \widetilde{A}\} = A^{1/2}\mathcal{U}(\mathbf{H})A^{-1/2} = \mathcal{U}(\mathbf{H}') .$$

where $\mathcal{U}(\mathbf{H})$ and $\mathcal{U}(\mathbf{H}')$ are the unitary groups of $\mathbf{H}$ and $\mathbf{H}'$.

In general, for $B \in \mathbf{C}_A$, the isotropy group $\mathcal{I}_B$, of $B$ is

$$\mathcal{I}_B = A^{1/2}\mathcal{I}_{\left(A^{-1/2}BA^{-1/2}\right)^{\wedge}}A^{-1/2}$$

with $\mathcal{I}_{\left(A^{-1/2}BA^{-1/2}\right)^{\wedge}}$ the isotropy group of (the positive invertible element of $\mathbf{L}(\mathbf{H})$) $(A^{-1/2}BA^{-1/2})^{\wedge}$ corresponding to the action $\mathbf{L} : \mathbf{GL}(\mathbf{H}) \times \mathbf{GL}(\mathbf{H})^+ \to \mathbf{GL}(\mathbf{H})^+$, $\mathbf{L}_V P = VPV^*$, $V \in \mathbf{GL}(\mathbf{H})^*$, $P \in \mathbf{GL}(\mathbf{H})^+$ and $P(V) = \mathbf{L}_V 1$. See [15].

The tangent space $(\mathcal{T}\mathcal{I}_A)_1$ coincides with $A^{1/2}\mathbf{L}(\mathbf{H})_{ah}A^{-1/2}$, where $\mathbf{L}(\mathbf{H})_{ah} = \{X \in \mathbf{L}(\mathbf{H}) : \ X^* = -X\}$, or also

$$(\mathcal{T}\mathcal{I}_A)_1 = \{Y \in \mathbf{L}(\mathbf{H}') : \ Y^{\#} = -Y\} = \mathbf{L}(\mathbf{H}')_{ah}$$

where $Y^{\#}$ is the adjoint of $Y$ in $\mathbf{L}(\mathbf{H}')$,

$$Y^{\#} = A^{1/2}X^*A^{-1/2} \ , \quad \text{if} \ \ Y = A^{1/2}XA^{-1/2} \ , \quad X \in \mathbf{L}(\mathbf{H}) .$$

For $\widetilde{B} \in \mathbf{R}_A$, the tangent space to $\mathbf{C}_A$ at $\widetilde{B}$, is

$$(\mathcal{T}\mathbf{C}_A)_{\widetilde{B}} = \{Y \in \mathbf{L}(\mathbf{H}_A, \mathbf{H}') : \; Y = A^{1/2} X \widetilde{A^{1/2}}, \; X = X^* \in \mathbf{L}(\mathbf{H})\}$$

with $\mathbf{L}(\mathbf{H})_h$ the subspace of hermitian elements of $\mathbf{L}(\mathbf{H})$, $\mathbf{L}(\mathbf{H})_h = \{X \in \mathbf{L}(\mathbf{H}) / X^* = X\}$.

Observe that $(\mathcal{T}\mathbf{C}_A)_{\widetilde{B}}$ is a (real) closed subspace of $\mathbf{L}(\mathbf{H}_A, \mathbf{H}')$ because

$$(\mathcal{T}\mathbf{C}_A)_{\widetilde{B}} = g(\mathbf{L}(\mathbf{H})_h) \; ,$$

$g : \mathbf{L}(\mathbf{H}) \to \mathbf{L}(\mathbf{H}_A, \mathbf{H}')$ the isometry defined before.

