HOMOTOPY OF STATE ORBITS[1]

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Abstract

Let M be a von Neumann algebra, φ a faithful normal state and denote by M^φ the fixed point algebra of the modular group of φ. Let U_M and U_{M^φ} be the unitary groups of M and M^φ. In this paper we study the quotient $\mathcal{U}_\varphi = U_M/U_{M^\varphi}$ endowed with two natural topologies: the one induced by the usual norm of M (called here usual topology of $\mathcal{U}_\varphi$), and the one induced by the pre-Hilbert C*-module norm given by the φ-invariant conditional expectation $E_\varphi : M \to M^\varphi$ (called the modular topology). It is shown that $\mathcal{U}_\varphi$ is simply connected with the usual topology. Both topologies are compared, and it is shown that they coincide if and only if the Jones index of E_φ is finite. The set $\mathcal{U}_\varphi$ can be regarded as a model for the unitary orbit $\{\varphi \circ Ad(u^*) : u \in U_M\}$ of φ, and either with the usual or the modular it can be embedded continuously in the conjugate space M^* (although not as a topological submanifold).

1 Introduction

Let M be a von Neumann algebra and φ a faithful normal state of M. Denote by U_M the unitary group of M, and by M^φ the centralizer of φ, that is $M^\varphi = \{x \in M : \varphi(xy) = \varphi(yx) \text{ for all } y \in M\}$. Let $\mathcal{U}_\varphi$ be the unitary orbit of φ, i.e.

$$\mathcal{U}_\varphi = \{\varphi \circ Ad(u) : u \in U_M\}$$

where $Ad(u)(x) = uxu^*$. The isotropy subgroup at φ (=the set of unitaries that leave φ fixed) is the unitary group U_{M^φ} of M^φ. In previous papers we introduced a homogeneous and reductive structure for $\mathcal{U}_\varphi$, by means of the natural identification

$$\mathcal{U}_\varphi \simeq U_M/U_{M^\varphi}.$$

We are not regarding $\mathcal{U}_\varphi$ with the norm topology of M^* (in [4] it was shown that in general $\mathcal{U}_\varphi$ is not a submanifold of M^*), but with the quotient topology induced by the usual norm of M. With this topology $\mathcal{U}_\varphi$ is a real analytic manifold. In particular, the map

$$\pi_\varphi : U_M \to \mathcal{U}_\varphi \ , \ \pi_\varphi(u) = \varphi \circ Ad(u^*)$$

is a (principal) fibre bundle, with fibre U_{M^φ}. In section 2 we use this fibration to prove that these orbits $\mathcal{U}_\varphi$ are always simply connected.

There is another natural topology in the set U_M/U_{M^φ}. Namely, let $E = E_\varphi$ be the unique φ-invariant conditional expectation $E : M \to M^\varphi$. This gives rise to

[1] 1991 Mathematics Subject Classification: 46L30, 58B05.

a natural pre-Hilbert C*-module structure for M, with M^φ valued inner product given by $< x, y >= E(x^*y)$ and norm $\|x\|_E = \|E(x^*x)\|^{1/2}$. It is well known that M is $\| \ \|_E$-complete if and only if the Jones index of E is finite ([8],[15]). The condition that E be of finite index is a rather strict requirement for φ. This implies that in general the usual norm and $\| \ \|_E$ define different topologies for M and for U_M/U_{M^φ}. We call them, respectively, the **usual** and the **modular** topology in $\mathcal{U}_\varphi$.

One has the following inequality, for $u, w \in U_M$:

$$\|\varphi \circ Ad(u^*) - \varphi \circ Ad(w^*)\| \leq 2\|u - w\|_E \leq 2\|u - w\|$$

where the first norm is the usual norm in M^*. If one replaces u, w by uv, wv', for $v, v' \in U_{M^\varphi}$ then $\varphi \circ Ad((uv)^*) = \varphi \circ Ad(u^*)$ and $\varphi \circ Ad((wv')^*) = \varphi \circ Ad(w^*)$. This implies that $\mathcal{U}_\varphi$, both in the usual and the modular topology, can be embedded in the conjugate space M^*. These matters are discussed in section 3. We present different models for $\mathcal{U}_\varphi$, inside the grassmannians of the basic extension of E (see below), and inside the interior tensor product $M \otimes_{M^\varphi} M$ of M regarded as a pre-Hilbert module M.

Finally, let us recall the basic extension of $E : M \to M^\varphi$. Denote by H_φ the completion of the pre-Hilbert space M with the inner product given by φ. Then E is bounded for this inner product, and therefore extends to a selfadjoint projection $e = e_\varphi$, called the Jones projection, whose range is the closure of M^φ in H_φ. Let $M_1 \subset B(H_\varphi)$ be the von Neumann algebra generated by M and e. We refer the reader to [11], [15] or [8] for the details of this construction. Some of the properties of e are:

- $eae = E(a)e, \ a \in M$
- $M \cap \{e\}' = M^\varphi$
- The map $x \mapsto xe$ is a *-isomorphism between M^φ and $M^\varphi e$.

