Extensions of cocommutative coalgebras and a Hochschild - Kostant - Rosenberg type theorem

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Introduction

Given a commutative algebra A over a field k, it is a well known result that the first Hochschild homology group $HH_1(A)$ is isomorphic to the A-module of Kähler differentials $\Omega^1(A)$, and is therefore a universal object for derivations from A into symmetric A-bimodules M. Dually, $HH^1(A) \cong Der_k(A) = \{f \in Hom_k(A, A) \,/ f(ab) = af(b) + bf(a)\ \forall a, b \in A\}$.

This object has been studied for a long while, as its description is connected with important properties of the algebra A. As one of the best-known examples, let us recall the Hochschild - Kostant - Rosenberg theorem [13], and its reciprocal statement [3, 2], which tells us that for a perfect field k the algebra $HH_*(A)$ (for A an essentially finite type commutative algebra) is isomorphic to the exterior algebra $\Lambda_A(\Omega^1(A|k))$ if and only if A is smooth.

If C is a topological cocommutative k-coalgebra (the topology may be the discrete one), we define an object Ω^1_C which is universal for coderivations of cosymmetric C-bicomodules into C. We show that this object is isomorphic to the first cohomology group $Hoch^1(C)$ associated to the coalgebra ([6], [17], [8]).

The behaviour of Ω^1_C with respect to localizations is studied. It turns out that $Hoch^1(C)$ localizes in more general situations than those described in [9]. For the higher cohomology groups, we obtain at this point that in certain cases there is an isomorphism with the degree n component of the graded-cocommutative coalgebra $\Lambda^*_C(\Omega^1_C)$ obtained from Ω^1_C, which is described in section 4. This fact gives also localization results.

Next we define homologically the concepts of smooth coalgebra, and study some properties of this kind of coalgebras.

Finally we prove:

Theorem If k is a field, $char(k) = 0$ and C is a cocommutative coalgebra satisfying one of the hypothesis below:

- C is a smooth algebraic coalgebra and $ke_i \wedge k.e_i$ is finite dimensional for every group-like element $e_i \in C$.
- C is a smooth local topological coalgebra provided of a topology verifying Proposition 1.2,
  $C = \cup_{n \in \mathbb{N}_0} \wedge^n (k.e)$ (e being the unique group-like element of C) and $k.e \wedge k.e$ is finite dimensional.

then $Hoch^*(C)$ is isomorphic to the exterior coalgebra on Ω^1_C.

The statement above is formulated both for usual and topological coalgebras, because it has been necessary for us to pass the boundary of the algebraic category, i.e. some of the proofs are given for topological coalgebras (incuded those with discrete topology). As a consequence, a result for this kind of coalgebras is obtained, under suitable conditions.

In fact, this theorem gives an answer to a problem which arises in [11] (Section 5). In this paper, they give a proof of Hochschild - Kostant - Rosenberg theorem using G-algebras and expecting that a dual version for coalgebras will hold. Their problem is the lack of a definition dualizing regular sequences. Such a definition appears here in the hypothesis of Theorem 6.5. It is worth to notice that they do not mention a key problem: *Hoch* theory does not localize well except under certain conditions; in section 8 we solve the localization problem for this situation.

The contents of the paper are as follows:

In section 1 we construct the universal object for coderivations of cosymmetric comodules, Ω^1_C. As we are working with topological coalgebras, we prove a result allowing us to calculate the topological version of $Hoch^*$ in terms of "resolutions". We then prove that Ω^1_C is isomorphic to $Hoch^1(C)$ and show how Ω^1_C behaves in an explicit example.

In section 2 we study the behaviour of the universal object Ω^1_C defined previously, with respect to localizations. As we prove that Ω^1_C is isomorphic to $Hoch^1(C)$, the last one commutes with localization.

The definition of the exterior coalgebra on Ω^1_C is given in section 3. We also study in detail the example of the coalgebra of distributions supported on a smooth compact manifold X, which suggests that a coalgebra version of a Hochschild - Kostant - Rosenberg type theorem exists. So, in section 4 we define what a "smooth coalgebra" is. Our definition is given in terms of extensions. When C is (cocommutative) smooth, Ω^1_C turns out to be, as expected, an injective cosymmetric C-comodule. Smoothness is also for coalgebras, a local concept.

Section 5 is devoted to the definition of local coalgebras in terms of grouplike elements and study its properties. A useful result is proven here: it is a "Nakayama's Lemma" for local coalgebras.

So, with these tools in hand, we return in section 6 to the comodule of "Kähler differentials" Ω^1_C, proving that in the smooth local case it is a free comodule. We state at the end of this section an equivalence between the structure of the graded coalgebra associated to C (which is now local) and the existence of a "Koszul resolution".

As we want to obtain this resolution, we describe in section 7 the structure of $gr(C)$. In fact, we prove a structure theorem for cocommutative local smooth coalgebras, describing them in terms of shuffle coalgebras (proposition 7.4. and 7.5).

Finally, in section 8 we give the arguments allowing us to pass from the global case to the local case, and we then prove our main theorem.

We will suppose that the field k is algebraically closed. As we are interested in the cohomology groups $Hoch^*$, this assumption is not restrictive in the sense that $Hoch^*(M, C|k) \otimes \overline{k} = Hoch^*(M \otimes \overline{k}, C \otimes \overline{k}|\overline{k})$ (with this notation, $|k$, resp $|\overline{k}$, means that the tensor products are taken over k, resp. over $\overline{k}$, and $\overline{k}$ denotes the algebraic closure of k). C will be always a topological coalgebra, unless the contary is stated, even if we use the algebraic notation. In particular, usual coalgebras will be considered as topological ones with the discrete topology, and he same will hold for comodules. All topologies considered will be Hausdorff.

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1 Universal object for coderivations

Given a field k and k-coalgebra (C, Δ, ϵ), there is an object L_C which is universal for k-coderivations of bicomodules M into C ([6, 8]). L_C is the cokernel of $\Delta : C \to C \otimes C$. The sequence

$$0 \longrightarrow C \xrightarrow{\Delta} C \otimes C \longrightarrow L_C \longrightarrow 0$$

is k-split by means of $s : C \otimes C \to C$, $s(c \otimes c') := \epsilon(c)c'$.

