THE STIEFEL BUNDLE OF A BANACH ALGEBRA

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Abstract. We introduce the Stiefel bundle associated to a given Banachable algebra and study the analytic properties of the resulting principal fiber bundle over the Grassmannian of equivalence classes of idempotents in the algebra. Our main application is when the algebra is that of the bounded linear operators of a Banach space. In particular, the problem of smooth parametrization of subspaces can then be reduced to one involving the smooth extension of sections.

1. Introduction

Grassmannians in finite and infinite dimensions provide a very useful framework in dealing with a variety of problems arising in systems–control theory (for instance in [6] [15]), the applications of matrix and operator–valued functions [12] [14], completely integrable systems [28] [31], besides in many other areas of mathematics. A particular problem may entail finding a suitable parametrization for families of linear subspaces of a given finite dimensional vector space or to determine the extensions of a given parametrization to match some order of differentiability. A family of linear subspaces is generally equivalent to the existence of a map having the right order of differentiability from a suitable domain into the corresponding Grassmannian. Consequently, there is a family of vector fields constituting a framing of the same differentiability class (an approach as adopted in e.g. [11]). The manifold of all such frames or possible bases, is traditionally referred to as the *Stiefel manifold*, a manifold which is the total space of a principal fiber bundle over the Grassmannian. The Lie–theoretic and topological properties of the Stiefel bundle are well understood in finite dimensions (see e.g. [17]). In order to generalize this procedure to accommodate subspaces of an infinite dimensional Banach space, there is clearly a problem with the correct idea of 'framing'. One approach is to formulate matters in terms of a lifting problem to a more general Stiefel manifold which is defined as a manifold of injective linear maps on a fixed splitting subspace, whose images split and are projection–wise in the same similarity class as this subspace. Then any frame can be chosen in the latter and the resulting map to the Stiefel manifold can then be seen as one defining the appropriate frame fields. For families of subspaces this problem was studied by Gohberg and Leiterer in [13] (see also [9] for certain cases). The problem can be formulated in the framework of the algebra of bounded linear maps on a Banach space anticipating a generalization to any Banach algebra and to some extent, to any topological algebra (see Remark 7.1).

In this paper we introduce the notion of a Stiefel bundle associated to a given Banach(able) algebra and study its geometric properties. As it is a new ingredient of the general theory, we include a concise and self–contained account of the essential ideas in the background. One aspect concerns the geometry of the space of

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idempotents of the algebra, along with the analytic structure of various associated Grassmannians as studied by other authors (cited below). Our viewpoint here is quite different and while it is more general, the proofs of our results are perhaps more elementary. Putting this into perspective, consider a topological algebra A, and let $P(A)$ denote the set of idempotents of A which we can consider to be abstract projection operators. We then define an equivalence relation "$\sim$" in $P(A)$ by $p \sim q$ if and only if $pq = q$ and $qp = p$. Note that if A is an algebra of linear operators on a vector space, then $p \sim q$ if and only if p and q have the same image. Thus it is reasonable to think of the space $\mathrm{Gr}(A) = P(A)/\sim$, as the generalization of the Grassmannian of subspaces of a given vector space. Of course one needs to show that $\mathrm{Gr}(A)$ is indeed a manifold and for a Banach algebra this was shown to be the case by Porta and Recht in [27] and the results therein were later applied to develop the differential geometry of $\mathrm{Gr}(A)$. One of the salient features involves proving that the quotient map of $P(A)$ onto $\mathrm{Gr}(A)$ is an open map. Here we shall include a short proof of this fact which together with a simple special case of a result in [10], will facilitate a rapid development of the necessary results and provide the Stiefel bundle for any Banach algebra.

Subsequent to [27], a number of authors have treated the differential geometry of $P(A)$ and related subsets of A as submanifolds of A (see [2] [3] [21] [22] [23] [24] [36] [37], as well as references therein). In discussing $P(A)$ and its submanifolds, we acknowledge some slight overlap between our results and those obtained by these authors at various stages of the development. We comment that in [22], there appears the notion of a Grassmannian of a 'Banach environment'. Although having interesting consequences for C*–algebra theory, the latter appears to be a Grassmannian only for the C*–algebra case where the restriction to the self–adjoint projections trivializes the relation "$\sim$". Thus in [22], the Grassmannian is defined to be a set of projections and so if "$\sim$ " is non–trivial, this does not capture the required notion of a set of spaces in the case where A is a general algebra of linear operators.

Our Stiefel manifold follows from introducing a certain analytic submanifold $V(A)$ of A, modeled on the partial isomorphisms of A whose domain and range are images of elements of A . In this way, $V(A)$ contains $P(A)$ and the natural image map $\mathrm{Im} : P(A) \to \mathrm{Gr}(A)$ extends to a well–defined map (denoted the same) $\mathrm{Im} : V(A) \to \mathrm{Gr}(A)$ which is an analytic fiber bundle with $V(A)$ and $\mathrm{Gr}(A)$ both analytic manifolds. The spaces $V(A)$ and $\mathrm{Gr}(A)$ have the same analytic structure as in [27]. Specializing to the similarity class $\mathrm{Sim}(p, A)$ of a fixed projection p in A, leads to the Stiefel bundle $V(p, A) \to \mathrm{Gr}(p, A)$ which is a locally trivial analytic principal bundle. Precise statements of these facts are to found in Theorems 6.1 and 7.1. The problem of extendability of a continuous family of subspaces to a globally smooth family parametrized by a certain subset of a Banach space, then has a general solution in terms of sections of $V(p, A)$, or more generally, in terms of sections of $V(A)$ (see Corollary 8.1). This follows from Theorem 7.1 together have further applications to such subjects as the study of invariant subspaces of matrices [12], the $(\mathcal{M}, \mathcal{M}^{\times})$–theory of Helton et al. [14] as well as the Cowen–Douglas theory [4]. Our development of ideas also embraces the structure of the 'restricted' Grassmannians over separable complex Hilbert spaces that feature in the infinite dimensional Hamiltonian theory of [28] [31] (see Example 6.2).

2. Some preliminaries

Since we shall be dealing with Banach manifolds and Banach algebras, we begin by recalling some of the basic definitions and terminology (see e.g. [5] [20]). If E and F are topological vector spaces, then $\mathcal{L}(E, F)$ denotes the vector space of continuous linear maps of E into F and where for $E = F$ we write $\mathcal{L}(E)$. Thus when E and F are Banach spaces, $\mathcal{L}(E, F)$ is a Banach space and $\mathcal{L}(E)$ is a Banach algebra. Given a topological vector space E, we say that E is a *Banachable space* when E is the underlying topological vector space of a Banach space. If $K \subset E$ is a closed linear subspace, we say that K *splits* E if K has a closed complementary subspace $L \subset E$, such that the addition map $K \times L \to E$ is a linear homeomorphism. In particular, given a Banach manifold M and for $x \in M$, the tangent space $T_x M$ is a Banachable space.

Recall that a map $f : M \to N$ of Banach manifolds is (of class) C^k if it is continuously k–times differentiable when $k \in \{0, 1, 2 \ldots, \infty, \omega\}$, where as usual, C^∞ means 'smooth' and C^ω means analytic. If f is a C^k–map of Banach manifolds, then we shall simply use $f'(x)$ to denote its derivative at x as a continuous linear map of tangent spaces. A useful concept here is the notion of a C^k–map on an arbitrary subset $X \subset M$ and the development in [26] for finite dimensional manifolds, extends in a straightforward manner to Banach manifolds as far as the basic definitions are concerned (see e.g. [18] and [10] where the latter provides a more categorical discussion of differentiabilty in the infinite dimensional setting). More specifically, if $x \in X$ and $Y \subset N$, we say that a map $f : X \to Y$ is *locally* C^k *at* x provided there is an open neighborhood U of x in M and a C^k–map $g : U \to N$, such that $f|X \cap U = g|X \cap U$. We call the map g *a local extension of* f . We say that f is a C^k–map if it is locally C^k at each point of X . Clearly, if X is open in M, then to say that f is C^k is the same as saying it is C^k in the ordinary sense, so there is no conflict with any prior convention. Now if $Z \subset M$ is also a subset with a map $h : Y \to Z$, if f is locally C^k at x and h is locally C^k at $y = f(x)$, then $h \circ f$ is locally C^k at x . Hence the composition of C^k–maps is also C^k . Since many of our maps will be restrictions of continuous linear maps, continuous bilinear maps or polynomials in Banach algebras, their analyticity is then automatic. In particular, the identity map of any subset of a Banach manifold is analytic. Note that if $M \subset N$ is a submanifold, $X \subset M$ a subset and there is a map f defined on X, then to say f is locally C^k at $x \in X$, is the same regardless of whether we consider X as a subset of M or of N, since by definition the submanifold splits locally. That is, x has an open neighborhood W in N that is diffeomorphic to a product $U \times V$, where U and V are open in Banach spaces and the diffeomorphism carries $W \cap M$ onto $U \times \{\text{pt}\}$.

If each point of X has an open neighborhood U in X which is C^k

M . In this case, we see from our previous remark that the idea of a C^k–map on X is the same whether we consider X as a manifold in its own right or as a subset of M . A given diffeomorphism of U onto an open subset of a Banach space provides a chart on U and the inverse diffeomorphism provides a parametrization of U . So if we assume here that $k > 0$, then for $x \in X$, the subspace T_xX is naturally a closed vector subspace of T_xM which splits T_xM . Also, if $Y \subset N$ is a subset of the Banach manifold N and $f : X \to Y$ is a C^k–map, then we have $f'(x) = g'(x)|T_xX$, for any local extension g of f . Moreover, the chain rule holds since the details in e.g. [26] [18] can be easily modified to the infinite dimensional case. In order to see this, we take h to be a local chart at x with $h(x) = 0$ and compose its local extension with the inverse parametrization to h, denoted by σ . We then obtain a C^k–local retraction $r : W \to W \cap X$ of some neighborhood W of $x \in M$. In this way T_xX can be naturally identified as the image of $\sigma'(0)$ in T_xM, and just as in [26], this subspace of T_xM can be seen to be independent of the choice of parametrization. Then $f \circ r$ and g are both local extensions of f with $f \circ r \circ \sigma = f \circ \sigma = g \circ \sigma$. In terms of such local extensions, the derivative is then given by

$$g'(x)|T_xX = f'(x) = (f \circ r)'(x)|T_xX \ . \tag{2.1}$$

As a self–map of W, we have $r^2 = r$, so by the chain rule we obtain a continuous linear map $r'(x)$ which is an idempotent and therefore a projection whose image is clearly T_xX, and whose kernel provides a complement to T_xX in T_xM .