2 The fundamental group of $\mathcal{U}_\varphi$ is trivial

Throughout this section $\mathcal{U}_\varphi$ is endowed with the usual topology (i.e. the quotient topology induced by the usual norm of M). Recall that a fibre bundle gives rise to an exact sequence of homotopy groups. In our case, the bundle π_φ yields the exact sequence

$$\cdots \pi_2(\mathcal{U}_\varphi) \to \pi_1(U_{M^\varphi}) \xrightarrow{i_*} \pi_1(U_M) \to \pi_1(\mathcal{U}_\varphi) \to \pi_0(U_{M^\varphi}) = 0,$$

where 1 is taken as base point for the homotopy groups of the unitary groups and φ is the base point for $\mathcal{U}_\varphi$. Here i_* denotes the homomorphism induced by the inclusion $i : U_{M^\varphi} \hookrightarrow U_M$. We can use then results by Handelman [10], Schröder [16], Breuer [7], as well as the classical result by Kuiper [13], computing the homotopy groups of the unitary group of a von Neumann algebra, in order to obtain information about $\mathcal{U}_\varphi$.

Note that the center $\mathcal{Z}(M^\varphi)$ of M^φ includes the center $\mathcal{Z}(M)$ of M. Suppose that $p \in \mathcal{Z}(M)$ is a projection. Then $\varphi_p = \varphi|_{Mp}$ is a faithful and normal state in Mp whose centralizer is the algebra $M^\varphi p$, and the canonical conditional expectation E_{φ_p} is the restriction of E to Mp. In other words, each projection in the center of M factorizes the unitary groups of M and M^φ and the orbit $\mathcal{U}_\varphi$:

$$U_M \simeq U_{Mp} \times U_{M(1-p)} \ , \ U_{M^\varphi} \simeq U_{(Mp)^{\varphi_p}} \times U_{(M(1-p))^{\varphi_{1-p}}} \ \text{ and } \mathcal{U}_\varphi \simeq \mathcal{U}_{\varphi_p} \times \mathcal{U}_{\varphi_{1-p}}.$$

Therefore, if one considers the central type decomposition projections of M, the study of the homotopy type of $\mathcal{U}_\varphi$ reduces to the case when M is of a definite type. We shall proceed to show that $\mathcal{U}_\varphi$ is simply connected through a series of lemmas, covering the possible types of M and M^φ.

By well known results [13] [7], the properly infinite part of M gives state orbits with vanishing π_1 group. Indeed,

Lemma 2.1 *If M is a properly infinite von Neumann algebra, then $\mathcal{U}_\varphi$ is simply connected.*

Proof. Since $\mathcal{U}_\varphi$ is connected, it remains to prove that $\pi_1(\mathcal{U}_\varphi)$ is trivial. This follows by appealing to the homotopy exact sequence, and the fact ([13], [7]) that U_M has trivial π_1 group. □

One is therefore constrained to the case when M is finite. Let us recall the following results (see [1]), which are based on the results of [10] (see also [16]). If M is of type II_1, Handelman proved that $\pi_1(U_M)$ is isomorphic to (the additive group) $\mathcal{Z}(M)_{sa}$ of selfadjoint elements of the center of M.

Lemma 2.2 *Let M be a type II_1 von Neumann algebra and Tr its center valued trace. Suppose that $N \subset M$ is a von Neumann subalgebra with the same unit, and denote by i the inclusion map $i : U_N \hookrightarrow U_M$. Then the image of the homomorphism $i_* : \pi_1(U_N) \to \pi_1(U_M) \simeq \mathcal{Z}(M)_{sa}$ is equal to the additive group generated by the set $\{Tr(p) : p$ projection in $N\}$.*

There is an analogous result for the type I case. If M is of type I_n, then $\pi_1(U_M) \simeq C(\Omega, \mathbb{Z})$, where Ω denotes the Stone space of the center of M.

Lemma 2.3 *Let M be a von Neumann algebra of type I_n, $N \subset M$, and Tr the center valued trace of M, then the image of i_* identifies with the group generated by the functions $\{Tr(p)\hat{} : p$ projection in $N\}$.*

In order to compute the π_1 group of $\mathcal{U}_\varphi$ we shall apply these results to the case $N = M^\varphi$. From these lemmas it is clear that one needs to compute $\{Tr(p) : p$ projection in $M^\varphi\}$.

The next step is to try further reductions using the type decomposition central projections of M^φ.

Lemma 2.4 *Suppose that M and M^φ are of type II_1. Then i_* is surjective. As a consequence, $\mathcal{U}_\varphi$ is simply connected.*

Proof. We claim that in this case the homomorphism i_* identifies with $Tr|_{\mathcal{Z}(M^\varphi)_{sa}} : \mathcal{Z}(M^\varphi)_{sa} \to \mathcal{Z}(M)_{sa}$. Then it is clear that i_* is surjective. In order to prove our claim, we shall see first that if Tr and Tr^φ denote respectively the center valued traces of M and M^φ, then $Tr \circ Tr^\varphi = Tr|_{M^\varphi}$. Indeed, let $x \in M^\varphi$, then $Tr^\varphi(x)$ is the (norm) limit of a sequence of elements $v_n x v_n^*$ in $co\{vxv^* : v \in U_{M^\varphi}\} \cap \mathcal{Z}(M^\varphi)$ (where $co\{vxv^* : v \in U_{M^\varphi}\}$ denotes the convex hull of $\{vxv^* : v \in U_{M^\varphi}\}$). Since $Tr(v_n x v_n^*) = Tr(x)$, it follows that $Tr(Tr^\varphi(x)) = Tr(x)$. Now let z be an element in $\mathcal{Z}(M^\varphi)$ with $0 \le z \le 1$, then there exists a projection $p \in M^\varphi$ such that $Tr^\varphi(p) = z$. Under the identification $\pi_1(U_{M^\varphi}) \simeq \mathcal{Z}(M^\varphi)_{sa}$, the class of the loop $\gamma(t) = e^{itp}$ in U_{M^φ} corresponds to the element z, i_* sends this element to the class of γ in U_M, that is, to $Tr(p)$ (see [10]). Therefore $i_*(z) = Tr(p) = Tr(Tr^\varphi(p)) = Tr(z)$. The fact that $\mathcal{U}_\varphi$ is simply connected follows from the exact sequence, where $\pi_1(\mathcal{U}_\varphi) = \pi_1(U_M)/Im(i_*) = 0$. □