The tangent map of $p$ at 1 is

$$(\mathcal{T}_p)_1 : \mathbf{L}(\mathbf{H}') \to (\mathcal{T}\mathbf{C}_A)_{\widetilde{A}}$$
$$(\mathcal{T}_p)_1 Y = Y\widetilde{A} + \widetilde{A}\psi(Y) \; .$$

A linear connection is defined on $\mathbf{C}_A$ by giving the following distribution of subspaces of $\mathbf{L}(\mathbf{H}')$ at each $W \in \mathbf{GL}(\mathbf{H}')$:

$$\mathcal{H}_1 = \mathbf{GL}(\mathbf{H}')_h = A^{1/2}\mathbf{L}(\mathbf{H})_h A^{-1/2}$$
$$\mathcal{V}_1 = (\mathcal{T}\mathcal{I}_A)_1 = \mathbf{GL}(\mathbf{H}')_{ah}$$

and

$$\mathcal{H}_W = W\mathcal{H}_1 \; , \qquad \mathcal{V}_W = W\mathcal{V}_1 \; .$$

and we have that

$$(\mathcal{T}_p)_1 \big|_{\mathcal{H}_1} : \mathcal{H}_1 \to (\mathcal{T}\mathbf{C}_A)_{\widetilde{A}}$$

is an isomorphism, $(\mathcal{T}_p)_1(Y) = 2Y\widetilde{A}$, $Y \in \mathcal{H}$, because $\widetilde{H}\psi(Y) = Y\widetilde{A}$ if $Y^{\#} = Y$.

Define

$$K_{\widetilde{A}} = \left( (\mathcal{T}_p)_1 \big|_{\mathcal{H}_1} \right)^{-1} : (\mathcal{T}\mathbf{C}_A)_{\widetilde{A}} \to \mathcal{H}_1$$

then

$$K_{\widetilde{A}}(Z) = \frac{1}{2} Z \widetilde{A}^{-1} \; .$$

Observe that if $Z \in (\mathcal{T}\mathbf{C}_A)_{\widetilde{A}}$ then $Z = A^{1/2} X \widetilde{A^{1/2}}$ for some hermitian $X$ in $\mathbf{L}(\mathbf{H})$, and $K_{\widetilde{A}}(Z) = \frac{1}{2} A^{1/2} X A^{-1/2} \in \mathbf{L}(\mathbf{H}')_h$.

Given $\gamma : [0,1] \to \mathbf{C}_A$ a smooth curve in $\mathbf{C}_A$, consider a lift $W : [0,1] \to \mathbf{GL}(\mathbf{H}')$, of $\gamma$, i.e., $W$ is a curve in $\mathbf{GL}(\mathbf{H}')$ such that $p(W(t)) = \gamma(t)$. $W$ is called horizontal if $\dot{W}(Y) \in \mathcal{H}_{W(t)}$ for all $t$ or, equivalently, if $W^{-1}\dot{W} \in \mathcal{H}_1$.

If

$$W(t) = A^{1/2} V(t) A^{-1/2} \qquad V(t) \subset \mathbf{GL}(\mathbf{H})$$

and

$$\gamma(t) = A^{1/2} P(t) \widetilde{A^{1/2}} \qquad\qquad P(t) \subset \mathbf{GL}(\mathbf{H})^+ \ ,$$

then $W$ is horizontal if and only if

$$V^{-1}\dot{V} \in \mathbf{L}(\mathbf{H})_h \ .$$

But $V^{-1}V \in \mathbf{L}(\mathbf{H})_h$ if and only if $V$ is an horizontal lift of $P(t)$.

Then $W$ is a horizontal lift of $\gamma$ if and only if $V$ is a horizontal lift of $P$, or equivalently if $V$ is a solution to the associated transport equation

$$\dot{V} = \frac{1}{2}\dot{P}P^{-1}V \ .$$

But then $W$ is a solution of the transport equation

$$\dot{W} = \frac{1}{2}\dot{\gamma}\gamma^{-1}W \ .$$

The transport equation induces a **covariant derivative** of a tangent field $X$ along a curve $\gamma$, $\frac{DX}{dt}$, and the field $X$ is **parallel** along $\gamma$ if $\frac{DX}{dt} = 0$. A curve $\gamma$ is a **geodesic** is $\dot{\gamma}$ is parallel along $\gamma$.

For $\widetilde{B}$ and $\widetilde{C} \in \mathbf{C}_A$ there exists a geodesic joining them, $\gamma_{\widetilde{B},\widetilde{C}}$, namely

$$\gamma_{\widetilde{B},\widetilde{C}}(t) = B^{1/2}\left(\left(B^{-1/2}CB^{-1/2}\right)^{\wedge}\right)^{t}\widetilde{B^{1/2}}$$

where, again, $\left(B^{-1/2}CB^{-1/2}\right)^{\wedge}$ is the extension of $B^{-1/2}CB^{-1/2}$ to an operator in $\mathbf{L}(\mathbf{H})^+$.