When C is a topological coalgebra ($\Delta : C \to C\widetilde{\otimes}C$), we consider complete C-bicomodules M for which the coactions $\rho_M^+ : M \to M\widetilde{\otimes}C$ and $\rho_M^- : M \to C\widetilde{\otimes}M$ are continuous (here $\widetilde{\otimes}$ stands for the completion of the algebraic tensor product w.r.t. a suitable topology on it). L_C is then a topological C-bicomodule with the quotient topology of $C\widetilde{\otimes}C/Im(\Delta)$, which is universal for continuous coderivations.

If C is a cocommutative topological coalgebra, the representations we shall take into account are complete topological C-bicomodules M which are C-cosymmetric (i.e. $\sigma_{12} \circ \rho_M^+ = \rho_M^-$, where σ denotes the permutation). In this context, a continuous map $f : M \to C$ is called a **coderivation** if

$$\Delta \circ f = (id \otimes f) \circ \rho_M^- + (f \otimes id) \circ \rho_M^+ = (1 + \sigma_{12}) \circ (id \otimes f) \circ \rho_M^-$$

where σ_{12} denotes the transposition of the first and second place. Let us take the subcomodule of L_C consisting of the cosymmetric elements and denote it Ω^1_C, i.e.

$$\Omega^1_C := Sym(C\tilde{\otimes}C/Im(\Delta))$$

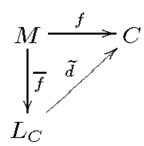
Proposition 1.1 *Ω^1_C is a universal object for coderivations in the category of cosymmetric bicomodules.*

Proof: Ω^1_C is by definition a cosymmetric C-bicomodule, we must exhibit a coderivation $d : \Omega^1_C \to C$. It is defined by

$$d([z]) := (\epsilon \otimes id - id \otimes \epsilon)(z) \qquad (z \in C\tilde{\otimes}C)$$

Observe that d is well defined on L_C because if $z \in Im(\Delta)$ then $(\epsilon\otimes id - id\otimes\epsilon)(z) = 0$.

It is easy to see that d is a coderivation. Given a cosymmetric C-bicomodule M and a coderivation $f : M \to C$, by the universal property of L_C ([6]), that remains true in the topological context, we have a commutative diagram



where $\overline{f}$ is a C-C-bicolinear map, defined by

$$\overline{f}(m) = (id \otimes f)\rho^-(m) = m_{-1} \otimes f(m_0)$$

and $\tilde{d}$ is defined by the same formula as d.

Since M is cosymmetric, $Im(\overline{f})$ is contained in Ω^1_C, so

$$Coder(M, C) = Com_{C^e}(M, L_C) =$$

$$= Com_{C^e}(M, \Omega^1_C) = Com_C(M, \Omega^1_C)$$

where $C^e := C\tilde{\otimes}C^{op}$, and the category of C-bicomodules is identified with the category of C^e-modules.

We shall call Ω^1_C the comodule of Kähler differentials of C.

We recall from [6, 8] that, in the algebraic case, given a k-coalgebra C there exist two cohomology theories associated to it, denoted $Hoch^*(C)$ and $H^*(C)$. The first is obtained as the derived functor of $-\square_{C^e}C$, and the other one is the derived functor of $Com_{C^e}(-, C)$. In the topological case, they are defined as the cohomology groups of the complexes corresponding to the canonical resolution of C as C^e-comodule, for example $Hoch^*(C)$ is the cohomology of the complex

$$0 \longrightarrow C \xrightarrow{b_0} C\widetilde{\otimes}C \xrightarrow{b_1} C\widetilde{\otimes}C\widetilde{\otimes}C \longrightarrow \cdots$$

$$b_0 = \Delta - \sigma_{12}\Delta$$

$$b_1 = \Delta \otimes id - id \otimes \Delta + \sigma_{132}(\Delta \otimes id)$$

$$b_n = \sum_{i=1^n} (-1)^{i-1}\Delta_i + (-1)^n \sigma_{1,n,n-1,\ldots,3,2}\Delta_1$$

where $\Delta_i = id_{C^{\otimes i-1}} \otimes \Delta \otimes id_{C^{\otimes n-i-1}}$ and σ_{132}, $\sigma_{1,n,n-1,\ldots,3,2}$ denote the cyclic permutations (132) and $(1, n, n-1, ..., 3, 2)$ respectively.

However, Taylor ([19], section 4) has shown that in certain cases (for example for nuclear and Fréchet algebras), the topological version of Hochschild homology behaves similarly to algebraic Hochschild homology. The same holds for coalgebras, namely:

Proposition 1.2 *Given a coalgebra C and an injective resolution of C as C^e-comodule $0 \to C \to X_0 \to X_1 \to X_2 \to ...$, if*

1. *C and X_i ($i \in \mathbb{N}_0$) are nuclear Fréchet spaces and M is Fréchet; or*
2. *C and X_i ($i \in \mathbb{N}_0$) are nuclear complete DF spaces and M is a complete DF space,*

the cohomology of the complex $X_*\square_{C^e}M$ is isomorphic to $Hoch^*(M, C)$ (here $\square_{C^e}$ is defined similarly to the purely algebraic case, using $\widetilde{\otimes}$ instead of $\otimes$).