Let us consider the following examples, which show that for M of type II_1, different types of M^φ can occur.

Examples 2.5 *1 Suppose that M is of type II_1, let p be a projection and τ a faithful tracial state. Consider $h = \frac{1}{2s}p + \frac{1}{2(1-s)}(1-p)$, with $s = \tau(p)$. Put $\varphi(x) = \tau(hx)$. Clearly φ is a faithful and normal state, with $E(x) = pxp + (1-p)x(1-p)$ and $M^\varphi = \{p\}' \cap M = pMp \oplus (1-p)M(1-p)$, which is also of type II_1. A similar example can be done with a family $\{p_n \in M\}_{n\in\mathbb{N}}$ of mutually orthogonal projections.*

2 Let again M be of type II_1, and let $\mathcal{A} \subset M$ be a maximal abelian subalgebra. Choose τ a faithful normal tracial state and h a positive operator without kernel (i.e. $ha = 0$ implies $a = 0$) that generates $\mathcal{A}$, normalized so that $\tau(h) = 1$. Note that $\mathcal{A}$ is purely non atomic. Consider the state $\varphi(x) = \tau(hx)$. Then $M^\varphi = \{h\}' \cap M = \mathcal{A}$. Another example can be obtained by tensoring M with $M_n(\mathbb{C})$ and φ with the usual trace t_n of $M_n(\mathbb{C})$. In this case $M_1 = M \otimes M_n(\mathbb{C})$ is of type II_1 and $M_1^{\varphi\otimes t} = M^\varphi \otimes M_n(\mathbb{C})^{t_n} = M_n(\mathcal{A})$ which is of type I_n.

Note that if M is a finite von Neumann algebra with a faithful normal state φ such that M^φ is abelian, then M^φ must be maximal abelian in M. Indeed, if $a \in (M^\varphi)' \cap M$, and $\varphi = \tau(h\ )$ for a tracial state τ, and $x \in M$, then $\varphi(ax) = \tau(hax) = \tau(ahx) = \tau(hxa) = \varphi(xa)$, i.e. $a \in M^\varphi$. In particular, if M is of type II_1, then the (necessarily non trivial) center of M^φ must be non atomic, and therefore the situation in the above example, part 2, is essentially the only possible one.

Lemma 2.6 *If M is of type II_1 and M^φ is abelian, then $\mathcal{U}_\varphi$ is simply connected.*

Proof. As before, one needs to show that if Tr is the center valued trace of M, then $\{Tr(q) : q \text{ projection in } M^\varphi\} = \mathcal{Z}(M)_{sa}$. First, pick $c \in \mathcal{Z}(M)_{sa}$ of the form $c = \sum_{i=1}^n \alpha_i p_i$ with p_i mutually orthogonal and $0 \leq \alpha_i \leq 1$. Given $\epsilon > 0$ sufficiently small, we claim that there exists projections $q_i \in M^\varphi$ such that $0 \leq (\alpha_i - \epsilon/n)p_i \leq Tr(q_i) \leq \alpha_i p_i$. Indeed, otherwise there would be a central projection p and an interval $(0, \lambda)$ such that between 0 and λp there are no values $Tr(q)$ with q projection in M^φ. In that case, put

$$\lambda_0 = \sup\{\lambda > 0 : \text{ there are no projections } q \in M^\varphi \text{ with } Tr(q) \leq \lambda p\}.$$

Clearly, $0 < \lambda_0 \leq 1$. Then one can find sequences $\lambda_n > \lambda_0$ and q_n, where λ_n decreases to λ_0 and q_n are projections in M^φ with $q_n \geq q_{n+1}$, such that $Tr(q_n) \leq \lambda_n p$. Then $q_0 = \wedge_n q_n$ is a projection in M^φ satisfying $Tr(q_0) = \lambda_0 p$. Suppose now that there exists a projection r in M^φ with $0 < r < q_0$. Let $e_n = \chi_{[0,\lambda_0 - 1/n)}(Tr(r))$ be the spectral projection of $Tr(r)$ associated to the interval $[0, \lambda_0 - 1/n)$, lying in $\mathcal{Z}(M)$. Then clearly $r_n = e_n r$ is a projection in M^φ satisfying that $Tr(r_n) \leq \lambda_0 - 1/n$. Since r_n increases to r, there exists n such that r_n is non zero, this implies a contradiction with the fact that λ is the supremum of the above set. Therefore no such r should exist, which in turn would imply that q_0 is a minimal projection in M^φ. Since M^φ is maximal abelian in M of type II_1, it has no minimal projections, and we arrive to a contradiction. Returning to our original central element c, it follows that we can find projections q_i in M^φ with $0 \leq (\alpha_i - \epsilon/n)p_i \leq Tr(q_i) \leq \alpha_i p_i$. Since $q_i \leq p_i$ and these projections are mutually orthogonal, it follows that $q = \sum_{i=1}^n q_i$ is a projection in M^φ. Moreover, we have that $c - \epsilon \leq Tr(q) \leq c$. Then we can construct an increasing sequence q_n of projections in M^φ such that $Tr(q_n)$ converges to c. Pick $q' = \vee q_n$, clearly $Tr(q') = c$. Now, if c is any element in $\mathcal{Z}(M)$ with $0 \leq c \leq 1$, let c_n be an increasing sequence of positive elements in $\mathcal{Z}(M)$ with finite spectrum, converging to c in norm. We can find projections q_n' in M^φ with $Tr(q_n') = c_n$. Then q_n' is an increasing sequence of projections, put $q_0' = \vee q_n'$. We obtain that $Tr(q_0') = c$, and the proof is complete.