Note that for $k > 0$, if E is a Banach space and K is a closed linear subspace of E which has no complement, then even though the inclusion $i : K \hookrightarrow E$ is analytic, K is not a C^k–submanifold of E . In fact the identity map of K is not a C^k–map of a subset of E into K . More precisely, the inverse of the identity of K is not a C^k–map on the subset K of E . Indeed, any local extension of the identity of K to a map of an open neighborhood of E into K would when differentiated, provide a continuous linear retraction of E on K whose kernel would be a closed complementary subspace to K .

Recall that a Banach (analytic) Lie group is a Banach analytic manifold which is also a group such that the operations of multiplication and inversion are analytic. A (Banach) Lie subgroup of a Banach Lie group is a closed subgroup which is also a Banach manifold (for the basics on Banach Lie groups, see e.g. [18] [34]). If G is a Banach Lie group and $g \in G$, then $L(g)$ denotes left translation by g, so for $h \in G$, we have $L(g)(h) = gh$. On differentiating, we find that $L(g)'(h) : T_gG \to T_{gh}G$, is a linear homeomorphism. We say that a vector field v on G is left invariant if $L(g)'(h)v(h) = v(gh)$, for all $g, h \in G$. Clearly, such a vector field on G is completely determined by its value at the identity $e \in G$. Following [34], we see that the left invariant vector fields are analytic and are complete; that is, their flows are one–parameter subgroups of self–diffeomorphisms of G . If v is a left–invariant vector field on G, let $\exp(tv)$ denote the one–parameter subgroup that it determines. If $w \in T_eG$, let $\exp(w)$ denote $\exp(v)(e)$, where v denotes the left invariant vector

The following proposition is a special case of [10] Theorem 5.1 which will be sufficient for our purposes here (see also [18] VIII and [34]).

Proposition 2.1. *Let G be a Banach Lie group acting smoothly (respectively, analytically, etc.) on a Banach manifold Y with $\pi_Y : Y \to Y/G$ the map to the orbit space. Suppose that (π_B, E, B) is a smoothly (resp. analytically, etc.) locally trivial principal G–bundle over B which is smooth (resp. analytic, etc.) with E and B Banach manifolds. Further, suppose that each point of Y has an open G–invariant neighborhood which is smoothly (resp. analytically, etc.) G–equivariantly diffeomorphic to E . Then $(\pi_Y, Y, Y/G)$ is a smoothly (resp. analytically, etc.) locally trivial principal G–bundle over Y/G and Y/G is a smooth (resp. analytic, etc.) Banach manifold modeled on the same Banach space as that of B . In fact, if f is a G–equivariant diffeomorphism of E onto an open set $V \subset Y$, then the induced map f/G on the quotient, is a diffeomorphism of B onto $p_Y(V)$, the image of V under the open map p_Y .*

Proof. The proof is an elementary and straightforward consequence of the fact that passing to the orbit space is a functor on G–spaces and that G–equivariant orbit maps to orbit spaces are always open maps. Further, if h is a G–equivariant self–diffeomorphism of E, then local smooth (resp. analytic, etc.) sections of (π_B, E, B) with the corresponding smoothness of π_B, induce the same smoothness of h/G as a self–diffeomorphism of B . □

3. The Grassmannian Gr(A)

Definition 3.1. We say that A is a *Banachable algebra* if A is an algebra whose underlying vector space structure is a Banach space in which multiplication is continuous.

Throughout we assume that A is a Banachable algebra. Let $P(A)$ be the set of idempotents in A . If A has an identity 1, then for $p \in P(A)$, we shall let $\hat{p} = 1 - p$. Thus the map sending $x \in A$ to $1 - x$, is an affine homeomorphic involution of A which maps $P(A)$ to itself. Let $G(A)$ be the group of units of A or of the algebra obtained by adjoining an identity to A should the latter not have one. We take $*$ to denote the inner automorphic action of $G(A)$, so $g * x = gxg^{-1}$, for $g \in G(A)$ and $x \in A$. Throughout, c, d, e, f, p, q are taken to denote elements of $P(A)$.

Remark 3.1. If E is a topological vector space and p is an idempotent in the algebra of continuous linear endomorphisms of E, with $pE = F$, then as a map of E onto F, p is a continuous open map. To see this, note that $q = 1 - p$ is also continuous, say with $qE = D$, so then (p, q) is a continuous linear map of E onto $F \times D$ which is an algebraic isomorphism of vector spaces. However, the inverse of (p, q) is just the restriction of the addition map to $F \times D$, and thus (p, q) is a linear homeomorphism. If $\pi_F : F \times D \to F$ is the first factor projection, then π_F is open and it follows that $\pi_F \circ (p, q) = p$ is also open onto its image F .

Next we introduce the Grassmannian of A and proceed to establish its basic analytic properties (cf. [2] [22] [27] [36]).

Definition 3.2. We have a natural partial order on $P(A)$ where we say that $p < q$ if

$\mathrm{Gr}(A)$ is a space with the quotient topology due to the natural quotient map

$$\mathrm{Im} = Q(A) : P(A) \to \mathrm{Gr}(A)\ , \tag{3.1}$$

where we decree $\mathrm{Im}(p)$ to be the image of p, which it is equivalent to under the left regular representation of A . Thus p and q are equivalent if and only if $pA = qA$, and $p < q$ if and only if pA is contained in qA .

Remark 3.2. We can always adjoin an identity to A if we wish and so obtain a Banachable algebra with identity. When A has an identity, the left (or right) regular representation of A on itself, denoted by L_A or simply L (resp., R_A or R) gives an isomorphism (resp., an anti–isomorphism) onto a closed subalgebra of $\mathcal{L}(A)$ the bounded linear operators on A, giving A an equivalent norm so that A is a Banach algebra with identity in the usual sense. But we will not need to consider the norm directly, and if $e \in P(A)$, then eAe is a Banachable algebra with identity e, but e may not have a unit norm, so eAe is not a Banach algebra in the usual sense. Since $[L, R] = 0$, we see that $L(c)$ and $R(e)$ are commuting idempotents in $\mathcal{L}(A)$, so $L(c)R(e)$ is an idempotent in $\mathcal{L}(A)$ whose range is cAe, hence cAe is a closed complemented subspace of A, even when A does not have an identity. Moreover, by Remark 3.1, there is an open continuous map $L(c)R(e) : A \to cAe$, and hence cAe has the quotient topology induced by $L(c)R(e)$. If $ce = 0 = ec$, then cAe is a subalgebra of A, in which multiplication reduces to 0, and hence for any $x \in cAe$, we have $1 + x = \exp(x)$. Then clearly $1 + x \in G(A)$ with its inverse equal to $1 - x$.

Observe that if A^- denotes the opposite algebra to A, then we have a corresponding quotient map

$$\mathrm{Im}^- = Q(A^-) : P(A) = P(A^-) \longrightarrow \mathrm{Gr}(A)\ , \tag{3.2}$$

for which $\mathrm{Im}^-(c) = \mathrm{Im}^-(d)$ if and only if $cd = c$ and $dc = d$, and from

$$\hat{c}\hat{d} = (1 - c)\hat{d} = \hat{d} + (cd - c)\ , \tag{3.3}$$

we see that $cd = c$ is equivalent to $\hat{c}\hat{d} = \hat{d}$. Thus $cd = c$ is the same as $\hat{d} < \hat{c}$, and $\mathrm{Im}^-(c) = \mathrm{Im}^-(d)$, implies that $\mathrm{Im}(\hat{c}) = \mathrm{Im}(\hat{d})$.

Now it is natural to think of $\mathrm{Im}(\hat{c})$ as the kernel of c for which we denote the latter by $\mathrm{Ker}(c)$, and subsequently we have a map $\mathrm{Ker} : P(A) \to \mathrm{Gr}(A)$. Let $\widehat{P} : A \to A$ be the map defined via the assignement $c \mapsto \hat{c}$ which is an involutive affine self–homeomorphism (thus it is an analytic diffeomorphism) of $P(A)$ onto itself. So $\mathrm{Gr}(A)$ inherits the quotient topology due to Ker, since $\mathrm{Ker} = Q(A) \circ \widehat{P}$. Also, we have $\mathrm{Im}(c) = \mathrm{Im}(e)$, if and only if $\mathrm{Im}^-(\hat{c}) = \mathrm{Im}^-(\hat{e})$, so $\widehat{P}$ induces a unique homeomorphism $\widehat{\mathrm{Gr}} : \mathrm{Gr}(A^-) \to \mathrm{Gr}(A)$, such that

$$\mathrm{Ker} = Q(A) \circ \widehat{P} = \widehat{\mathrm{Gr}} \circ Q(A^-)\ . \tag{3.4}$$

Henceforth, we identify $\mathrm{Gr}(A)$ and $\mathrm{Gr}(A^-)$ via $\widehat{\mathrm{Gr}}$, as spaces so that $\mathrm{Ker} = Q(A^-)$. Below we shall establish that both $P(A)$ and $\mathrm{Gr}(A)$ are (Banach) analytic manifolds and $Q(A)$ is analytic with local analytic sections, so that $\widehat{P}$ is an analytic map. This implies that $\widehat{\mathrm{Gr}}$ is also analytic and the identification of $\mathrm{Gr}(A)$ with $\mathrm{Gr}(A^-)$ is

$\mathrm{Ker}(e)$, then $c = ce = e$, so we have an injective map $(\mathrm{Im}, \mathrm{Ker}) : P(A) \to \mathrm{Gr}(A) \times \mathrm{Gr}(A)$.