We shall need the following Lemma, which is completely analogous to Proposition 2.7 of [19]:

Lemma 1.3 *Let $\mathcal{M}$ be a class of (topological) C-bicomodules containing $C\widetilde{\otimes}M\widetilde{\otimes}C$ and the cokernel of $\rho_M : M \to C\widetilde{\otimes}M\widetilde{\otimes}C$ whenever $M \in \mathcal{M}$. Let K_p be a sequence of covariant functors from $\mathcal{M}$ into vector spaces such that*

1. *$K_0(M) = Hoch^0(M, C)$ for all $M \in \mathcal{M}$.*

2. *$K_p(C\tilde{\otimes}M\tilde{\otimes}C)=0$ for all $p>0$ and $M\in\mathcal{M}$.*

3. *Each short exact sequence $0\to M_1\to M_2\to M_3\to 0$ ($M_i\in\mathcal{M}$) induces maps $\delta_p: K_p(M_3)\to K_{p+1}(M_1)$ such that*

$$..\to K_p(M_3)\to^{\delta} K_{p+1}(M_1)\to K_{p+1}(M_2)\to K_{p+1}(M_3)\to ...$$

is exact.

Then $K_p(M)\cong Hoch^p(M,C)$ for all $M\in\mathcal{M}$.

Proof of the proposition: We shall see that the conditions of the Lemma 1.3 are fulfilled. Take $\mathcal{M}$ as the collection of Fréchet C-bicomodules and, with the notations of the above Lemma, $K_p(M):=H_p(X_*\square_{C^e}M)$ where $0\to C\to X_*$ is a relatively injective $\mathbb{C}$-split resolution of C as C^e-comodule and each X_i is a nuclear Fréchet relative injective C^e-comodule (for example, take the standard resolution).

If $M\in\mathcal{M}$, as C is Fréchet and nuclear, then $C\tilde{\otimes}M\tilde{\otimes}C\in\mathcal{M}$ as well (see for example [12]). Concerning $Coker(M\to C\tilde{\otimes}M\tilde{\otimes}C)$, it is a quotient of a Fréchet space by a closed subspace, then it is Fréchet.

Also if $Y\in\mathcal{M}$,

$$0\to C\tilde{\otimes}Y\to X_0\tilde{\otimes}Y\to X_1\tilde{\otimes}Y\to ...$$

is exact, as C and the X_i's are nuclear Fréchet spaces, so that $K_p(C\tilde{\otimes}M\tilde{\otimes}C)=0$ (this follows from $X\square_{C^e}(C\tilde{\otimes}M\tilde{\otimes}C)\cong X\tilde{\otimes}M$).

It follows by diagram chasing that $Hoch^0(M)=K_0(M)$. Finally, as X_i is a relative injective C-bicomodule, it is a direct sumand of $C^e\tilde{\otimes}V_i$ for some topological vector space V_i, and so given a $\mathbb{C}$-split exact sequence $0\to M_1\to M_2\to M_3\to 0$ then $0\to X_i\square_{C^e}M_1\to X_i\square_{C^e}M_2\to X_i\square_{C^e}M_3\to 0$ is a short exact sequence. The desired long exact sequence is the well-known long exact sequence on cohomology associated to the last short exact sequence of complexes.

Next we want to show that Ω^1_C is isomorphic to $Hoch^1(C)$, turning then the latter into an alternative universal object for coderivations when the coalgebra C is not cocommutative.

Proposition 1.4 *If C is a cocommutative coalgebra, then $Hoch^1(C)\cong\Omega^1_C$.*

Proof: We first define $\gamma:\Omega^1_C\to Hoch^1(C)$ by

$$\gamma([z]):=z-(\Delta\otimes\epsilon)(z)$$

where $[z] \in \Omega_C^1$ is the class of some $z \in C\widetilde{\otimes}C$. The map γ is well defined because if $z = \Delta(c)$, then $\Delta(c) - (\Delta \otimes \epsilon)(\Delta(c)) = \Delta(c) - \Delta(c) = 0$. On the other hand, we can see that $Im(\gamma) \subseteq Hoch^1(C) = Ker(b_1)$ as follows:

Let $[z] \in \Omega_C^1$, then

$$0 = \rho^-([z]) - \sigma_{312}\rho^+([z]) = (id \otimes \pi)(\Delta \otimes id)(z) - (id \otimes \pi)\sigma_{312}(id \otimes \Delta)(z)$$

where $\pi : C \otimes C \to L_C$ is the canonical projection, so we see that $(\Delta \otimes id)(z) - \sigma_{312}(id \otimes \Delta)(z) \in Im(id \otimes \Delta)$, and

$$\begin{aligned}(\Delta \otimes id)(z) - \sigma_{312}(id \otimes \Delta)(z) = & \ (id \otimes \Delta)(id \otimes id \otimes \epsilon)(\Delta \otimes id)(z) - \sigma_{312}(id \otimes \Delta)(z) \\ = & \ (id \otimes \Delta)(\Delta \otimes \epsilon)(z) - (id \otimes \Delta)\sigma_{12}(z)\end{aligned}$$

If we compute $b_1(z - (\Delta \otimes \epsilon)(z))$ we obtain

$$(\Delta \otimes id)(z) - (id \otimes \Delta)(z) + \sigma_{132}(\Delta \otimes id)(z) - (\Delta \otimes id)(\Delta \otimes \epsilon)(z)$$

which is zero, from the above equation via the permutation $a \otimes b \otimes c \mapsto b \otimes c \otimes a$.

The inverse of γ is given by:

$$\gamma' : Hoch^1(C) \to \Omega_C^1$$

$$z \mapsto [z]$$

checking that $Im(\gamma') \in \Omega_C^1$, we immediately have that they are inverses.

Example: Consider an algebraically closed field k of characteristic 0 and $A = k[x]$, which is a Hopf algebra with comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$ and antipode $x \mapsto -x$. Although A^* is not a bialgebra, there is an object denoted A^0 which is the biggest subset of A^* such that it is a Hopf algebra with the dual structure of $k[x]$ [16]. So $A^0 \subseteq k[x]^* = k[|x|]$. Defining a morphism, which is an isomorphism,

$$k[|x|] \to k[|s|]$$

$$\sum a_n x^n \mapsto \sum a_n n! s^n$$

one gets the identification $A^0 = k[s, e^{\lambda s}]_{\lambda \in k} = \oplus_{\lambda \in k} k[s]e^{\lambda s} \subset k[|s|]$, where $\Delta(s) = s \otimes 1 + 1 \otimes s$ and the algebra structure is the usual one in $k[|s|]$ viewed as formal power series (the exponentials are forced to be group-like).