□

Lemma 2.7 *Let M be a von Neumann algebra of type II_1 and φ a faithful and normal state such that M^φ is of type I. Then $\mathcal{U}_\varphi$ is simply connected.*

Proof. Let p_n be the projections of the center of M^φ decomposing it in its type I_n parts, $n < \infty$. Pick $c \in \mathcal{Z}(M)$, and put $c_n = cp_n$. Suppose that for each n we can find q_n in $p_n M^\varphi \subset M^\varphi$ with $Tr(q_n) = c_n$. Then $q = \sum_n q_n$ is a projection in M^φ such that $Tr(q) = c$. Therefore it remains to prove our statement in the case M^φ of type I_n. Indeed, note that $p_n M^\varphi$ is the centralizer of the (faithful and normal) state φ_n of $p_n M p_n$, which is simply the restriction of φ to $p_n M p_n$.

Let now e be a minimal abelian projection in M^φ. Again, pick $0 \leq c \leq 1$ in $\mathcal{Z}(M)$. Now eMe is of type II_1, and the state φ_e of eMe given by the restriction of φ to this algebra has centralizer equal to $eM^\varphi e$. By the lemma above, there exists a projection $q \in eM^{\varphi}e \subset M^{\varphi}$ such that

$$Tr_e(q) = ec,$$

where Tr_e is the center valued trace of eMe, i.e. $Tr_e(exe) = eTr(exe)$. Since M^{φ} is of type I_n, it follows that $Tr(e) = 1/n$. Taking trace in the above equality yields $(1/n)Tr(q) = (1/n)c$ and the statement follows. □

Finally, the case when M is of type I is dealt in a similar way.

Lemma 2.8 *If M is a finite type I von Neumann algebra, and φ is a faithful and normal state, then $\mathcal{U}_{\varphi}$ is simply connected.*

Proof. As remarked before, one can restrict to the case when M is of type I_n, for a fixed $n < \infty$. In this case, $\pi_1(U_M)$ equals $C(\Omega, \mathbb{Z})$, where the isomorphism is implemented by the map sending the class of the curve $\alpha(t) = e^{2\pi i t p}$ to the continuous map $Tr(p)\hat{}$, for p a projection in M. In other words, $\pi_1(U_M)$ identifies with elements $c \in \mathcal{Z}(M)_{sa}$ which are of the form $c = \sum_{i=1}^{k} m_i p_i$, with p_i mutually orthogonal in $\mathcal{Z}(M)$ and m_i are integers. The proof follows, recalling that $\mathcal{Z}(M) \subset \mathcal{Z}(M^{\varphi})$, and therefore $Tr(p_i) = p_i$, i.e. c lies in the image of i_*. □

We may state now our theorem:

Theorem 2.9 *Let M be a von Neumann algebra, and φ a faithful and normal state. The unitary orbit of φ, $\mathcal{U}_{\varphi} = \{\varphi \circ Ad(u) : u \in U_M\}$ regarded with the (usual) quotient topology $U_M/U_{M^{\varphi}}$ is simply connected.*

Proof. As noted at the beginning of the section, it suffices to prove the statement in the case when M is of a definite (finite) type. Type I case was dealt in 2.8. Suppose that M is of type II_1. Then M^{φ} is finite, and there exist two projections p_I, p_{II} in $\mathcal{Z}(M^{\varphi})$ such that $p_I + p_{II} = 1$, $p_I M^{\varphi}$ is of type I and $p_{II} M^{\varphi}$ is of type II_1. In [2] it was shown that if p is a projection in a von Neumann algebra M, then the unitary orbit $\{upu^* : u \in U_M\}$ is simply connected. This unitary orbit is the base space of a fibration of U_M with fibre U_N where $N = \{p\}' \cap M = \{pxp + (1-p)x(1-p) : x \in M\}$. In other words, the quotient U_M/U_N is simply connected. In our case we have that

$$U_M / \left(U_{p_I M p_I} \times U_{p_{II} M p_{II}}\right)$$

is simply connected. The inclusion $U_{M^{\varphi}} \subset U_M$ can be factorized

$$U_{M^{\varphi}} = U_{p_I M^{\varphi}} \times U_{p_{II} M^{\varphi}} \subset U_{p_I M p_I} \times U_{p_{II} M p_{II}} \subset U_M.$$