Suppose $h : A \to B$ is a continuous homomorphism of Banachable algebras. Clearly, there is an induced map $P(h) : P(A) \to P(B)$ that satisfies $P(hg) = P(h)P(g)$ and $P(\mathrm{id}) = \mathrm{id}$ and so $P(h)$ defines a functor. Moreover, if $p < q$ in $P(A)$, then $h(p) < h(q)$ in $P(B)$ and so $P(h)$ induces a continuous map $\mathrm{Gr}(h) : \mathrm{Gr}(A) \to \mathrm{Gr}(B)$. In particular, when A is Banachable, $\mathrm{Aut}(A)$ acts analytically on $P(A)$, and hence when A has an identity, so does the subgroup of inner automorphisms which likewise induce actions on $\mathrm{Gr}(A)$. It is important to observe that Im is actually an open map in order to ensure that the action regarded as a map $\mathrm{Aut}(A) \times \mathrm{Gr}(A) \to \mathrm{Gr}(A)$, is continuous. But this indeed the case as will be seen in Proposition 4.1 in the next section.

4. The map $\mathrm{Im} : P(A) \to \mathrm{Gr}(A)$

We start with some preliminary observations which will facilitate the proof of Proposition 4.1 below. Firstly, let $i : A \hookrightarrow B$ be the inclusion homomorphism of A as a closed subalgebra of B . Then there is an induced inclusion $P(i) : P(A) \hookrightarrow P(B)$, and $\mathrm{Gr}(i)$ is clearly injective. In fact, if A is a right ideal of B, then $\mathrm{Gr}(i)$ is itself an inclusion, since for c equivalent to an idempotent $b \in B$, means that $b = cb$, so necessarily $b \in A$. Thus $P(A)$ is the inverse image of $\mathrm{Gr}(A)$ as a subset of $\mathrm{Gr}(B)$ under $\mathrm{Im}_B = Q(B)$ on $P(B)$. Since the pullbacks of open maps are open and Im_B is open (see Proposition 4.1), then it follows that $\mathrm{Im}_B|P(A)$ is an open map onto $\mathrm{Gr}(A) \subset \mathrm{Gr}(B)$. Hence $\mathrm{Gr}(i) : \mathrm{Gr}(A) \to \mathrm{Gr}(B)$ is a homeomorphism onto its image. In particular, as A is a two–sided ideal in B where we take B to be the algebra obtained by adjoining an identity to A, we see that $\mathrm{Im}_A : P(A) \to \mathrm{Gr}(A)$ is always an open map if it can be shown to be open when A has an identity.

Suppose then that A has an identity denoted by 1 . If $c, d \in P(A)$, let

$$g(c,d) = cd + \hat{c}\hat{d} \ , \qquad g(c,c) = 1 \ . \tag{4.1}$$

Note that g is the restriction of a continuous bilinear map of $A \times A \to A$ and so is analytic. Thus by the continuity of g, by the fact that $G(A)$ is open in A and that inversion in $G(A)$ is continuous (indeed, analytic), we know that there exists an open neighborhood $U(c)$ of $c \in P(A)$, such that for any pair $d, e \in U(c)$, we have $g(d,e) \in G(A)$ and $g(d,e)$ can be as close to the identity as is wished. But we have $d(g(d,e)) = (g(d,e))e$, and hence if $g(d,e) \in G(A)$, then $g(d,e) * e = d$. So for c fixed, $g(d,c)$ defines a local analytic section of the inner automorphic action of $G(A)$ on $P(A)$ over $U(c)$.

Next, let $\mathcal{R}$ denote a relation and for a given set V, let $\mathcal{R}V = \{ \, x : x\mathcal{R}v \ , \ v \in V \, \}$. We say that $\mathcal{R}$ is continuous if $\mathcal{R}V$ is open whenever V is open. Observe that in the case where $\mathcal{R}$ is an equivalence relation, $\mathcal{R}$ is open if and only if the natural map onto the quotient, is an open map.

Proposition 4.1. (cf. [27]) *If $\mathcal{R}$ is a relation in $P(A)$ which is preserved by the* *are both discrete spaces, and all orbits under the $G(A)$–action are both open and closed.*

Proof. It is enough to assume that A has an identity 1 and a norm making it a Banach algebra with identity. Consider $\mathcal{R}$ to be any relation on $P(A)$ preserved by all inner automorphisms of A . Firstly, let us take an open subset $W \subset P(A)$ and consider $c\mathcal{R}f$ with $f \in W$. Next, we choose an open neighborhood V of $1 \in G(A)$, such that for any $g \in V$, we have $g * f \in W$. For $c, d, e \in P(A)$, let $U(c)$ be the neighborhood as defined above such that $g(d, e) \in V$, whenever $d, e \in U(c)$.

Since $d \in U(c)$, then on choosing $e = c$ we have $g(d, c) * c = d$, and hence the relation $d\mathcal{R}(g(d, e)*f)$. This implies that $\mathcal{R}W$ is open and therefore $\mathcal{R}$ is continuous. Thus on setting $\mathcal{R} = \sim$, we find that $\mathrm{Im} = Q(A)$ is an open map and the actions on $P(A)$ induce continuous actions on $\mathrm{Gr}(A)$.

On the other hand, the description of $U(c)$ also shows that it is an open set contained in the orbit of c under the inner automorphic action of $G(A)$, and hence these orbits are all open in $P(A)$. Since the natural quotient map onto an orbit space of a topological group action is always open, it follows that the orbit space $P(A)/G(A)$ is discrete and since Im is an open map, it follows that the orbits of the induced inner automorphic action on $\mathrm{Gr}(A)$ are also open and so the resulting orbit space $\mathrm{Gr}(A)/G(A)$ is also discrete. □

For $g \in G(A)$ and $x \in \mathrm{Gr}(A)$, we take gx to be the result of the inner automorphic action of $G(A)$ on $\mathrm{Gr}(A)$, so that we have $\mathrm{Im}(g * c) = g\mathrm{Im}(c)$. If J is a closed two–sided ideal of A and $i : J \hookrightarrow A$ the inclusion, then there are induced inclusions $P(i) : P(J) \hookrightarrow P(A)$ and $\mathrm{Gr}(i) : \mathrm{Gr}(J) \hookrightarrow \mathrm{Gr}(A)$, as before. As J is invariant under inner automorphisms, we see that $P(i)$ and $\mathrm{Gr}(i)$ are open maps and $P(J)$ and $\mathrm{Gr}(J)$ are thus disconnected from their complements in $P(A)$ and $\mathrm{Gr}(A)$, respectively.

Proposition 4.2. *The fibers of* $\mathrm{Im} : P(A) \to \mathrm{Gr}(A)$ *are convex and contractible analytic submanifolds of A . For a given $p \in P(A)$, the fiber over* $\mathrm{Im}(p)$ *is equal to* $p + pA\hat{p}$.

Proof. Assuming that A still has an identity, observe that if $w \in pA\hat{p}$, then $pw = w$ and $wp = 0$. It follows that $w^2 = 0$ and $p+w$ is an idempotent in $P(A)$. Moreover, setting $c = p+w$, we have $cp = p$ and $pc = c$, so p and c are equivalent. In particular, we have $\mathrm{Im}(p) = \mathrm{Im}(c)$.

Conversely, if $\mathrm{Im}(c) = \mathrm{Im}(p)$, then $cp = p$, so $(c - p)p = 0$ and thus $c - p \in A\hat{p}$. Since $pc = c$, we have $c-p \in pA$, and so $c-p \in pA \cap A\hat{p} = pA\hat{p}$, from which it follows that $c \in p + pA\hat{p}$. Thus as $pA\hat{p}$ is a complemented subspace of A, its complement being $\hat{p}A + pAp$, it follows that each fiber of Im is an analytic submanifold of A which is a translate of complemented closed linear subspaces of A, so it is affine and in particular, it is convex and contractible.

If A does not possess an identity, then with R denoting the regular right representation on itself, we find that $p + \widehat{R}(p)(pA)$ is the inverse image of $\mathrm{Im}(p)$ under Im . Since $\widehat{R}(p)$ is a projection in $\mathcal{L}(A)$, we find that $\widehat{R}(p)(pA)$ is complemented

5. The spaces $W(p,A)$ and $V(p,A)$

Firstly, we recall a well–known property :

Definition 5.1. We say that $u \in A$ is a *partial isomorphism* if there exists a $v \in A$ such that $uvu = u$ and $vuv = v$, in which case we call v an *pseudoinverse* for u . In general such a pseudoinverse is not unique.

Let $W(A)$ denote the set of all partial isomorphisms of A . If $u \in W(A)$ has a pseudoinverse v, then clearly $v \in W(A)$ with pseudoinverse u, and it is easy to see that both vu and uv belong to $P(A)$. If $p \in P(A)$, then we take $W(p,A) \subset W(A)$ to denote the subset of all partial isomorphisms of A having a pseudoinverse v satisfying $vu = p$. With regards to the opposite algebra A^-, we have $W(p,A) = W(A^-,p)$ (the notation reflects the nature of the domain and codomain respectively). Now for $p,q \in P(A)$, we set

$$
\begin{aligned}
W(p,A,q) &= W(p,A) \cap W(A,q) \\
&= \{\ u \in qAp : \exists\ v \in pAq\ ,\ vu = p \text{ and } uv = q\ \}\ .
\end{aligned}
\tag{5.1}
$$

It will be convenient to use the following notation in order to specify a *unique* pseudoinverse. If $u \in W(p,A,q)$, then a unique pseudoinverse $v \in W(q,A,p)$ for u is denoted by

$$
v = u^{-(p,q)}\ . \tag{5.2}
$$

Remark 5.1. One might well think of q as the 'image projection' of u in $W(A,q)$, but this is not well defined because even though $W(A)$ is the union of all $W(A,q)$ for $q \in P(A)$, this union is not disjoint. However, it is straightforward to show that on defining $\mathrm{Im}(u) = \mathrm{Im}(q)$, we obtain a well–defined surjective map $\mathrm{Im} : W(A) \to \mathrm{Gr}(A)$.