Identifying $k[x] \otimes k[x] \cong k[x, y]$ in the standard way, then $I = Ker(m : k[x] \otimes k[x] \to k[x])$ is identified to $< x - y >$, so $I/I^2 = k[x].(x - y)$.

By definition, $\Omega^1_{A^\circ} = Sym(A^0 \otimes A^0/Im(\Delta)) \subseteq \frac{k[s,e^{\lambda s}]_{\lambda\in k}\otimes k[t,e^{\lambda t}]_{\lambda\in k}}{Im(\Delta)}$. We show that $\Omega^1_{A^\circ} \cong k[s, e^{\lambda s}]_{\lambda\in k}$ defining $\widetilde{f} : k[s,t,e^{\lambda s}, e^{\mu t}]_{\lambda,\mu\in k} \to k[s,e^{\lambda s}]_{\lambda \in k}$ by

$$\begin{cases} \widetilde{f}(1) = 0 & \\ \widetilde{f}(s) = 1 & \\ \widetilde{f}(t) = -1 & \\ \widetilde{f}(s^n) = ns^{n-1} & \text{if } n > 0 \\ \widetilde{f}(s^n t) = -s^n & \text{if } n > 0 \\ \widetilde{f}(s^n t^m) = 0 & \text{if } n \geq 0,\ m > 1 \\ \widetilde{f}(e^{\lambda s}) = \lambda e^{\lambda s} & \\ \widetilde{f}(e^{\mu . t}) = -1 & \end{cases}$$

$\widetilde{f}$ derives elements depending on s, and so $Im(\widetilde{f})$ contains all polynomials and exponentials in s. Also, as $(s+t)^n = \sum_{k=0}^n \binom{n}{k} s^{n-k}t^k$, $\widetilde{f}((s+t)^n) = ns^{n-1} - ns^{n-1} = 0$, so $\widetilde{f}$ vanishes on $Im(\Delta)$ and we have a well defined map $f : \frac{k[s,t,e^{\lambda s},e^{\mu t}]_{\lambda,\mu\in k}}{Im(\Delta)} \to k[s,e^{\lambda s}]_{\lambda\in k}$.

Restricting f to the cosymmetric elements $s(s+t)^n$ with $n \geq 0$, we have $f(s(s+t)^n) = f(\left(\sum_{k=0}^n \binom{n}{k} s^{n+1-k}t^k\right) = f(s^{n+1} + ns^n t + o(t^2)) = (n+1)s^n - ns^n = s^n$. Then, as $f(s.e^{\lambda(s+t)}) = e^{\lambda s}$, polynomials divisible by s and exponentials belong to $Im(f)$ and we obtain the result describing $\Omega^1_{A^\circ}$.

In this case, $\Omega^1_{A^\circ} = \left(\Omega^1(A|k)\right)^0$, where on the right side, $(-)^0$ means the A-module which has the universal property:

$$Hom_A(\Omega^1(A|k), X^*) \cong Com_{A^\circ}(X, (\Omega^1(A|k))^0) \quad (X \text{ a } C\text{-comodule})$$

2 Behaviour of Ω^1_C with respect to localization

Given a complete topological cocommutative coalgebra C (we consider in particular the discrete case) and a multiplicative subset $S \subset C'$, the localization $C_{[S]}$ is another topological coalgebra provided of a morphism $\pi : C_{[S]} \to C$, universal for coalgebra morphisms $f : D \to C$ from topological cocommutative coalgebras, such that the elements of S define, by the C'-action, invertible endomorphisms in D, or, equivalently $s \circ f$ is invertible in the algebra D' for all $s \in S$.

The construction of $C_{[S]}$ and of localization of comodules $M_{[S]}$ has been carried out in [18] for linear topologies and in [9] for the locally convex case.

Given C and S as above, we want to establish the relation between the bicomodules $\Omega^1_{C_{[S]}}$ and $\left(\Omega^1_C\right)_{[S]}$. In fact, we shall prove that they are isomorphic, at least when C has a topology with the same or less open sets as the topology induced from C'' via the canonical evaluation map (for example when C is reflexive, or when C is the dual of some other space).

We shall begin by proving that a coderivation $D : M \to C$ from a cosymmetric C-bicomodule M, induces a coderivation $D_{[S]} : M_{[S]} \to C_{[S]}$ such that the following diagram is commutative:

$$\begin{array}{ccc} M_{[S]} & \xrightarrow{D_{[S]}} & C_{[S]} \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_C} \\ M & \xrightarrow{D} & C \end{array}$$

In order to do so, we first define a coderivation $M_{[S]} \to M''_{[S]} \to C''_{[S]}$ which is analogous to the Leibniz rule "$d(\frac{f}{g}) = \frac{1}{g}df - \frac{f}{g^2}dg$". More precisely:

$$M_{[S]} \to M''_{[S]} \to C''_{[S]}$$

$$\{m_s\}_{s\in S} \mapsto \{c_t\}_{t\in S}$$

where

$$c_t(--) = m_t(--\circ D) - m_{t^2}(--.(t\circ D))$$

The family $\{c_t\}_{t\in S} \in \prod_{t\in S} C''$. In order to see that it is an element of $C''_{[S]}$ we have to verify that $r.c_{tr} = c_t,\ \forall r, t \in S$.

$$\begin{aligned} r.c_{tr}(f) = \;& c_{tr}(r.f) = m_{tr}(rf\circ D) - m_{t^2r^2}(rf.(tr\circ D)) \\ =\;& m_{tr}(r.(f\circ D)) + m_{tr}(f.(r\circ D)) - m_{t^2r}(f.(tr\circ D)) \\ =\;& m_t(f\circ D) + m_{tr}(f.(r\circ D)) - m_{t^2r}(f.t.(r\circ D)) - m_{t^2r}(f.r.(t\circ D)) \\ =\;& m_t(f\circ D) - m_{t^2}(f.(t\circ D)) \\ =\;& c_t(f) \end{aligned}$$

Here we have used the formula

$$(h.g)\circ D = h.(g\circ D) + g.(h\circ D) \quad (h,\ g\in C')$$

Notice that M' is a C'-module by means of the structure map of M, namely $M'\otimes C' \to (M\otimes C)' \to^{\rho^*} M'$ and $h\circ D$ is an element of M'. This formula is a straightforward consequence of the definition of coderivation.