The inclusion $U_{p_I M^{\varphi}} \subset U_{p_I M p_I}$ induces an epimorphism of the π_1 groups, by lemma 2.7. The same happens with the inclusion $U_{p_{II} M^{\varphi}} \subset U_{p_{II} M p_{II}}$, by lemma 2.4. The last inclusion $U_{p_I M p_I} \times U_{p_{II} M p_{II}} \subset U_M$ also induces an epimorphism of the π_1 groups by the remark above. Therefore $\mathcal{U}_{\varphi}$ is simply connected also in this case.
□

3 Topologies in $\mathcal{U}_\varphi$

In the previous section we considered in $\mathcal{U}_\varphi$ the quotient topology U_M/U_{M^φ} with $U_{M^\varphi} \subset U_M$ endowed with the norm topology of M. In this section we shall consider in M also the norm $\| \ \|_E$ given by $\|x\|_E = \|E(x^*x)\|^{1/2}$. That is, the norm of M regarded as a pre-C*-module over M^φ, with the M^φ valued inner product $< x, y >= E(x^*y)$. It is known [8] that M is complete with this norm if and only if the index of E is finite. We shall denote the corresponding topologies induced in $\mathcal{U}_\varphi \simeq U_M/U_{M^\varphi}$ as the usual topology and the modular topology. Let us recall some facts

Remark 3.1 *If the index of E is finite, then both topologies coincide, because both norms are equivalent in M in this case.*

In general ([4]) $\mathcal{U}_\varphi$ with the modular topology is naturally homeomorphic to the orbit

$$U_M(e) = \{ueu^* : u \in U_M\} \subset M_1$$

via the map $\varphi \circ Ad(u^*) \mapsto ueu^*$. Here $U_M(e)$ is considered with the norm topology of M_1. This orbit is a subset of the grassmannians (=projections) of M_1. It is a submanifold of the grassmannians if and only if the index of E is finite. It follows that $\mathcal{U}_\varphi$ is in general a metric space, and a complete metric space in the finite index case.

Remark 3.2 *The condition that the centralizer expectation E of a state φ be of finite index is rather strong. It implies that M must be finite. Moreover, if M is a factor, it happens if and only if the Radon-Nikodym derivative of φ with respect to the trace of M is a (bounded) operator with finite spectrum (see [4]). If this condition holds, then $\mathcal{U}_\varphi$ is simply connected with the modular topology as well.*

The following results establish that the finite index case is the only situation in which both topologies in $\mathcal{U}_\varphi$ coincide.

Proposition 3.3 *Let $F : M \to N \subset M$ be a faithful conditional expectation of infinite index. Then the norm of M and the norm $\| \ \|_F$ induced by F define topologies in U_M/U_N which are not equivalent.*

Proof. Since the index of F is infinite [8], [9], there exist elements $a_n \in M$ with $0 \le a_n \le 1$, $\|a_n\| = 1$ and $F(a_n) \to 0$ as n tends to infinity. It is straightforward to verify that the distance $d(a_n, N) = \inf\{\|a_n - b\| : b \in N\}$ does not tend to zero with n. Let $u_n \in U_M$ be unitaries such that $1 - a_n = \frac{u_n + u_n^*}{2}$. Then

$$\|u_n - 1\|_F^2 = \|2 - F(u_n + u_n^*)\| = 2\|F(a_n)\| \to 0.$$

Therefore the sequence of the classes of the elements u_n tends to the class of 1 in the modular topology. We claim that $[u_n]$ does not tend to $[1]$ in the usual topology (induced by the norm of M). Suppose not. Then there exist unitaries $v_n \in U_N$ such that $u_n v_n \to 1$. Then

$$\|u_n - v_n^*\|^2 = \|(u_n - v_n^*)(u_n^* - v_n)\| = \|2 - u_n v_n - v_n^* u_n^*\| \to 0.$$

This implies that $d(u_n, N) \to 0$, and therefore $d(a_n, N) \to 0$, an absurd. □

Corollary 3.4 *The usual and the modular topologies coincide in $\mathcal{U}_\varphi$ if and only if the index of E is finite.*

In [3] it was shown that when the index of a conditional expectation $F : M \to N$ is finite then the mapping $U_M \ni u \mapsto ufu^* \in U_M(f) = \{ufu^* : u \in U_M\}$ is a (principal) fibre bundle (where f denotes the Jones projection of F). Using the result above it can be shown that also the converse is true:

Corollary 3.5 *Let $F : M \to N$ be a conditional expectation and f the Jones projection of F. Then the mapping*

$$U_M \ni u \mapsto ufu^* \in U_M(f) = \{ufu^* : u \in U_M\}$$

has continuous local cross sections if and only if the index of F is finite.