Let $G(p) = G(pAp)$. Clearly, if $u \in W(p,A,q)$ and $v \in W(q,A,e)$, then $vu \in W(p,A,e)$, and in particular, $W(p,A,p) = G(p)$. Thus $G(p)$ acts on the right of $W(p,A,q)$ via the restriction of multiplication and so the action is analytic. Likewise, there is an analytic left action of $G(q)$ on $W(p,A,q)$, again by restricting multiplication. Also, we have $W(p,A^-,q) = W(q,A,p)$. If $u \in G(p)$, then its unique pseudoinverse can be written as $u^{-p} = u^{-(p,p)}$.

Next we define

$$
W(c,A) = \bigcup_{e \in P(A)} W(c,A,e)\ , \qquad W(A) = \bigcup_{c \in P(A)} W(c,A)\ . \tag{5.3}
$$

Observe that $G(p)$ acts on the right of $W(p,A)$ by right multiplication and hence this action is analytic since it is the restriction of a continuous bilinear map, and likewise for the action on the left of $W(A,p)$.

Proposition 5.1. *There is a well-defined map* $\mathrm{Im} = \mathrm{Im}_A : W(A) \longrightarrow \mathrm{Gr}(A)$ *extending* $Q(A)$ *defined by taking* Im *to be constant with value* $\mathrm{Im}(e)$ *on* $W(c,A,e)$ *for each* $e \in P(A)$ *. Also, the restriction* $\mathrm{Im}|W(p,A)$ *induces a unique injection of* $W(p,A)/G(p)$ *into* $\mathrm{Gr}(A)$.

have

$$ef = uvuw = upw = uw = f \ , \tag{5.4}$$

thus $e > f$ and likewise $f > e$, so then $\mathrm{Im}(e) = \mathrm{Im}(f)$.

Conversely, if $\mathrm{Im}(e) = \mathrm{Im}(f)$, then $ef = f$ and $fe = e$, so $e \in W(e, A, f)$ and $f = e^{-(e,f)}$. Also, if $u \in W(p, A, e)$ and $v \in W(p, A, f)$ with $\mathrm{Im}(e) = \mathrm{Im}(f)$, then on choosing $w = v^{-(f,p)}$, we have $weu \in W(f, A, p)W(e, A, f)W(p, A, e)$ which is contained in $W(p, A, p) = G(p)$. Thus setting $g = weu$, we see that

$$vg = vweu = feu = eu = u \ , \tag{5.5}$$

thus u and v are in the same $G(p)$–orbit. On the other hand, if $g \in G(p)$ and $u \in W(p, A)$, then clearly $\mathrm{Im}(ug) = \mathrm{Im}(u)$ which implies that $\mathrm{Im}|W(p, A)$ induces a unique injection of $W(p, A)/G(p)$ into $\mathrm{Gr}(A)$. □

Again assuming that A has an identity, let $G = G(A)$ be as above.

Definition 5.2. We say that $x, y \in A$ are *similar* if x and y are in the same orbit under the inner automorphic action of $G(A)$ on A .

Thus for $p \in P(A)$, we call the orbit of p under the inner automorphic action of G on $P(A)$, *the similarity class of* p and denote this by $\mathrm{Sim}(p, A)$ where we observe that $\mathrm{Sim}(p, A) = G(A) * p$.

Definition 5.3. Let $u \in W(A)$. We call u a *proper partial isomorphism* if for some $W(p, A, q)$, we have $u \in W(p, A, q)$ where p and q are similar.

Now we proceed to define the space $V(p, A)$ which is central to our study.

Definition 5.4. Let $V(A)$ be the set of all partial isomorphisms of A . If $p \in P(A)$, then we take $V(p, A)$ to denote the set of all proper partial isomorphisms of A having a pseudoinverse $v \in W(q, A, p)$ with $q \in \mathrm{Sim}(p, A)$. With regards to (5.1) this is expressed by

$$V(p, A) = \bigcup_{q \in \mathrm{Sim}(p,A)} W(p, A, q) \ . \tag{5.6}$$

We also define $V(A, p) = V(p, A^-)$. Thus $V(p, A)$ is a $G(p)$–invariant subset of $W(p, A)$ and the natural map

$$V(p, A)/G(p) \longrightarrow \mathrm{Gr}(A) \ , \tag{5.7}$$

is injective. Likewise, $V(A, p)$ is $G(p)$–invariant in $W(A, p)$ under the left $G(p)$–action.

Next we introduce the Grassmannian naturally associated to $V(p, A)$. Let $\mathrm{Gr}(p, A)$ denote the image of $\mathrm{Sim}(p, A)$ under Im_A . Otherwise stated, $\mathrm{Gr}(p, A)$ is the image of the map in (5.7). For if $g \in G$, then $gp \in W(p, A, q)$ where $q = g * p = gpg^{-1}$, since pg^{-1} is the pseudoinverse of $gp \in W(q, A, p)$, and therefore the image of $V(p, A)/G(p)$ is all of $\mathrm{Gr}(p, A)$. Consequently, the map in (5.7) is bijective.

Proposition 5.2. *With regards to* (5.1) *and* (5.7), *we have the following properties :*

(1) *If* $u \in W(p, A, q)$ *and* $g \in G(A)$, *then* $gu \in W(p, A, g * q)$ *with pseudoinverse*

(2) $\mathrm{Im}(gu) = g\mathrm{Im}(u)$ *for any* $u \in W(A)$ *and* $g \in G(A)$. *In particular,*

$$\mathrm{Im} = \mathrm{Im}_A : W(A) \longrightarrow \mathrm{Gr}(A) \ ,$$

is a $G(A)$*–equivariant map.*

(3) *For any* $c \in P(A)$ *and* $g \in W(A)$, *we have* $\mathrm{Im}(gc) = g\mathrm{Im}(c) = \mathrm{Im}(g * c)$.

(4) *If* $u \in W(p, A, q)$ *and* $v \in W(c, A, d)$ *with* $pc = 0 = cp$ *and* $dq = 0 = qd$, *then*

$$(u + v)^{-(p+c,q+d)} = u^{-(p,q)} + v^{-(c,d)} \ .$$

(5) $\mathrm{Im}|V(p, A)$ *is* $G(A)$*–equivariant* .

Proof. Statements (1) – (4) follow essentially from the definitions. Observe that $gc \in W(c, A, g * c)$ and by definition, Im is constant on $W(c, A, g * c)$ with constant value $\mathrm{Im}(g * c)$. This means that $V(p, A)$ is $G(A)$–invariant and so $G(A)$ also acts analytically on $V(p, A)$ via left multiplication, and $\mathrm{Im}|V(p, A)$ is $G(A)$–equivariant. □

Lemma 5.1. *The left action of* $G(A)$ *on* $V(p, A)$ *is transitive.*

Proof. Consider the left action of $G = G(A)$ on Ap . For $u \in V(p, A)$, we can choose $v \in pAq$ for some $q \in \mathrm{Sim}(p, A)$, such that $u \in qAp$, $vu = p$ and $uv = q$. Thus $u \in W(p, A, q)$ and its pseudoinverse v belongs to $W(q, A, p)$. Of course, $p \in V(p, A)$ and $G(p)$ is indeed contained in $V(p, A)$ because $G(p) = W(p, A, p)$.

Since p and q are similar, we can choose $g \in G$ such that $q = g * p = gpg^{-1}$, so $gp = qg$. Then by Proposition 5.2 parts 1. and 4., we have $\hat{q}g\hat{p} \in W(\hat{p}, A, \hat{q})$ and so $u + \hat{q}g\hat{p} \in G(A)$ with pseudoinverse $v + \hat{p}g\hat{q}$. Taking $a = u + \hat{q}g\hat{p}$, we see that $ap = u$ and therefore $Gu = Gp = V(p, A)$. Thus the left action of $G(A)$ on $V(p, A)$ is transitive. □

Proposition 5.3. *The set* $V(p, A)$ *is a Banach analytic submanifold of* A . *Furthermore, the map* $\mathrm{Im}|V(p, A)$ *is a continuous open map and induces a homeomorphism* $V(p, A)/G(p) \cong \mathrm{Gr}(p, A)$.

Proof. Firstly, observe that since $R(p)$ is an idempotent continuous linear map of A into itself and onto Ap, then by Remark 3.1, it follows that $R(p)$ is an open map onto Ap . Now since $G(A)$ is open in A, it follows that $R(p)|G(A) : G(A) \to V(p, A)$ is an open map. Also, we have $V(p, A) = Gp = R(p)(G)$, so $V(p, A)$ is an open subset of Ap and is therefore an analytic submanifold of A . Thus to see that Im is continuous on $V(p, A)$, it suffices to show that its composition with $R(p)|G(A)$ is continuous. But for $g \in G(A)$, we have

$$\mathrm{Im} \circ R(p)(g) = \mathrm{Im}(gp) = g\mathrm{Im}(p) = \mathrm{Im}(g * p) \ , \tag{5.8}$$

and for fixed p this composition is continuous as the action of $G(A)$ on $P(A)$ is already continuous. It is in fact analytic and $\mathrm{Im} : P(A) \to \mathrm{Gr}(A)$ is continuous. Thus $\mathrm{Im}|V(p, A)$ is continuous for each $p \in P(A)$.