We have to show now that $Im(D_{[S]}|_{M_{[S]}}) \subseteq C_{[S]}$. To see that $\phi \in C''$ is in C, we must find an element $c_\phi \in C$ such that $\phi(f) = f(c_\phi)$ for all $f \in C'$. In our case,

$$c_t(f) = m_t(f \circ D) - m_{t^2}(f.(t \circ D)) =$$

$$= (f \circ D)(m_t) - (f.(t \circ D))(m_{t^2}) = f(D(m_t)) - f((1 \otimes (t \circ D))\rho(m_{t^2}))$$

as a consequence $c_t = D(m_t) - (1 \otimes (t \circ D))\rho(m_{t^2}) \in C$. The map $\{m_s\}_{s \in S} \mapsto \{c_t\}_{t \in S}$ that we have finally found is continuous because we are assuming that the topology of C is induced by the inclusion $C \to C''$.

The universal coderivation $d : \Omega^1_C \to C$ thus gives a coderivation $d_{[S]} : (\Omega^1_C)_{[S]} \to C_{[S]}$, and by the universal property of $\Omega^1_{C_{[S]}}$ a $C_{[S]}$-bicomodule map $(\Omega^1_C)_{[S]} \to \Omega^1_{C_{[S]}}$.

On the other hand, $\pi_C : C_{[S]} \to C$ induces a C-bicomodule map $\Omega^1_{C_{[S]}} \to \Omega^1_C$ and therefore a $C_{[S]}$-bicomodule map $\Omega^1_{C_{[S]}} = (\Omega^1_{C_{[S]}})_{[S]} \to (\Omega^1_C)_{[S]}$. Next we shall show that these maps are inverse to each other.

Proposition 2.1 *$\Omega^1_{C_{[S]}}$ is isomorphic to $(\Omega^1_C)_{[S]}$.*

Proof: We denote by ϕ and ψ the maps:

$$\phi : (\Omega_C)_{[S]} \to \Omega_{C_{[S]}}$$

$$\{m_t\}_{t \in S} \mapsto \overline{(id \otimes d_{[S]}) \circ \rho^-(\{m_t\}_{t \in S})}$$

where $d : \Omega_C \to C$ is the universal coderivation $d = \epsilon \otimes Id_C - Id_C \otimes \epsilon$, and

$$\psi : \Omega_{C_{[S]}} \to (\Omega_C)_{[S]}$$

$$\sum \overline{\{x_t\}_{t \in S} \otimes \{y_s\}_{s \in S}} \mapsto \left\{ \frac{1}{r} . \sum \overline{x_1 \otimes y_1} \right\}_{r \in S}$$

both extended by continuity.

Let us see that $\phi \circ \psi = id_{\Omega_{C_{[S]}}}$: it is enough to see, by the universal property of $\Omega^1_{C_{[S]}}$, that

$$d_{C_{[S]}} \circ \phi \circ \psi = d_{C_{[S]}}$$

Again it is sufficient to see that

$$\pi_C \circ d_{C_{[S]}} \circ \phi \circ \psi = \pi_C \circ d_{C_{[S]}}$$

and this is true because of the following four equalities that we will prove in turn:

$$\pi \circ d_{C_{[S]}} = d_C \circ \widetilde{\pi} \tag{1}$$

where $\widetilde{\pi}$ is the morphism $\Omega_{C_{[S]}} \to \Omega_C$ induced by the coalgebra morphism π_C : $C_{[S]} \to C$.

$$\widetilde{\pi} \circ \phi = \pi_{\Omega_C} \tag{2}$$

$$\pi_{\Omega_C} \circ \psi = \widetilde{\pi} \tag{3}$$

$$d_C \circ \widetilde{\pi} = \pi_C \circ d_{C_{[S]}} \tag{4}$$

(1):

$$\pi \circ d_{C_{[S]}}(\{m_t\}_{t\in S}) = d(m_1) - (1 \otimes (\epsilon \circ d)) \circ \rho(m_1) = d(m_1)$$

because $\epsilon \circ d = 0$

(2):

$$\pi_{\Omega_C}(\{\overline{m_t}\}_{t\in S}) = \overline{\pi \otimes \pi(\{m_t\}_{t\in S})}$$

$$\widetilde{\pi} \circ \phi(\{\overline{m_t}\}_{t\in S}) = \overline{(\pi \otimes \pi) \circ (id \otimes d_{[S]}) \circ \rho_{(\Omega_C)_{[S]}}(\{m_t\}_{t\in S})} =$$

$$= \overline{(\pi \otimes \pi \circ d_{[S]}) \circ \rho_{(\Omega_C)_{[S]}}(\{m_t\}_{t\in S})} = \overline{(id \otimes d) \circ (\pi \otimes \pi_{\Omega_C}) \circ \rho_{(\Omega_C)_{[S]}}(\{m_t\}_{t\in S})}$$

We have $(\pi \otimes \pi_{\Omega_C}) \circ \rho_{(\Omega_C)_{[S]}}(\{m_t\}_{t\in S}) = \rho_{\Omega_C} \circ \pi_{\Omega_C}(\{m_t\}_{t\in S})$.