Proof. It only remains to prove that if the above mapping has local cross sections, then the index of F is finite. The existence of local cross sections implies that the bijective and continuous map induced in the quotient,

$$U_M/U_N \to U_M(f)$$

is open, and therefore a homeomorphism. On the other hand it holds in general ([4]) that this same bijection is a homeomorphism between $U_M(f)$ and the modular topology in U_M/U_N. It follows by the proposition above, that the index of F is finite. □

Next we show that $\mathcal{U}_\varphi$ with the modular topology, can be presented as a subset of the interior tensor product $M \otimes_{M^\varphi} M$ of the pre-C*-module M over M^φ with itself (see [14] for the particulars of this construction). The inner product and the norm of $M \otimes_{M^\varphi} M$ are given by: if $x_i, y_i \in M$, $i = 1, 2$ then

$$< x_1 \otimes y_1, x_2 \otimes y_2 >= E(y_1^* E(x_1^* x_2) y_2)$$

and

$$\|x_1 \otimes y_1\| = \|E(y_1^* E(x_1^* x_1) y_1)\|^{1/2}.$$

Consider the set

$$\mathcal{D}_\varphi = \{u \otimes u^* : u \in U_M\} \subset M \otimes_{M^\varphi} M$$

Clearly the map $U_M \to \mathcal{D}_\varphi$, $u \mapsto u \otimes u^*$ induces a well defined bijection

$$\delta : U_M/U_{M^\varphi} \to \mathcal{D}_\varphi \qquad \delta([u]) = u \otimes u^*.$$

Proposition 3.6 *The map δ is continuous both in the usual and modular topology of $\mathcal{U}_\varphi$. It is a homeomorphism in the modular topology.*

Proof. Since the space considered is homogeneous and the action of the unitary group is continuous, it suffices to consider continuity at the class [1] of 1. First, note that $u_\alpha \otimes u_\alpha^* \to 1 \otimes 1$ in the norm topology of $M \otimes_{M^\varphi} M$ if and only if $E(u_\alpha)E(u_\alpha^*) \to 1$ in M^φ. Then it is clear that the map $U_M \ni u \mapsto u \otimes u^* \in M \otimes_{M^\varphi} M$ is continuous in the norm topology of M. Therefore the map induced in the quotient U_M/U_{M^φ}, i.e. δ, is continuous in what we are calling the usual topology of the quotient.

In the modular topology, as noted above, $\mathcal{U}_\varphi$ is homeomorphic to the orbit $U_M(e) = \{ueu^* : u \in U_M\} \subset M_1$ in the norm topology. Therefore $[u_\alpha] \to [1]$ if and only if $u_\alpha e u_\alpha^* \to e$. This implies that $eu_\alpha e u_\alpha^* e = E(u_\alpha)E_\alpha(u_\alpha^*)e \to e$. Since the mapping $M^\varphi \to M^\varphi e$, $x \mapsto xe$ is a *-isomorphism, it follows that $E(u_\alpha)E(u_\alpha^*) \to 1$, that is $u_\alpha \otimes u_\alpha^* \to 1 \otimes 1$ in $M \otimes_{M^\varphi} M$.

In order to see that δ is a homeomorphism with the modular topology, suppose that $u_\alpha \in U_M$ such that $E(u_\alpha)E(u_\alpha^*) \to 1$. Therefore there exists α_0 such that for $\alpha \geq \alpha_0$ $E(u_\alpha)E(u_\alpha^*)$ is invertible in M^φ. Since M^φ is finite, it follows that also $E(u_\alpha)E(u_\alpha^*)$ is invertible, which implies that $E(u_\alpha)$ is invertible. Then the unitary part v_α of $E(u_\alpha^*)$ ($E(u_\alpha^*) = v_\alpha \left(E(u_\alpha)E(u_\alpha^*)\right)^{1/2}$) satisfies that $E(u_\alpha)v_\alpha \to 1$. Indeed, note that $v_\alpha = E(u_\alpha^*)\left(E(u_\alpha)E(u_\alpha^*)\right)^{-1/2}$, and then $E(u_\alpha)v_\alpha = E(u_\alpha)E(u_\alpha^*)(E(u_\alpha)E(u_\alpha^*))^{-1/2} \to 1$. On the other hand, $[u_\alpha] \to [1]$ in $\mathcal{U}_\varphi = U_M/U_{M^\varphi}$ in the modular topology if and only if there exist unitaries $w_\alpha \in M^\varphi$ such that $\|u_\alpha w_\alpha - 1\|_E \to 0$. Put $w_\alpha = v_\alpha$ as above, then

$$\begin{aligned}\|u_\alpha v_\alpha - 1\|_E^2 &= \|E\left((v_\alpha^* u_\alpha^* - 1)(u_\alpha v_\alpha - 1)\right)\| \\ &\leq \|1 - E(u_\alpha)v_\alpha\| + \|v_\alpha^* E(u_\alpha^*) - 1\|\end{aligned}$$

which tend to zero, and therefore δ is a homeomorphism in the modular topology of $\mathcal{U}_\varphi$. □

The following result will be useful in the study of these topologies

Proposition 3.7 *Let u and w be unitaries in M, then*

$$\|\varphi \circ Ad(u^*) - \varphi \circ Ad(w^*)\| \leq 2\|u - w\|_E \leq 2\|u - w\|. \tag{3.1}$$

Proof. The second inequality is obvious. In order to prove the first note that for any $x \in M$,

$$|\varphi(u^* x u) - \varphi(w^* x w)| \leq |\varphi(u^* x(u - w))| + |\varphi((u^* - w^*) x w)|.$$