Let $\tau_p : G \to P(A)$, be the analytic evaluation map defined by $g \mapsto g * p$. Then the previous argument implies the following composition of maps :

$$(\mathrm{Im}|V(p, A)) \circ (R(p)|G) = Q(A) \circ \tau_p = (Q(A)|\mathrm{Sim}(p, A)) \circ \tau_p \ . \tag{5.9}$$

are open and all are surjective once we consider the codomain of $\mathrm{Im}|V(p,A)$ and $Q(A)|\mathrm{Sim}(p,A)$ to be $\mathrm{Gr}(p,A)$. In particular, this means that the bijection of $V(p,A)/G(p)$ onto $\mathrm{Gr}(p,A)$ as induced by $\mathrm{Im}|V(p,A)$, is a homeomorphism. □

In the following section we will pursue the relationship between $V(p,A)$ and $\mathrm{Gr}(p,A)$, which motivates calling $V(p,A)$ the 'Stiefel bundle' or 'manifold of framings' of A .

6. $V(p,A) \to \mathrm{Gr}(p,A)$ as an analytic principal bundle

Fixing $p \in P(A)$, we let $B = G(p) \times \hat{p}Ap$ and $X = \hat{p}Ap$. Next we define a map

$$\Lambda : B \to X \ , \qquad \Lambda(x,y) = yx^{-p} \ . \tag{6.1}$$

We see that $G(p)$ acts naturally on the right of B coordinatewise by multiplication, so for $(x,y) \in B$ and $u \in G(p)$, we have $(x,y)u = (xu,yu)$. These are clearly analytic maps since $G(p)$ is a Banach (analytic) Lie group and the restrictions of continuous bilinear maps yield analytic maps.

Lemma 6.1. *The triple (Λ, B, X) is an analytically trivial principal $G(p)$–bundle over X .*

Proof. If $\Lambda(x,y) = \Lambda(a,b)$, then we have $yx^{-p} = ba^{-p}$, and therefore $y = ba^{-p}x$. But $a^{-p}x \in G(p)$ and since we have $(x,y) = (a,b)a^{-p}x$, it follows that (x,y) and (a,b) are in the same orbit of the $G(p)$–action. In order to see that Λ is an open map, we define a map $w : B \to B$ by $(x,y) \mapsto (x, yx^{-p})$. It is clear that w is continuous and the map $\tilde{w} : B \to B$ defined by $(x,y) \mapsto (x,yx)$, is the inverse of w . Moreover, w is a homeomorphism and w composed with the second factor coordinate projection, which is clearly an open map, results in the map Λ . It follows that Λ is the orbit map of B under the right $G(p)$–action. Let

$$Z = \{\ (a,b) \in B \times B : \Lambda(a) = \Lambda(b)\ \} = B \times_X B \ , \tag{6.2}$$

be the Whitney sum of B with itself fibered over Λ . If $\sigma : Z \to G(p)$ is the map defined by $((u,v),(x,y)) \mapsto u^{-p}x$, then on setting $a = (u,v)$ and $b = (x,y)$, we have $a\sigma(a,b) = b$, so σ is the transition map of this action which is clearly a free action. If $k : B \to G(p)$ is the first factor coordinate projection, then k is analytic. So the Whitney sum with itself being the restriction of $k \times k$ on $B \times B$, is analytic. Denoting this restriction by μ, then

$$\mu : Z \to G(p) \times G(p) \ , \tag{6.3}$$

is an analytic map. Moreover, if $\nu : G(p) \times G(p) \to G(p)$ is defined by $(u,x) \mapsto u^{-p}x$, then ν is analytic since $G(p)$ is a Banach Lie group. This implies that $\nu\mu : Z \to G(p)$, and clearly $\sigma = \nu\mu$. Thus $\Lambda : B \to X$ is an analytic principal $G(p)$–bundle. On defining $s : X \to B$ by $s(y) = (p,y)$, we see that s is obviously an analytic section of this bundle and so (Λ, B, X) is analytically trivial. □

Suppose we have a map $h : B \to P(A)$ defined by $h(x,y) = p + yx^{-p}$. In order to

Lemma 6.2. *Define a map $H : B \to Ap$ by $H(x, y) = x + y$. Then H defines an analytic diffeomorphism of B onto an open set U which is contained in $V(p, A)$. In particular, H provides an analytic trivialization of $V(p, A)$ as a principal $G(p)$–bundle over the image of U under the orbit map.*

Proof. We claim that H is analytic since addition is linear on $A \times A \to A$, and H is the restriction of a continuous linear map. Thus if $T : (pAp) \times (\hat{p}Ap) \to Ap$ is the addition map and since p and $\hat{p}$ are idempotents with $p + \hat{p} = 1$, the map T is a linear isomorphism which is continuous and hence is a homeomorphism by the open mapping theorem. In fact, the pseudoinverse $T^{-p}(a) = (pap, \hat{p}ap)$ is clearly continuous, so we need not appeal to the open mapping theorem as remarked earlier. Since B is open in $(pAp) \times (\hat{p}Ap)$, it follows that $T|B$ is a homeomorphism of B onto an open subset $U \subset Ap$. But clearly, we have $T|B = H$ and thus $H : B \to U$ is an analytic diffeomorphism. Observe also that $U = G(p) + \hat{p}Ap$.

Now for $(x, y) \in B$, we have as before $x^{-p}y = 0$ because $x^{-p} \in pAp$ and $y \in \hat{p}Ap$. It follows that $x^{-p}(x + y) = p$, whereas

$$(x + y)x^{-p} = p + yx^{-p} = h(x, y) , \tag{6.4}$$

which is an idempotent. On setting $q = h(x, y)$, we see that $x^{-p} \in pAq$ and $x + y \in qAp$. Hence we must have $x + y \in W(p, A, q)$ and x^{-p} is its pseudoinverse in $W(q, A, p)$. Let $t = 1 + yx^{-p}$. Again by $x^{-p}y = 0$, it follows that

$$(1 + yx^{-p})(1 - yx^{-p}) = 1 = (1 - yx^{-p})(1 + yx^{-p}) , \tag{6.5}$$

implying that $t \in G(A)$ and $1 - yx^{-p}$ is the pseudoinverse of t . Since $py = 0$, it is straightforward to see that

$$tp = (1 + yx^{-p})p = p + yx^{-p} = q , \tag{6.6}$$

and

$$qt = (p + yx^{-p})(1 + yx^{-p}) = p + yx^{-p} = q , \tag{6.7}$$

which amounts to showing that p and q are in the same similarity class. On the other hand, we have

$$tpt^{-p} = (1 + yx^{-p})p(1 - yx^{-p}) = (p + yx^{-p})p = p + yx^{-p} = q , \tag{6.8}$$

and therefore $x + y \in W(p, A, h(x, y))$ which implies that $x + y \in V(p, A)$. So using t, we can define a map $t : B \to G(A)$, by $t(x, y) = 1 + yx^{-p}$. Observe that t is an analytic map such that pointwise $tpt^{-p} \in P(A)$ with $tp \in W(p, A, tpt^{-p})$, and pt^{-p} is its pointwise inverse in $V(tpt^{-p}, A, p)$ on B .

Thus the map $H : B \to U \subset Ap$ indeed has its image contained in $V(p, A)$. Now both $V(p, A)$ and B have right analytic G(p)–actions and

$$H((x, y)u) = H(xu, yu) = xu + yu = (x + y)u = (H(x, y))u , \tag{6.9}$$

so the map H is a $G(p)$–equivariant diffeomorphism. Thus H provides an analytic trivialization of $V(p, A)$ as a principal $G(p)$–bundle over U under the orbit map. ☐

Theorem 6.1. *The Grassmannian* $\mathrm{Gr}(p, A)$ *is a Banach analytic manifold modeled on* $X = \hat{p}Ap$ *and*

$$G(p) \to V(p, A) \longrightarrow \mathrm{Gr}(p, A) \;, \tag{6.10}$$

is a locally trivial analytic principal $G(p)$*–bundle. Furthermore,* $\mathrm{Gr}(A)$ *is a Banach analytic manifold.*

Proof. Using the associative law of multiplication in A, we find that the left action of G is by G(p)–equivariant diffeomorphisms which are all analytic as multiplication in A is analytic via continuous bilinear maps. This means that we can take U to be the image of B under H, which by Lemma 6.2 we know to be open in $V(p, A)$ and analytically $G(p)$–equivariantly diffeomorphic to B . Thus for any $u \in V(p, A)$, we can choose an element $g \in G$ with $u \in gU$, and hence show that gH is an analytic $G(p)$–equivariant diffeomorphism of B onto gU . This implies that $V(p, A)$ can be covered by open sets that are analytically $G(p)$–equivariantly diffeomorphic to B which we already know to be an analytic and analytically trivial principal $G(p)$–space. Therefore $V(p, A)$ is a locally trivial analytic principal $G(p)$–bundle over its orbit space, $V(p, A)/G(p)$, where the latter is necessarily an analytic manifold modeled on $\hat{p}Ap$ via the homeomorphism $V(p, A)/G(p) \to \mathrm{Gr}(p, A)$ as induced by $\mathrm{Im}|V(p, A)$. Consequently, $\mathrm{Gr}(A)$ is an analytic manifold, since each $G(A)$–orbit, $\mathrm{Gr}(p, A)$, is both open and closed in $\mathrm{Gr}(A)$. □

Thus we refer to $V(p, A)$ as *the Stiefel bundle of* A, in the sense of a principal $G(p)$–bundle defined over the Grassmannian $\mathrm{Gr}(p, A)$. The following example in the spatial case gives a further justification of its title.