Computing $\overline{(id \otimes d) \circ \rho_{\Omega_C} \circ \pi_{\Omega_C}(\{m_t\}_{t\in S})} =$

$$= \overline{(id \otimes d) \circ \rho_{\Omega_C}(m_1)} =$$

$$= \overline{(id \otimes (\epsilon \otimes 1) - (1 \otimes \epsilon)) \circ \rho_{\Omega_C}(m_1)} =$$

$$= \overline{(id \otimes (\epsilon \otimes 1)) \circ \rho_{\Omega_C}(m_1)} - \overline{((id \otimes (1 \otimes \epsilon)) \circ \rho_{\Omega_C}(m_1)}$$

As $(\epsilon \otimes 1) \circ \rho = id$, we have to see that the other term equals zero, or equivalently that $(id \otimes (1 \otimes \epsilon)) \circ \rho_{\Omega_C}(m_1) \in Im(\Delta_C)$. If $m = x \otimes y$ this is true because $id \otimes (1 \otimes \epsilon)) \circ \rho(x \otimes y) = \Delta(x)\epsilon(y)$; it is also true when m is a linear combination of elementary tensors, and finally it is also true when m belongs to the closure of the algebraic span of elementary tensors, because of continuity and of the fact that $Im(\Delta)$ is closed.

(3):

$$\begin{aligned}\pi_{\Omega_C}\psi\left(\sum\overline{\{x_t\}_{t\in S}\otimes\{y_s\}_{s\in S}}\right) &= \pi_{\Omega_C}\left(\left\{\tfrac{1}{r}.(\sum\overline{x_1\otimes y_1})\right\}_{r\in S}\right)\\ &= \pi_{\Omega_C}\left(\{\sum\overline{x_r\otimes y_1}\}_{r\in S}\right)\\ &= \sum\overline{x_1\otimes y_1} = \widetilde{\pi}\left(\sum\overline{\{x_t\}_{t\in S}\otimes\{y_s\}_{s\in S}}\right)\end{aligned}$$

(4):

$$d\left(\widetilde{\pi}\left(\sum\overline{\{x_t\}_{t\in S}\otimes\{y_s\}_{s\in S}}\right)\right) = d\left(\sum\overline{x_1\otimes y_1}\right) = \sum\epsilon(x_1)y_1-\epsilon(y_1)x_1$$

On the other hand

$$\pi d_{C_{[S]}}\left(\sum\overline{\{x_t\}_{t\in S}\otimes\{y_s\}_{s\in S}}\right) = \pi\left(\sum\overline{\epsilon(\{x_t\}_{t\in S})\{y_s\}_{s\in S}-\{x_t\}_{t\in S}\epsilon(\{y_s\}_{s\in S})}\right) =$$

$$= \pi\left(\sum\epsilon(x_1)\{y_s\}_{s\in S}-\{x_t\}_{t\in S}\epsilon(y_1)\right) = \sum\epsilon(x_1)y_1-\epsilon(y_1)x_1$$

The proof that the composition in the other sense is the identity uses the same equalities:

$$\psi\circ\phi = id_{(\Omega_C)_{[S]}} \Leftrightarrow \pi_{\Omega_C}\circ\psi\circ\phi = \pi_{\Omega_C}$$

By (3): $\pi_{\Omega_C}\circ\psi = \widetilde{\pi}$ and using (2) $\widetilde{\pi}\circ\phi = \pi_{\Omega_C}$, then

$$(\pi_{\Omega_C}\circ\psi)\circ\phi = \widetilde{\pi}\circ\phi = \pi_{\Omega_C}$$

Example: ($k[x]^0$ revisited) We have already described the Hopf algebra $k[x]^0$, in fact, $k[x]^0$ is the topological dual of $A = k[x]$ with respect to the following linear topology:

Given $\lambda\in k$ and $n\in\mathbb{N}$, take the k-vector space $V_{n,\lambda} = \langle(x-\lambda).x^m,\ m\geq n\rangle$ and $\{V_{n,\lambda}\}_{(n,\lambda)\in\mathbb{N}\times k}$ as a basis of neighbourhoods of 0. Then the elements of the continuous dual of A are sequences $\{a_n\}_{n\in\mathbb{N}_0}$ in $k[|x|]$ such that there exists $\lambda\in k$ and $n\in\mathbb{N}_0$ with $a_{m+1} = \lambda a_m$, $\forall m\geq n$, i.e. they differ modulo a polynomial from the series representing $\frac{1}{1-\lambda x}$, and these series are exactly those ones giving exponentials when we apply the isomorphism $k[|x|]\cong k[|s|]$ of the preceding paragraphs. As $\overline{k} = k$, $\{(x-\lambda)\}_{\lambda\in k}$ is nothing but the set of irreducible polynomials in

$k[x]$, which correspond to semisimple (finite dimensional) representations of $k[x]$, the collection of them allowing the computation of $k[x]^0$. If k is not algebraically closed, one replaces the family of polynomials $\{(x-\lambda)\}_{\lambda\in k}$ by the family of irreducible polynomials of $k[x]$.

Consider $\{V\subset A:\ \exists I\subset V,\ I$ a finite codimensional ideal $\}$, then $A^0=A'$ (the topological dual with respect to the topology defined by this family of subspaces). Since

$$Hom_{cont}(V,W')\cong Hom_{cont}(W,V')$$

$$f\mapsto f'\circ i_W$$

we have that $\underset{\leftarrow_i}{lim}\ V_i'=\left(\underset{\rightarrow_i}{lim}\ V_i\right)'$, so $A^0_{[S]}=(A_S)'$. However, the topology in A_S in order to compute this dual is the direct limit topology, which agrees with the final topology of the canonical map $A\to A_S$, but it need not be the same topology giving $(A_S)^0$.

Nevertheless, if $A=k[x]$ and $S=\{1,x,x^2,x^3,...\}$, $C=k[x]^0=k[s,e^{\lambda s}]_{\lambda\in k}=\oplus_{\lambda\in k}k[s]e^{\lambda s}$, we obtain by a direct computation that $C_{[S]}=k[s,e^{\lambda s}]_{\lambda\in k-0}=\oplus_{\lambda\in k-0}k[s]e^{\lambda s}$, as the element $x\in S$ induces the derivation operator in C, which is an isomorphism when restricted to the components corresponding to exponentials $e^{\lambda s}$ with $\lambda\neq 0$. The canonical map $C_{[S]}\to C$ is the inclusion $k[s,e^{\lambda s}]_{\lambda\in k-0}\to k[s,e^{\lambda s}]_{\lambda\in k}$. Given another coalgebra D and a map $f:D\to C$ such that $.x$ is an isomorphism in D, then polynomials cannot belong to $Im(f)$, and then f factorizes through $C_{[S]}$. On the other hand, $A_S=k[x,x^{-1}]$, and $A^0_S=k[s,e^{\lambda s}]_{\lambda\in k-0}$.