Observe that the geodesic $\gamma_{\widetilde{B},\widetilde{C}}$, joining $\widetilde{B}$ and $\widetilde{C}$ only depends on $\widetilde{B}$ and $\widetilde{C}$, and not on $A$.

Also, if $\widetilde{B} = A^{1/2}P_1\widetilde{A^{1/2}}$ and $\widetilde{C} = A^{1/2}P_2\widetilde{A^{1/2}}$ with $P_1$ and $P_2$ in $\mathbf{GL}(\mathbf{H})^+$ then $A^{-1/2}\gamma_{B,C}\left(\widetilde{A^{1/2}}\right)^{-1}$ is the geodesic in $\mathbf{GL}(\mathbf{H})^+$ joining $P_1$ and $P_2$: if $A^{-1/2}B^{1/2} = P_1^{1/2}U$ and $A^{-1/2}C^{1/2} = P_2^{1/2}V$, with $U$ and $V$ in $\mathcal{U}(\mathbf{H})$, then

$$A^{-1/2}\gamma_{B,C}(t)\left(\widetilde{A^{1/2}}\right)^{-1} = P_1^{1/2}U\left(\left(B^{-1/2}CB^{-1/2}\right)^{\wedge}\right)^{t}U^{-1}P_1^{1/2} \ .$$

and

$$\left(B^{-1/2}CB^{-1/2}\right)^{\wedge} = U^{-1}P_1^{-1/2}P_2P_1^{-1/2}U \ ,$$

therefore

$$\left(\left(B^{-1/2}CB^{-1/2}\right)^{\wedge}\right)^{t} = U^{-1}\left(P_1^{-1/2}P_2P_1^{-1/2}\right)^{t}U$$

and then

$$A^{-1/2}\gamma_{B,C}\left(\widetilde{A^{1/2}}\right)^{-1} = P_1^{1/2}\left(P_1^{-1/2}P_2P_1^{-1/2}\right)^t P_1^{1/2}$$
$$= \gamma_{P_1,P_2}(t)\ .$$

Finally, we define a Finsler structure by setting a norm $\|\ \|_{\widetilde{B}}$ on the tangent space $(\mathcal{T}\mathbf{C}_A)_{\widetilde{B}}$ : if $Y \in (\mathcal{T}\mathbf{C}_A)_{\widetilde{B}}$

$$\|Y\|_{\widetilde{B}} = \|Y\|_{L(\mathbf{H}_B,\mathbf{H}'_B)} = \|B^{-1/2}Y\left(\widetilde{B^{1/2}}\right)^{-1}\|_{L(\mathbf{H})}$$

where $\mathbf{H}'_B$ denotes the Hilbert space $\left(\mathbf{R}(B^{1/2}), \langle\ ,\ \rangle_{B^{-1}}\right)$.

Observe that this metric coincides with the metric defined for positive invertible operators (see [15]), because in this case, if $X \in \mathbf{L}(\mathbf{H})_h$

$$\|X\|_A = \|A^{-1/2}XA^{-1/2}\| \qquad\qquad A \in \mathbf{GL}(\mathbf{H})^+\ .$$

The length of a curve $\gamma : [0,1] \to \mathbf{C}_A$ is

$$L(\gamma) = \int_0^1 \|\dot{\gamma}\|_\gamma dt$$

and if $\gamma(t) = A^{1/2}P(t)\widetilde{A^{1/2}}$, $P(t) \subset \mathbf{GL}(\mathbf{H})^+$ then $L(\gamma_{(t)}) = L(P(t))$.

Then as a consequence of the results for invertible positive operators, if $\gamma : [0,1] \to \mathbf{C}_A$ is a curve with $\gamma(0) = \widetilde{B}$ and $\gamma(1) = \widetilde{C}$ then

$$L(\gamma) \geq L\left(\gamma_{\widetilde{B},\widetilde{C}}\right) = \|\log(B^{-1/2}CB^{-1/2})^\wedge\|\ .$$

where $\gamma_{\widetilde{B},\widetilde{C}}$ is the geodesic joining $\widetilde{B}$ and $\widetilde{C}$. See [12].