Example 6.1. Let E be a Banach space (real or complex) with $A = \mathcal{L}(E)$ the bounded linear operators on E and let W be a closed complemented subspace of E . Without loss of generality we may assume that for $p \in P(A)$, $W = \mathrm{Im}\ p$ and $W' = \mathrm{Ker}\ p$, so that

$$E = W \oplus W' \;, \qquad W \cap W' = \{0\} \;,$$

is a decomposition of closed subspaces. In other words, W is a closed splitting subspace for E with complementary subspace W' (see e.g. [33]). In this case, let $\widetilde{V}(p, A) = \widetilde{V}(W, E) \subseteq \mathcal{L}(W, E)$, consist of injective linear maps with closed images which split E . So if $T \in \widetilde{V}(W, E)$, then T is injective and there is a (continuous) projection $q \in P(E)$ such that for $x \in W$, $T(x) = q(x)$, and

$$E = \mathrm{Im}\ T \oplus \mathrm{Ker}\ T \;, \qquad \mathrm{Im}\ T \cap \mathrm{Ker}\ T = \{0\} \;.$$

For a fixed $p \in \mathcal{L}(E)$, let $F = p(E)$. In [9] we introduced the subspace $\mathcal{V}(p, E) \subset \mathcal{L}(F, E)$ defined by

$$\mathcal{V}(p, E) := \{\ T \in \widetilde{V}(W, E) : T(W) = q(W) \;,\ q \in \mathrm{Sim}\ (p, \mathcal{L}(E))\ \} \;.$$

The assignment $T \mapsto \mathrm{Im}\ T = W$, defines a locally trivial K–principal bundle

$$K \to \mathcal{V}(p, E) \longrightarrow \mathrm{Gr}(W, E) \;,$$

where $K = \mathrm{GL}(W)$ and $\mathrm{Gr}(W, E)$ is *the Banach Grassmannian* of closed subspaces subspace of E similar to F, whereas $u \in V(p, \mathcal{L}(E))$ belongs to $\mathcal{L}(E)$. In other words, $V(p, \mathcal{L}(E))$ restricted to F results in $\mathcal{V}(p, E)$. It follows that the restriction map which sends u to its restriction to F, defines the analytic diffeomorphism of $V(p, \mathcal{L}(E))$ onto $\mathcal{V}(p, E)$, where the inverse is simply the composition with p . For $\mathcal{V}(p, E)$, the map to $\mathrm{Gr}(W, E)$ is the assignment of T to its image $W = \mathrm{Im}\, T$, as we have noted. Whereas for $V(p, \mathcal{L}(E))$, we have the map Im onto $\mathrm{Gr}(p, A)$ where the images are identified. In either event, the base space is the same Banach Grassmannian $\mathrm{Gr}(W, E) = \mathrm{Gr}(p, \mathcal{L}(E))$. We leave it as a short exercise for the reader to see that for e.g. $E = \mathbb{C}^n$, the definitions reduce to those for the usual Stiefel bundle of k–frames over the Grassmannian $G_k(\mathbb{C}^n)$ of k–planes in $\mathbb{C}^n$, for some pair (k, n) (see e.g [17]).

Example 6.2. Let $\mathcal{H}$ be a separable complex Hilbert space with an orthogonal direct sum decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Here the $\mathcal{H}_\pm$ are closed subspaces and the decomposition is specified by a unitary operator $J : \mathcal{H} \to \mathcal{H}$ such that $J|\mathcal{H}_+ = 1$ and $J|\mathcal{H}_- = -1$. Let W be a closed splitting subspace commensurable with $\mathcal{H}_+$ (that is, for which $W \cap \mathcal{H}_+$ has finite codimension in both W and $\mathcal{H}_+$). The restricted Grassmannians considered in [28] [31] that are submanifolds of $\mathrm{Gr}(W, \mathcal{H})$, provide examples of $\mathrm{Gr}(p, A)$ where $A = \mathcal{B}_J(\mathcal{H})$ is the Banach algebra of bounded operators $T : \mathcal{H} \to \mathcal{H}$ such that $[J, T]$ is a Hilbert–Schmidt operator. There is a norm $\| \, \|_J$ defined by $\|T\|_J = \|T\| + \|[J, T]\|_2$ and with the topology induced by $\| \, \|_J$, the group of units $G(A)$ is a complex Banach Lie group. Indeed for such closed splitting subspaces W there is the notion of 'admissible bases' [28] [31] for which there is a well–defined Stiefel manifold (see [25]). Such a manifold can be realized as a submanifold of our $V(p, A)$ and the resulting Stiefel bundle provides another interesting example of (6.10). Further development of this particular relationship with integrable systems remains a topic for future investigation.

7. Existence of analytic sections

Next we investigate the analytic geometry of the principal bundle (6.10). The first step is to show that (6.10) admits special local analytic sections.

Lemma 7.1. *Let $S(p, A) = V(p, A) \cap P(A)$. Then* $\mathrm{Im}|S(p, A)$ *is an analytic diffeomorphism of $S(P, A)$ onto an open subset of* $\mathrm{Gr}(p, A)$ *. Furthermore,* $\mathrm{Im} = Q(A)$ *admits local analytic sections through each point of $P(A)$.*

Proof. Suppose that $c, e \in P(A) \cap V(p, A)$. Since $G(A)$ acts transitively on $V(p, A)$, we have $V(p, A) = Gp$ with $G = G(A)$, and so there is a $g \in G$, with $c = gp$ and clearly $cp = c$. Then we have

$$gp = c = cc = gpgp \;\Rightarrow\; p = pgp = pc \,, \tag{7.1}$$

and thus $pc = p$, which together with $cp = c$, implies that $\mathrm{Ker}(c) = \mathrm{Ker}(p) = \mathrm{Ker}(e)$.

Now recall that if $\mathrm{Im}(c) = \mathrm{Im}(e)$, then we have $c = e$ showing that $\mathrm{Im}|S(p, A)$ is an injective map. Also recall that the section s of (z, B, X) is given by $s(y) = (p, y)$, so $H \circ s(y) = p + y$, which is clearly in $S(p, A)$ as we have $H \circ s(y) = h(p, y)$. Thus $S(p, A)$ contains a relatively open neighborhood of p that is the image of image is a relatively open neighborhood of $p \in S(p, A)$. Since $gp = c$, we have $g^{-1}c = p$, and therefore $Gp = Gc$, or $V(c, A) = V(p, A)$; consequently, $S(c, A) = S(p, A)$. So applying this to c, we find that $\mathrm{Im}|V(p, A)$ has a local analytic section whose image is a relatively open neighborhood of $c \in S(c, A) = S(p, A)$. It follows that Im $S(p, A)$ is an analytic diffeomorphism of S(p,A) onto an open subset of $\mathrm{Gr}(p, A)$. Of course, this tacitly uses the fact that the analytic manifold structure given to $G(A) \cdot \mathrm{Im}(p) = \mathrm{Gr}(p, A)$, by the induced homeomorphism of $V(p, A)/G(p)$, is the same as that obtained from $V(c, A)/G(c)$, but we will see later that this is straightforward. Now since $S(p, A) \subset P(A)$, then for $p \in S(p, A)$ and

$$(\mathrm{Im}|V(p, A))|S(p, A) = Q(A)|S(p, A) \ , \tag{7.2}$$

we conclude that $\mathrm{Im} = Q(A)$ admits local analytic sections through each point of $P(A)$. ☐

Now we observed that $\mathrm{Ker}(c) = \mathrm{Ker}(p)$, for each $c \in S(p, A)$. On the other hand, as Ker is effectively $Q(A^-)$, we conclude that the inverse image of $\mathrm{Ker}(p)$ under Ker is $p + \hat{p}Ap$, which was already seen to be contained in $S(p, A)$. Therefore $S(p, A)$ is just the fiber of the map $Q(A^-) = \mathrm{Ker}|P(A)$ over $\mathrm{Ker}(p)$. Applying this to A^-, it follows symmetrically that $S(A, p) = p + pA\hat{p}$, which we already know to be the fiber of $Q(A) = \mathrm{Im}$ over p, is in fact the intersection $S(A, p) = P(A) \cap V(p, A)$, recalling that $V(p, A) = V(A^-, p)$. Clearly, $S(p, A)$ and $S(A, p)$ intersect only at the single point p itself. Moreover, the tangent spaces of both of these at p, are the complementary subspaces $\hat{p}Ap$ and $pA\hat{p}$ whose sum (see below) will be the tangent space $T_pP(A)$. Consequently, both sections intersect transversally at p . Because we have

$$S(p, A) = p + \hat{p}Ap \ , \tag{7.3}$$

the analytic diffeomorphism $\mathrm{Im}|S(p, A)$ is a parametrization of an open neighborhood of $\mathrm{Im}(p)$ in $\mathrm{Gr}(A)$ and its inverse can then be seen to represent a canonical coordinate chart.

Proposition 7.1. *The map* $\mathrm{Im}|P(A) : P(A) \to \mathrm{Gr}(A)$ *is an analytic map which admits local analytic sections.*

Proof. Let $g : U(p) \times U(p) \to G(A)$ be the map as defined by

$$g((c, d)) = cd + \hat{c}\hat{d} \ . \tag{7.4}$$

Since the map on $A \times A \to A$ given by $(x, y) \mapsto xy + (1 - x)(1 - y)$ is a continuous bilinear map, we see that g is analytic, in fact it is a quadratic map. Also, we have $g(c, d)d = cg(c, d)$, so $c = g(c, d) * d$, and hence in particular, $g(c, p) * p = c$. Hence it follows that

$$\mathrm{Im}(c) = \mathrm{Im}(g(c, p) * p) = g(c, p)\mathrm{Im}(p) \ . \tag{7.5}$$

But $G(A)$ now acts analytically on $\mathrm{Gr}(A)$ because the map $\mathrm{Im} : V(p, A) \to \mathrm{Gr}(p, A)$ is an open, analytic and $G(A)$–equivariant map for each $p \in P(A)$. Also, the action on $\mathrm{Gr}(p, A)$ is induced by the analytic left action of $G(A)$ on $V(p, A)$. More

Let $\#$ denote the left action of $G(A)$ on $V(p,A)$ by left multiplication. Then if $\mathrm{Gr}(\#)$ denotes the action on $\mathrm{Gr}(p,A)$ by $\mathrm{Im}|V(p,A)$ as induced from $\#$, we have

$$\mathrm{Gr}(\#)\cdot(\mathrm{id}\times \mathrm{Im}|V(p,A)) = (\mathrm{Im}|V(p,A))(\#)\ . \tag{7.6}$$