Proposition 2.2 *Given a topological algebra A, $C=A'$ and S a multiplicative subset of $Z(A)$, then*

1. $C_{[S]}=(A_S)'$.
2. $\Omega^1_C=(\Omega^1(A))'$.
3. $(\Omega^1_C)_{[S]}=(\Omega^1(A)')_{[S]}=(\Omega^1(A)_S)'=(\Omega^1(A_S))'=\Omega^1_{C_{[S]}}$.

Proof:

1. This is proved in [18] for linear topologies and in [9] for the locally convex case.

In order to see 2., let us be given a C-comodule M:

$$Com_C(M,\Omega^1_C)=Coder(M,C)=Coder(M,A')=$$

$$= Der(A, M') = Hom_A(\Omega^1(A), M') = Com_C(M, \Omega^1(A)')$$

Finally, one of the equalities of 3. states that $\Omega^1(A)$ localizes for a topological algebra A. The proof of this fact is similar to the coalgebra case, but no assumptions concerning the topology of A are needed, simply define maps $\frac{adb}{s} \mapsto \frac{a}{s}d(\frac{b}{1})$ and $\frac{a}{s}d(\frac{b}{t}) \mapsto \frac{a}{s}(\frac{1}{t}db - \frac{b}{t^2}dt)$. The other equalities have been already proved.

The category of topological C-bicomodules is not an abelian category, so we cannot use the cohomological machinery to prove that $Hoch^*(C)$ localizes.

Despite, we are able to construct long exact sequences for $Hoch^*$, but as we shall see, this argument cannot be used because we don't know if localization is exact and we don't even have a general argument to assure that localization of Fréchet (or DF-spaces) are Fréchet (resp. DF-spaces) except in particular situations.

We shall next make explicit the isomorphisms between $Hoch^1(C_{[S]})$ and $Hoch^1(C)_{[S]}$.

We first define

$$j : Hoch^1(C_{[S]}) \to Hoch^1(C)_{[S]}$$

$$\{x_s \otimes y_t\}_{s,t \in S} \mapsto \{x_s \otimes y_1\}_{s \in S}$$

and

$$\mu : Hoch^1(C)_{[S]} \to Hoch^1(C_{[S]})$$

$$\{x_s \otimes y\}_{s \in S} \mapsto \{x_{rs} \otimes y + \Delta(x_{rs^2}s(y))\}_{r,s \in S}$$

We will see through the computations, that μ is well-defined. The composition $j \circ \mu$ gives:

$$(j \circ \mu)(\{x_s \otimes y\}_{s \in S}) = j(\{x_{rs} \otimes y + \Delta(x_{rs^2})s(y)\}_{r,s \in S}) = \{x_r \otimes y + \Delta(x_r)\epsilon(y)\}_{r \in S}$$

Since $x_r \otimes y_r \in Hoch^1(C)$,

$$x_{r,1} \otimes x_{r,2} \otimes y - x_r \otimes y_1 \otimes y_2 + x_{r,2} \otimes y \otimes x_{r.1} = 0$$

so $\Delta(x^r)\epsilon(y) = 0$, implying $j \circ \mu = id_{Hoch^1(C)_{[S]}}$.

The composition $\mu \circ j$ gives:

$$(\mu \circ j)(\{x_s \otimes y_t\}_{s,t \in S}) = \mu(\{x_s \otimes y_1\}_{s \in S}) = \{x_{rs} \otimes y_1 + \Delta(x_{rs^2})s(y_1)\}_{r,s \in S}$$

In order to see that $\{x_s \otimes y_t\}_{s,t\in S}$ coincides with the last expression, we embed $C_{[S]}$ in $C''_{[S]} = ((C')_S)'$ and evaluate both expressions on elements of $A_S = C'_S$. Then:

$$\{x_{rs} \otimes y_1 + \Delta(x_{rs^2})s(y_1)\}_{r,s\in S}(\frac{a}{t} \otimes \frac{b}{u}) = x_{tu}(a)y_1(b) + x_{tu^2}(ab)s(y_1)$$

As $\frac{a}{t} \otimes \frac{b}{u} - \frac{a}{tu} \otimes \frac{b}{1} + \frac{ab}{tu^2} \otimes \frac{u}{1} = 0$ in $HH_1(A_S)$, then:

$$\{x_s \otimes y_t\}_{s,t\in S}(\frac{a}{t} \otimes \frac{b}{u}) = \{x_s \otimes y_t\}_{s,t\in S}(\frac{a}{tu} \otimes \frac{b}{1} - \frac{ab}{tu^2} \otimes \frac{u}{1}) =$$

$$= x_{tu}(a)y_1(b) + x_{tu^2}(ab)s(y_1)$$

as we wanted to show.

In particular $Im(\mu) \subseteq Hoch^1(C_{[S]})$ and then it follows that μ is well defined.

We remark that we have used that the map $C \to C''$ is injective, which is true if and only if C is a Hausdorff space. We have also used that $Hoch^1(C)$ is Hausdorff, but this is true when C is such a space, as $Hoch^1(C)$ is a Kernel in $C\tilde{\otimes}C$ and so a (closed) subspace of a Hausdorff space. In higher degrees, $Hoch^n(C)$ need not be Hausdorff. This is the obstruction not allowing the use of the long exact sequence argument.