Note that if v is unitary, by the Cauchy-Schwarz inequality we have that $|\varphi(zv)| \leq \varphi(zz^*)^{1/2}$ and $|\varphi(v^* z)| \leq \varphi(z^* z)^{1/2}$. Applying these inequalities we obtain

$$|\varphi(u^* x(u - w))| \leq \varphi((u^* - w^*) x^* x(u - v))^{1/2} = \varphi \circ E((u^* - w^*) x^* x(u - v))^{1/2},$$

and

$$|\varphi((u^* - w^*)xw)| \le \varphi \circ E((u^* - w^*)xx^*(u - w))^{1/2}.$$

Note that $(u^* - w^*)x^*x(u - v) \le \|x\|^2(u^* - w^*)(u - v)$, and analogously for the other term. Thus we obtain

$$\begin{aligned} |\varphi(u^*xu) - \varphi(w^*xw)| &\le 2\|x\| \; \varphi \circ E((u^* - w^*)(u - v))^{1/2} \\ &\le 2\|x\| \; \|E((u^* - w^*)(u - v))\|^{1/2}. \end{aligned}$$

□

Proposition 3.8 *$\mathcal{U}_\varphi$ is complete in the usual topology (induced by the usual norm of M)*

Proof. One has continuous local cross sections $\sigma_{[u]} : \mathcal{V}_{[u]} \subset \mathcal{U}_\varphi \to U_M$ defined on a neighborhood $\mathcal{V}_{[u]}$ of $[u] \in \mathcal{U}_\varphi$. If $[u_n]$ is a Cauchy net in $\mathcal{U}_\varphi$, choose $[u_{k_0}]$ such that for $n \ge k_0$, $[u_n] \in \mathcal{V}_{[u_{k_0}]}$. Then since $\sigma = \sigma_{[u_{k_0}]}$ is continuous, $\sigma([u_n])$ is a Cauchy sequence in U_M in the norm topology, therefore convergent to a unitary v in M. Then clearly $[u_n]$ converges to $[v]$. □

Let us consider the same question for the modular topology. Of course, if the index of E is finite, one obtains that $\mathcal{U}_\varphi$ is closed (and complete). In the general case, one expects to obtain other elements in the closure of $\mathcal{U}_\varphi$ with the modular topology.

Denote by X_M the completion of the pre-Hilbert C*-module M to a Hilbert C*-module. Note that M acts on X_M by left multiplication, and can be viewed as a closed subalgebra of the algebra of adjointable operators of X_M.

Suppose that u_n is a sequence of unitaries converging to an element $x \in X_M$. Note that this implies that $< x, x >= 1$, i.e. x lies in the unit sphere of X_M. Put $\varphi_x \in M^*$, $\varphi_x(a) = \varphi(< x, ax >)$. Clearly φ_x is a state of M.

Proposition 3.9 *All elements in the closure of $\mathcal{U}_\varphi$ with the modular topology are of the form φ_x for some x in the unit sphere of X_M. Such states φ_x are normal. If M is finite, then φ_x is also faithful.*

Proof. An argument similar to the one in the previous proposition, shows that a Cauchy sequence $[u_n]$ for the modular topology yields another sequence of unitaries u'_n in U_M such that $[u_n] = [u'_n]$ and u'_n form a Cauchy sequence in U_M for the norm $\| \; \|_E$ (this is clear using the continuous cross sections available for this topology as well). Therefore u'_n converge to some element x in the unit sphere of X_M. Now using the above result,

$$\|\varphi \circ Ad(u_n'^*) - \varphi \circ Ad(u_k'^*)\| \le 2\|u'_n - u'_k\|_E,$$

and therefore $\varphi \circ Ad(u_n'^*)$ is a Cauchy sequence in $M_* \subset M^*$. Then $\varphi \circ Ad(u_n'^*)$ converges to a normal state ψ. Then $\varphi(u_n'^* a u'_n)$ converges to $\psi(a)$. On the other hand, $\varphi(u_n'^* a u'_n) = \varphi \circ E(u_n'^* a u'_n) = \varphi(< u'_n, a u'_n >)$, which converges to $\varphi_x(a)$ by the continuity of the scalar product.

Suppose now that M is finite, and fix a faithful tracial and normal state τ. Pick φ_x with x a limit of unitaries u_n as above. Then if $\varphi_x(a^*a) = 0$, one has that $< x, a^*ax >=< ax, ax >= 0$, which implies that au_n tends to zero in the norm $\| \|_E$. In other words, $E(u_n^*a^*au_n) \to 0$. Note that $\tau \circ E = \tau$, and therefore $\tau(E(u_n^*a^*au_n)) = \tau(u_n^*a^*au_n) = \tau(a^*a) = 0$, i.e. $a = 0$. □

We do not know if these states in the closure of $\mathcal{U}_\varphi$ for the modular topology are in general faithful. There are other cases other than the finite case in which this happens. To prove our result we need the following lemma, which was proven in [1].

Lemma 3.10 *The Jones projection e associated to E is finite in M_1.*

If $E : M \to N \subset M$ is a conditional expectation, the normalizer $\mathcal{N}(E)$ is the group

$$\mathcal{N}(E) = \{u \in U_M : Ad(u^*) \circ E \circ Ad(u) = E\}.$$

Proposition 3.11 *If $\mathcal{N}(E)$ generates M as a von Neumann algebra, then the states φ_x in the closure of $\mathcal{U}_\varphi$ in the modular topology are faithful.*