Thus the preceding formulas imply that $\mathrm{Gr}(\#) = \mathrm{Gr}(*)$ is analytic on $G(A)\times V(p,A)$ and $Q(A) = \mathrm{Im}$ is analytic on $P(A)$, and therefore $\mathrm{Im}|P(A) = Q(A)$ is an analytic map of $P(A)$ onto $\mathrm{Gr}(A)$ which admits local analytic sections. □

From Proposition 4.2, we know that $p + pA\hat{p}$ is the set of idempotents equivalent to p and that $p+\hat{p}Ap$ is the image of the analytic section of $V(p,A)$ defined above. Consequently, $p+\hat{p}Ap$ is an open neighborhood of p in $S(p,A)$. As pointed out before, $1+x = \exp(x)$ for $x\in \hat{p}Ap$, and $p+x = (1+x)p$. So it follows that $p+\hat{p}Ap = (\exp(\hat{p}Ap))p$. Moreover, we have seen that $p+pA\hat{p}$ is the fiber of Im over p, the idempotents equivalent to p under the order relation. Regarding $P(A)$ as a submanifold of A defined by $x^2 = x$, it follows that the tangent space at p, is given by

$$T_pP(A) = \hat{p}Ap + pA\hat{p}\ . \tag{7.7}$$

By our earlier discussion, the relation $\exp(x) = 1+x$ for $x\in pA\hat{p}$, implies that $p(\exp(x)) = p+x$ for $x\in pA\hat{p}$. The latter suggests defining a map $\Psi : (\hat{p}Ap)\times(pA\hat{p})\to A$ given by

$$\begin{aligned}\Psi(y,z) &= (1+y)(p+z)(1-y) = (\exp(y))p(\exp(z))(\exp(-y))\\ &= (\exp(y))(\exp(-z))p(\exp(z))(\exp(-y))\\ &= \exp(y)*\exp(-z)*p\end{aligned} \tag{7.8}$$

In effect, $p+z$ is an idempotent equivalent to p, that is, we have $\mathrm{Im}(p) = \mathrm{Im}(p+z)$, and $\Psi(y,z)$ is just the result of applying to $p+z$ the inner autmorphism induced by $1+y\in G(A)$.

Proposition 7.2. *The map $\Psi : (\hat{p}Ap)\times(pA\hat{p})\to A$ is an injective analytic map which is a local diffeomorphism at* 0 *onto a relatively open subset of $P(A)$ containing p . In particular, $P(A)$ is an analytic submanifold of A .*

Proof. It is straightforward to see that Ψ is an injective map and clearly takes its values in $P(A)$, and it is an analytic map since it is actually a polynomial of degree three. The injectivity of Ψ follows from the fact that A is expressible as the sum of four complementary subspaces :

$$A = (pAp)\oplus(\hat{p}Ap)\oplus(pA\hat{p})\oplus(\hat{p}A\hat{p})\ . \tag{7.9}$$

Now if c is close enough to $p\in P(A)$, then from the proof of Proposition 4.1, c and p are in the same orbit of the inner automorphic action of of $G(A)$. Hence we have $\mathrm{Im}(c)\in \mathrm{Gr}(p,A)$ and in fact, $\mathrm{Im}(c)\in U/G(p)$ where the latter is the open domain of the section s as above. Moreover, $s\circ \mathrm{Im}(c)$ is of the form $p+y$ for some $y\in\hat{p}Ap$, since the image of this section is $p+\hat{p}Ap$ by (7.3). On the other hand, we have $p+y = (1+y)p(1-y)$. As Im is $G(A)$–equivariant under the inner automorphic action of $G(A)$, this means that

We therefore have

$$\mathrm{Im}((1-y)*c) = \mathrm{Im}(p)\ , \tag{7.11}$$

and so there exists a unique $z \in pA\hat{p}$, satisfying $p + z = (1-y)c(1+y)$, and furthermore

$$c = (1+y)(p+z)(1-y) = \Psi(y,z)\ . \tag{7.12}$$

Thus Ψ is an analytic injective map onto a neighborhood of $p \in P(A)$. As functions of $c \in U(p)$, where $U(p)$ is a sufficiently small neighborhood, we have $y(c) = s \circ \mathrm{Im}(c) - p$, and $z(c) = (1-y(c))c(1+y(c))$. Hence as the section s and $\mathrm{Im} = Q(A)$ are both analytic, then so too are y and z, and (y,z) is now the inverse of Ψ on $U(p)$. □

On expanding $\Psi(y,z)$, we find that

$$\begin{aligned}\Psi(y,z) &= (1+y)(p+z)(1-y) = (p+y+z+yz)(1-y) \\ &= p+y+z = [y,z] - yzy\ .\end{aligned} \tag{7.13}$$

Consequently, the derivative at $(0,0)$ is given by linear terms

$$\Psi'(0,0)(u,v) = u+v\ . \tag{7.14}$$

So $\Psi'(0,0)$ is just the addition map of $(\hat{p}Ap) \times (pA\hat{p}) \to A$, which is a linear homeomorphism on $\hat{p}Ap + pA\hat{p}$, where the latter is a closed complemented subspace of A . Thus Ψ is an analytic diffeomorphism onto an open neighborhood of $p = \Psi(0,0) \in P(A)$. It follows that $P(A)$ is itself an analytic submanifold of A, since the orbit of $p \in P(A)$ is modeled on $\hat{p}Ap + pA\hat{p}$, which by (7.7) is the tangent space $T_pP(A)$ at p regarded as a subspace of A .

In summarizing, we combine Theorem 6.1 with Propositions 5.2, 6.1, 7.1 and 7.2 to establish the main result of this section.

Theorem 7.1. *The map* $\mathrm{Im} : V(A) \to \mathrm{Gr}(A)$ *is the projection of an analytic fiber bundle. The spaces* $V(A)$ *and* $P(A)$ *are analytic submanifolds of* A*,* $\mathrm{Gr}(A)$ *is an analytic manifold, and* $\mathrm{Gr}(p,A)$ *is open and closed in* $\mathrm{Gr}(A)$ *. For* $q \in P(A)$ *with* $\mathrm{Im}(q) \in \mathrm{Gr}(p,A)$ *and for each* $p \in P(A)$*, the restriction of* Im *to* $P(A) \cap V(p,A)$ *is an analytic diffeomorphism onto an open subset of* $\mathrm{Gr}(p,A)$ *which thus provides natural analytic local sections for the map* $\mathrm{Im} : P(A) \to \mathrm{Gr}(A)$ *passing through each point of* $P(A)$ *.*

Finally, we show that $\mathrm{Gr}(p,A)$ and $G/G(\mathrm{Im}(p))$ agree as homogeneous spaces possessing the same analytic structure as shown by other authors (cf. [27] [29] [36]).

Proposition 7.3. *As Banach homogeneous spaces,* $\mathrm{Gr}(p,A) = G/G(\mathrm{Im}(p))$ *. Furthermore, they are equivalent as Banach analytic manifolds.*

Proof. If w is taken to be an inner automorphism of A, then w restricts to an analytic diffeomorphism $V(p,A) \to V(w(p),A)$, via the assignment $p \mapsto w(p)$. It also restricts to an isomorphism $G(pAp) \cong G(qAq)$, where $q = w(p)$. Thus an analytic bundle map can be defined over the induced map

$$V(p,A)/G(p) \longrightarrow V(q,A)/G(q)\ , \tag{7.15}$$

equivalent. Furthermore, the resulting Banach analytic manifold structure on the orbit of $\mathrm{Im}(p)$ under the inner automorphism group is independent of the particular representative in the similarity class of p . Thus we obtain a Banach analytic manifold structure on each inner automorphic orbit of $\mathrm{Gr}(A)$.

Let $A[p]$ denote the commutant of p in A . The isotropy subgroup of p under the inner automorphic action of G is then $G(A[p])$. Thus $G(A[p])$ is an analytic Banach Lie subgroup of G regarded as a submanifold, since $A[p] \cap G = G(A[p])$. In fact, $A[p] = pAp + \hat{p}A\hat{p}$ which is complemented in A, where the complement is $\hat{p}Ap + pA\hat{p}$. Thus $G/G(A[p])$ is an analytic manifold, and if we can find a local analytic section of the inner automorphic action about a single point of $\mathrm{Sim}(p, A)$, then $\mathrm{Sim}(p, A)$ is an analytic submanifold of A . But this we had shown in Proposition 6.1 above. However, $(tpt^{-p})H^{-p}$ is a local section over U that is analytic, and since Im admits local analytic sections, the same is true for the action of G induced on $\mathrm{Gr}(p, A)$ when endowed with the analytic structure as given by Lemma 6.2. Hence this analytic structure must agree with the analytic structure as a homogeneous space $G/G(i(p))$, where $G(i(p))$ denotes the isotropy subgroup of $\mathrm{Im}(p)$ in $\mathrm{Gr}(p, A)$.