Then if

$$d(\widetilde{B},\widetilde{C}) = \inf\{L(\gamma) : \gamma[0,1] \to \mathbf{C}_A,\quad \gamma(0) = \widetilde{B}, \gamma(1) = \widetilde{C}\}$$

is the geodesic metric, then

$$d(\widetilde{B},\widetilde{C}) = \mathbf{L}(\gamma_{\widetilde{B},\widetilde{C}}) = \|\log(B^{-1/2}CB^{-1/2})^\wedge\|\ .$$

But $\|\log(B^{-1/2}CB^{-1/2})^\wedge\|$ coincides with the Thompson metric $d_T(B,C)$. See [12].

Therefore the geodesic metric coincides with the Thompson metric defined in each component,

$$d(\widetilde{B},\widetilde{C}) = d_T(B,C) = \|\log(B^{-1/2}CB^{-1/2})^\wedge\|$$

In the general case, if $A \in \mathbf{L}(\mathbf{H})^+$ is not necessary injective, $\mathbf{C}_A$ can be considered as a differential submanifold of $\mathbf{L}(\mathbf{M}_A, \mathbf{M}')$; $\mathbf{M}_A = \left(\overline{\mathbf{R}(A)}, \langle\ ,\ \rangle_{(A|_\mathbf{M})^{-1}}\right)$, $\mathbf{M} = \overline{\mathbf{R}(A)}$, as in Section 3, and as in the injective case, we obtain a homogeneous space with a Finsler structure.

The results of this section can be put in the context of an abstract $C^*$-algebra.

We have described the spacial case because it does not hide the geometrical notions.

## 6. Geometry of density operators.

In a series of papers, A. Uhlmann and others [39], [40], [41], [19] made a differential geometric study of the set of density operators in a Hilbert space $\mathbf{H}$. We describe their main results in order to compare them with our results of section 2. A *density operator* on $\mathbf{H}$ is a positive trace class operator with trace 1:

$$\Omega = \{\rho \in \mathbf{L}(\mathbf{H})^+ : tr(\rho) = 1\} .$$

The "extended space" $\mathbf{H}^{\text{ext}}$ is

$$\mathbf{H}^{\text{ext}} = \{\omega \in \mathbf{L}(\mathbf{H}) : tr(\omega\omega^*) < +\infty\} ,$$

i.e. the space of Hilbert-Schmidt operators on $\mathbf{H}$, with scalar product given by

$$< \omega_1, \omega_2 >= tr(\omega_1^*\omega_2) = tr(\omega_2\omega_1^*) .$$

Consider the map $\mathbf{H}^{\text{ext}} - \{0\} \to \Omega$

$$\omega \mapsto \rho_\omega = \omega\omega^*/tr(\omega\omega^*) .$$

An element $\omega \in \mathbf{H}^{\text{ext}} - \{0\}$ is called a *purification*; $\omega$ is a *standard purification* of $\rho$ if $\rho = \omega\omega^*$.

There is not a unique way of constructing standard purifications: observe that if $u$ is unitary and $\omega$ is a standard purification of $\rho$ then $\omega u$ is also standard.

Suppose that $\rho : [0,1] \to \Omega$ is a smooth curve of density operators. A *parallel purification* of $\rho$ is a smooth curve $\omega : [0,1] \to \mathbf{H}^{\text{ext}}$ such that

$$\omega^*\dot{\omega} - \dot{\omega}^*\omega = 0 .$$

A solution of this equation can be found by solving

(6.1) $$\dot{\rho} = g\rho + \rho g$$

and then

$$\dot{\omega} = g\omega .$$

The solution $g$ of (6.1) is automatically Hermitian.

Consider the following variational problem: given a curve $\rho : [0,1] \to \Omega$ find

$$\inf \int_0^1 < \dot\omega, \dot\omega >^{1/2} dt$$

where the infimum is taken over all standard purifications $\omega$ of $\rho$. The Euler equation of this problem is precisely (6.1). Thus, $\omega$ is a parallel purification if and only if $\omega$ solves the variational problem above.