Proof. Represent M and M^φ in H_φ as in the basic construcion. As in the proposition above, one needs to show that if u_n are unitaries in M converging to $x \in X_M$ in the norm $\| \|_E$, and $a \in M$ such that $E(u_n^*a^*au_n) \to 0$, then it must be $a = 0$. We claim that under this hypothesis au_n tends to zero in the strong operator topology. Note that au_ne tends to zero in norm, and therefore $au_nbe = au_neb \to 0$ for all $b \in M^\varphi$. On the other hand, if $v \in \mathcal{N}(E)$ then also $au_nve \to 0$ in the norm of M_1. Indeed, $(au_nve)^*au_nve = ev^*u_n^*a^*au_nve = E(v^*u_n^*a^*au_nv)e = v^*E(u_n^*a^*au_n)ve \to 0$. Since M^φ and $\mathcal{N}(E)$ generate M, it follows that au_nxe tends to zero in norm for any $x \in M$. The claim is proven, using that the sequence is bounded in norm, and the fact that Me is dense in H_φ. By the previous lemma, e is finite, and therefore $(au_n)^*e \to 0$ in the strong operator topology (see [12]). Again using that the sequence is bounded, one has that $aa^*e = (au_n)(au_n)^*e \to 0$ strongly, that is $aa^*e = 0$. Then $E(aa^*)e = eaa^*e = 0$, which implies that $E(aa^*) = 0$, and therefore $a = 0$. □

There is an easy example of this situation.

Example 3.12 *Take $M = B(\ell^2(\mathbb{Z}))$ and E the conditional expectation onto the subalgebra of diagonal matrices in the canonical basis of $\ell^2(\mathbb{Z})$. This subalgebra is the centralizer of a state φ, which can be constructed by means of a density operator a chosen as a diagonal trace class positive matrix, with different non zero entries in the diagonal, and with trace 1. The bilateral shift and its integer powers normalize this expectation, and it is straightforward to verify that the diagonal matrices together with the powers of the bilateral shift generate $B(\ell^2(\mathbb{Z}))$.*

The inequality 3.1 implies that one can embed $\mathcal{U}_\varphi$, both with the usual and the modular topologies, in the state space of M (with the norm topology of M^*).

However the usual and the modular topologies of $\mathcal{U}_\varphi$ do not coincide with the norm topology of the state space. This fact is clear in the following example.

Example 3.13 *Let M and $\varphi = Tr(a\ .)$ as in the preceeding example. Let $t = I \oplus \sigma$ act on $\ell^2(\mathbb{Z}) = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, where σ is the unilateral shift. Denote by q_n the $n \times n$ Jordan nilpotent, and u_n the unitary operator on $\ell^2(\mathbb{N})$ having the unitary matrix $q_n + {q_n^*}^{n-1}$ on the first $n \times n$ corner, and the identity matrix afterwards. Finally put $w_n = 1 \oplus u_n \in B(\ell^2(\mathbb{Z}))$. It is straightforward to verify that $w_n a w_n^* \to tat^*$ in the trace norm of $B(\ell^2(\mathbb{Z}))$. This means that $\varphi \circ Ad(w_n^*) \to \varphi_t$ in the norm of $B(\ell^2(\mathbb{Z}))^*$. But by the proposition above, it is clear that φ_t does not belong to the closure of $\mathcal{U}_\varphi$ either in the modular or usual topology, since it is not faithful ($\ker t^*$ is not trivial).*

References

[1] E. Andruchow, G. Corach and D. Stojanoff, Geometry of the sphere of a Hilbert module, Math. Proc. Cambridge Phil. Soc. 127 No. 2 (1999), 295-315.

[2] E. Andruchow, G. Corach and D. Stojanoff, Projective space of a C*-module, (preprint).

[3] E. Andruchow and D. Stojanoff, Geometry of conditional expectations and finite index, International J. Math. Vol. 5, No. 2 (1994), 169-178.

[4] E. Andruchow and A. Varela, Weight centralizer expectations with finite index, Math. Scand. 84 No.2 (1999), 243-260.

[5] C. Apostol, L.A. Fialkow, D.A. Herrero and D.V. Voiculescu, Approximation of Hilbert space operators, Vol 2, Pitman, Boston, 1984.

[6] H. Araki, M. Smith and L. Smith, On the homotopical significance of the type of von Neumann algebra factors, Commun. Math. Phys. 22 (1971),71-88.

[7] M. Breuer, A generalization of Kuiper's theorem to factors of type II_∞, J. Math. Mech. 16 (1967), 917-025.

[8] M. Baillet, Y. Denizeau and J.-F. Havet, Indice d'une esperance conditionelle, Compositio Math. 66 (1988), 199-236.

[9] M. Frank and E. Kirchberg, On conditional expectations of finite index, preprint (1996).

[10] D. E. Handelman, K_0 of von Neumann algebras and AFC*-algebras, Quart. J. Math. Oxford (2) 29 (1978), 429-441.

[11] V.F.R. Jones, Index for subfactors, Invent. Math. 72 (1983), 1-25.

[12] R.V. Kadison, J.R. Ringrose, Fundamentals of the theory of operator algebras, Vol. II, Academic Press, New York, 1986.

[13] N. Kuiper, The homotopy type of the unitary group of Hilbert space, Topology 3 (1965), 19-30.

[14] E.C. Lance, Hilbert C*-modules / A toolkit for operator algebraists, LMS Lecture Notes Series 210, Cambridge University Press, Cambridge, 1995.

[15] M. Pimsner and S. Popa, Entropy and index for subfactors, Ann. Sci. Ecole Norm. Sup. 19 (1986), 57-106.

[16] H. Schröder, On the homotopy type of the regular group of a W*-algebra, Math. Ann. 267 (1984), 271-277.

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