Now if $g \in G$ and $g\mathrm{Im}(p) = \mathrm{Im}(p)$, then gpg^{-1} and p are equivalent, so we have $gpg^{-1}p = p$ and $pgpg^{-1} = gpg^{-1}$. Thus $pg^{-1}p = g^{-1}p$ and $pgp = gp$, or equivalently, $\hat{p}g^{-1}p = 0$ and $\hat{p}gp = 0$, so it follows that $g \in A[p] + pA\hat{p}$ which is complemented by $\hat{p}Ap$ in A . In other words, $g \in pAp + A\hat{p}$, and since $(A\hat{p})(pA) = 0$, it also follows that $pAp + A\hat{p}$ is a subalgebra of A . Now since the latter contains 1, its group of units $G(\mathrm{Im}(p))$ is an analytic Lie subgroup (submanifold) of G, and hence the analytic structure on $\mathrm{Gr}(p, A)$ must agree with that of the Banach homogeneous space $G/G(\mathrm{Im}(p))$. $\square$

Remark 7.1. More generally, suppose A is taken to be a continuous inverse algebra, meaning that A is any topological algebra with (jointly) continuous multiplication, with the property that $G(A)$ is a nonempty open subset and inversion is continuous (see [35] p. 87). Let TIA denote the category of such algebras with continuous algebra homomorphisms. If A is such an algebra and $p \in P(A)$ an idempotent, then following Remark 3.1, pAp is also such an algebra with identity p . Clearly, TIA has finite products and up to TIA–isomorphism, the coordinate projections are just maps of the form $a \mapsto pap$, and so they are simple polynomial maps. Let $\mathcal{D}$ denote the category of pairs (X, A) where A is in TIA, $X \subset A$ is a subset and for which $f : (X, A) \longrightarrow (Y, B)$ is a $\mathcal{D}$–map, provided that up to TIA-isomorphisms on either end, f is locally the restriction of a (noncommutative) several variable polynomial map in the variables $(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1})$. There is an obvious forgetful functor to the category TOP of spaces and continuous maps, and we can form an example of abstract differentiability as in [10] (Example 2.1). In this case, the maps might reasonably be called (noncommutative) rational maps. Then as in [10], we can speak of rational Lie groups, rational principal bundles, etc. In fact, we point out that all results so far are equally valid for A just taken to be a continuous inverse algebra. As a consequence, the corresponding Stiefel bundle of A is actually a rational bundle in this noncommutative sense. In particular, this means that the Banach algebra Stiefel bundle $V(p, A)$ is actually a noncommutative rational

8. Application to global smoothing of sections

We commence by stating a modification of the smooth approximation theorem of [32] 6.7 (see also [16] V.4.1) that is valid for infinite dimensions.

Theorem 8.1. *Let $\mathcal{E} = (\pi, P, X)$ be a smooth bundle, (meaning only that $\pi : P \to X$ is a smooth map), which is smoothly locally trivial with fiber Y a Banach manifold modeled on a Banach space F, and where X regarded as a subset of some Banach manifold, admits smooth partitions of unity. Let $C \subset X$ be a closed subset and $U \subset X$ an open subset containing C . Given a continuous section s of $\mathcal{E}$, such that $s|U$ is smooth, then for an open subset $T \subset P$ containing $s(X)$, there exists a smooth section t of $\mathcal{E}$ extending s, such that $t(X) \subset T$.*

Remark 8.1. Note that the hypothesis regarding smooth partitions of unity is satisfied when X is taken to be a subset of a separable Hilbert space (see [20] II-3).

Proof. Firstly, we remark that if the bundle $\mathcal{E}$ is trivial with fiber an open ball in a Banach space, then the result would follow from any partition of unity argument. In this case, it is enough to cover the section with with fiberwise convex open tubes all of which are contained in U . We can piece together constants with a partition of unity and then piece these onto the section with a 'halo' homotopy as in ([32] 6.7).

Next, for each $x \in X$, we can choose an open set $U(x)$ containing x, such that $\mathcal{E}$ is smoothly trivial over $U(x)$, together with a smooth trivialization

$$h(x) : \mathcal{E}|U(x) \to U(x) \times Y \ . \tag{8.1}$$

Let $g(x) : U(x) \times Y \to Y$ be the second factor projection. Shrinking $U(x)$ if necessary, we may assume that $g(x) \circ h(x) \circ s(U(x)) \subset B(x)$, where $B(x) \subset Y$ is an open subset diffeomorphic to the open unit ball B in F .

Now let $T(x) = h(x)^{-1}(U(x) \times B(x))$, so that $T(x)$ is an open subset of P containing $s(U(x))$. Using the paracompactness of X, we can choose a σ–discrete open refinement $\{V_\alpha\}_{\alpha \in \mathbf{N}}$, of the open cover $\{U(x)\}_{x \in X}$, where each V_α is a disjoint collection of open subsets of X (see e.g. [7]). Let U_α be the union of sets in V_α . We can then use the above trivialization and the diffeomorphisms onto B to assemble the obvious diffeomorphism of $U_\alpha \times B$, onto an open subset $T_\alpha \subset P$ containing $s(U_\alpha)$. This gives a smooth trivialization of $(p_\alpha, T_\alpha, U_\alpha)$ onto the trivial bundle whose fiber is B, where $p_\alpha = p|T_\alpha$. After passing to a further refinement, if necessary, we can assume that the open cover $\{U_\alpha\}_{\alpha \in \mathbf{N}}$, is locally finite. Now for each $x \in X$, let

$$D(x) = \bigcap_{x \in U_\alpha} \{P_x \cap T_\alpha\} \ , \qquad T_0 = \bigcup_{x \in X} D(x) \ . \tag{8.2}$$

It follows that $s(X)$ is contained in T_0, and since the cover is locally finite, $s(X)$ is in fact contained in the interior of T_0 . Effectively, if x is any point of X, it is possible to choose an open neighborhood W which meets only a finite number of the U_α . Consequently, for each $y \in W$, the set obtained by intersecting the sets T_α for which U_α meets W, will be an open neighborhood of $s(x)$ contained in T_0 .

Just as in the argument of [32] 6.7, we need to shrink the open cover three times and at each stage in the inductive modifications, we will require the construction of the sections to be such that the latter remain within the topological interior of T_0 . Let A_α be the closure of the α–th open set in the smallest shrinking, C_α the closure of the open set in the next smallest and K_α the closure of the open set in the first shrinking. Then we have the sequence of inclusions :

$$A_\alpha \subset \text{Int } C_\alpha \subset C_\alpha \subset \text{Int } K_\alpha \subset K_\alpha \subset U_\alpha \,. \tag{8.3}$$

Thus each of these sequences of closed sets is a locally finite family whose interiors cover X, and as K_α is closed in X, it follows that K_α is paracompact and so admits smooth partitions of unity as X does by hypothesis. In constructing a smooth section over K_α that approximates a previous section via successive modification and assuming that there is a section inside T_0, we can make our approximation locally to lie within a small local tube around the image by working in $K_\alpha \times B$, since the local tubes around a continuous section form the neighborhood basis at each point along the existing section. As we remarked at the start of the proof, by using a partition of unity argument, we can piece together a smooth approximation to our pre–existing section over K_α, whose image lies inside the local tubes and therefore inside the interior of T_0 . Then the halo homotopy can be applied to piece the section onto the pre–existing section.

Finally, invoking the Stone–Weierstrass theorem as in [32], is here replaced by the existence of smooth partitions of unity on X . At each stage, the section image is constrained to lie within the interior of T_0 . Consequently, the successive modifications together define a globally smooth section whose image is contained in T_0 and therefore in T, and it thus extends the restriction of s to C . ☐

As a consequence of Theorems 7.1 and 8.1, we obtain the following result for global smoothing of sections of $V(A) \to \text{Gr}(A)$, when pulled back by maps $f : X \to \text{Gr}(A)$.

Corollary 8.1. *Let X and C be as in Theorem 8.1 and consider smooth maps $f : X \to \text{Gr}(A)$ and $g : C \to V(A)$, satisfying $\text{Im} \circ g = f|C$. Suppose there exists a continuous map $h_0 : X \to V(A)$ extending the map g smoothly on a neighborhood of C and lifting f through the map $\text{Im} : V(A) \to \text{Gr}(A)$; that is, $f = \text{Im} \circ h_0$. Then there exists a smooth map $h : X \to V(A)$, also extending g and satisfying $f = \text{Im} \circ h$.*

The same result applies when $V(A)$ is restricted to $V(p, A)$ with the induced restriction of $\text{Gr}(A)$ to $\text{Gr}(p, A)$ as in 6.10.

Proof. In view of Theorem 7.1, the pullback $f^*V(A) \to X$ of $V(A)$ by the map f is a smooth locally trivial fiber bundle. Consider now the restriction $f^*V(A)|C$. We thus view g as a smooth section $g : C \to f^*V(A)|C$, which is extended continuously to X by h_0 regarded as a section $h_0 : X \to f^*V(A)$, with the above properties. The result then follows directly from Theorem 8.1 . ☐

Remark 8.2. Given a Banach space E and $A = \mathcal{L}(E)$, Corollary 8.1 generalizes [13] Theorem 1.1 where in the latter case, X is taken to be a compact subset of $\mathbb{R}^n$.

with the above modification of the Steenrod smoothing approximation theorem (cf. *the section extension property* of [8]). Corollary 8.1 is also a generalization of [9] Theorem 4.3 which in turn generalizes the finite dimensional case in [11] IV–1–2.

Example 8.1. Again taking E to be a Banach space and $A = \mathcal{L}(E)$, Corollary 8.1 provides a useful criterion when dealing with the parametrization of smooth bases for the kernel and image for suitable classes of operator or matrix–valued functions (e.g. in the latter case when $A = \mathcal{L}(E) \otimes \mathcal{M}_n$). For X as above, an operator–valued function $f_0 : X \to \mathcal{L}(E)$ whose pointwise image is T, induces a map $f : X \to \mathrm{Gr}(p, A) = \mathrm{Gr}(W, E)$, by taking p to be the projection onto $W = \mathrm{Im}\ T$ (see Example 6.1). Suppose that $W = \mathrm{Im}\ T$ is a closed splitting subspace of E such that by Example 6.1, the following

$$E = \mathrm{Im}\ T \oplus \mathrm{Ker}\ T\ , \qquad \mathrm{Im}\ T \cap \mathrm{Ker}\ T = \{0\}\ ,$$

decomposes E into closed complemented subspaces. Then it is clear that smooth bases for Im T (or for Ker T) could be parametrized in terms of those maps from X that satisfy the hypotheses of Corollary 8.1. In other words, such maps f provide smooth lifts from X to $V(p, \mathcal{L}(E))$ where the latter is regarded as the manifold of bases for Im T, as is required. The smooth parametrization of the bases is effectively the resulting map $h : X \to V(p, A)$ of Corollary 8.1. As special instances, note that the existence of global bases is immediate when X is contractible or when $K = \mathrm{GL}(W)$ in Example 6.1, is the group of invertibles in a separable complex Hilbert space (and hence is contractible by [19]).

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