Uhlmann asserts that the computation can be done in general, however, the proofs are done only in the finite dimensional case. It should be said that $\Omega$ is neither a smooth manifold nor a manifold with boundary when $\mathbf{H}$ is infinite dimensional.

We proceed now to compare the study made by Uhlmann with the results of sec-
tion 2.

Recall the left action $L_g : \mathbf{GL(H)}^+ \to \mathbf{GL(H)}^+$ of $\mathbf{GL(H)}$ over $\mathbf{GL(H)}^+$. The horizontal lift $\omega$ of a curve $\rho$ in $G^+$ is the solution of the transport equation

$$\dot\omega\omega^{-1} = \frac{1}{2}\dot\rho\rho^{-1} \ .$$

If $\omega$ satisfies $\omega\omega^* = \rho$ then $\omega$ is a horizontal lift if and only if $\omega$ satisfies

$$\dot\omega\omega^* - \omega\dot\omega^* = 0 \ .$$

Observe the resemblance of the last equation with (6.1). Let us modify our construction of section 2 in order to obtain a coincidence between horizontal lifts and parallel lifts. We insist that Uhlmann's constructions are done in $\Omega$ with essentially singulars operators meanwhile, we work only with invertible operators.

For $a \in \mathbf{GL(H)}^+$ let $\pi_a : \mathbf{GL(H)} \to \mathbf{GL(H)}^+$ be defined by

$$\pi_a(\omega) = \omega^* a \omega = L_{\omega^*} a \ .$$

Abbreviate $\pi = \pi_1$.

The *horizontal space* $\mathcal{H}_1$ in $I \in \mathbf{L(H)}$ is

$$\mathcal{H}_1 = \{X \in \mathbf{L(H)} : X^* = X\} = \mathbf{L(H)}_h \ .$$

In general, for $\omega \in \mathbf{GL(H)}$ define

$$\mathcal{H}_g = \mathcal{H}_1 \, g \ .$$

Notice the difference between the horizontal space of section 2.

A *lift* $\omega$ of a smooth curve $\alpha$ in $\mathbf{GL(H)}^+$ is a smooth curve $\omega$ in $G$ such that

$$\pi(\omega) = \alpha \ , \text{ i.e. } , \omega^*\omega = \alpha \ .$$

As before, we can prove that for every smooth curve $\alpha : [0,1] \to \mathbf{GL(H)}^+$ and every $\omega_0 \in \mathbf{GL(H)}$ such that $\omega_0^*\omega_0 = \alpha(0)$ there exists a unique horizontal lift $\omega : [0,1] \to \mathbf{GL(H)}$ of $\alpha$. It is the unique solution of the transport equation

$$\begin{cases} \omega^{-1}\dot{\omega} & = \frac{1}{2}\alpha^{-1}\dot{\alpha} \\ \omega(0) & = \omega_0 \ . \end{cases}$$

Given smooth curve $\rho : [0,1] \to \mathbf{GL(H)}^+$ the curve $\omega : [0,1] \to \mathbf{GL(H)}^+$ is called a *purification* of $\rho$ if $\rho = \omega\omega^*$ .

The curve $\omega$ is *parallel* if

$$\omega^*\dot{\omega} - \dot{\omega}^*\omega = 0 \ .$$

We have that if $\omega$ is a purification a $\rho$, then $\omega$ is parallel if and only if $\omega$ is a horizontal lift of $\alpha = \omega^*\omega$, and as a consequence the following

**Proposition 6.2.** *For a smooth curve $\rho$ in $\mathbf{GL(H)}^+$ with a purification $\omega$ in $\mathbf{GL(H)}$, it holds that $\omega$ is parallel if and only if it satisfies*

$$\dot{\rho} = \rho\dot{\omega}\omega^{-1} + \dot{\omega}\omega^{-1}\rho$$

**Remark.** *It can be proved that the connection determined by the horizontal spaces $\mathcal{H}_g$ coincides, when $A$ is the matrix algebra $M_n(\mathbb{C})$, with the distribution defined by Dittmann and Rudolph [19].*

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