3 The exterior coalgebra on Ω^1_C and higher degrees of $Hoch^*$

In this section we construct the "exterior coalgebra" on Ω^1_C, considering the particular case of C being the dual of a topological algebra A. Then we analize in detail the example $C = \mathcal{D}(X)$ (distributions on a compact smooth manifold), whose the n-th component of the exterior coalgebra turns out to be isomorphic to $Hoch^n(C)$, which, as a consequence, localizes in a more general situation that the case studied in [9]. Moreover, the isomorphism between $Hoch^n(C)$ and Ω^n_C in this case suggested us to compare both objects for any $n \in \mathbb{N}$, in following sections for arbitrary coalgebras.

Le C be a topological coalgebra and M a topological bicomodule. The construction of the tensor coalgebra $T_k M$ of M over k in a purely algebraic context is carried out in detail in [16]. Let us define $T_C M$ as the space $\prod_{n\in\mathbb{N}_0} M^{\square_C n}$, where

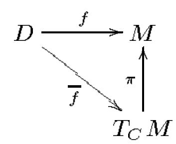
$M^{\square_C 0} = C$, $M^{\square_C 1} = M$ and $M^{\square_C (n+1)} = M \square_C (M^{\square_C n})$. The coproduct in $T_C M$ is obtained as the transpose of the product in the tensor algebra, namely

$$\Delta(m_1, ..., m_k) := \sum_{i=0}^{k} (m_1, ..., m_i) \otimes (m_{i+1}, ..., m_k)$$

where $m_0 \otimes (m_1, ..., m_k)$ stands for $\rho_M^- \otimes id_{M^{\otimes k-1}}(m_1, ..., m_k)$ and similarly with $i = k$ and ρ_M^+. Notice that the coalgebra map is defined like in the "algebraic" $T_k M$ over the elementary tensors, extended by linearity and next by continuity to the completion, and finally restricted to $T_C M$ which is a subobject of the "topological" $T_k M$.

The coalgebra $T_C M$ has the following universal property:

Given a coalgebra D over C (i.e., a coalgebra together with a coalgebra map $D \to C$) and a C-bicomodule map $f : D \to M$, then there exists a unique C-coalgebra map $\overline{f} : D \to T_C M$ such that the following diagram commutes



Let C be cocommutative. If M is a $\mathbb{Z}_2$-graded comodule, then so is $T_C M$. For M a cosymmetric C-bicomodule, we consider it as graded with $deg(m) = 1$ $\forall m \in M$, and let $\Lambda_C M$ be the **biggest graded-cocommutative subcoalgebra** of $T_C M$. It is characterized by a similar universal property with respect to graded-cocommutative coalgebras which are (graded)cosymmetric as C-bicomodules (C is considered as graded by $deg(c) = 0$, $\forall c \in C$).

Remark: If $C = A^0$, then $\Lambda_C \Omega_C^1 \cong (\Lambda_A \Omega^1(A))^0 = \Omega^*(A)^0$, as C-comodules. On the other hand, $\bigoplus_{n \in \mathbb{N}_0} Hoch^n(C)$ can be provided of a structure of graded coalgebra as follows:

Let $(\mathcal{C}^*(C), b)$ be the standard complex whose homology computes $Hoch^*(C)$; we begin by defining the comultiplication $\Delta : \mathcal{C}^*(C) \to \mathcal{C}^*(C) \tilde{\otimes} \mathcal{C}^*(C)$ in low degrees by:

For $c \in C = \mathcal{C}^0(C)$, $\Delta(c) = \Delta_C(c) \in C \tilde{\otimes} C = \mathcal{C}^0(C) \tilde{\otimes} \mathcal{C}^0(C)$.

For $x, y \in C$, $\Delta(x \otimes y) = (\Delta(x) \otimes y, \sigma_{132} \Delta(x) \otimes y) \in (\mathcal{C}^0(C) \tilde{\otimes} \mathcal{C}^1(C)) \oplus (\mathcal{C}^1(C) \tilde{\otimes} \mathcal{C}^0(C))$.

For $x, y, z \in C$, $\Delta(x \otimes y \otimes z)=$

$$((\Delta\otimes 1\otimes 1)(x\otimes y\otimes z), \sigma_{1432}(\Delta\otimes 1\otimes 1)(x\otimes y\otimes z), \sigma_{23}(\Delta\otimes 1\otimes 1)(x\otimes y\otimes z)-\sigma_{23}(\Delta\otimes 1\otimes 1)(x\otimes z\otimes y))$$

considered as an element of $(\mathcal{C}^0(C)\widetilde{\otimes}\mathcal{C}^2(C)) \oplus (\mathcal{C}^2(C)\widetilde{\otimes}\mathcal{C}^0(C)) \oplus (\mathcal{C}^1(C)\widetilde{\otimes}\mathcal{C}^1(C))$. This definition may be generalized to arbitrary degrees as follows:

The component in $\mathcal{C}^i(C)\widetilde{\otimes}\mathcal{C}^{n-i}(C)$ of $\Delta(x\otimes x_1\otimes ... x_n)$ is defined in three steps; first make the sum $\sum_{\sigma\in S_{i,n-i}} sgn(\sigma)x \otimes x_{\sigma(1)} \otimes ... \otimes x_{\sigma(n)} \in C^{\widetilde{\otimes} n+1}$ where $S_{i,n-i}$ denotes the $(i, n-i)$-shuffles of S_n, then comultiplicate the first element x and finally carry the second coordinate up to the the $(i+1)$-th place, i.e. compose with $\sigma_{(2,i+2,i+i,i,i-1,\ldots,4,3)}$.

As $(b\otimes 1 + 1\otimes b)\Delta = \Delta b$, we obtain a differential graded coalgebra and hence a coalgebra structure in cohomology, which turns out to be graded-cocommutative because $\mathcal{C}^*(C)$ is graded-cocommutative.

As a consequence of the graded-cocommutative coalgebra structure of $Hoch^*(C)$, we have that the map $Hoch^*(C) \to Hoch^1(C) \cong \Omega^1_C$ lifts to a coalgebra map

$$Hoch^*(C) \to \Lambda_C \Omega^1_C$$

Next we shall study the behaviour of $\Omega^n(A)$ and $\Omega^n_{A'}$ for a commutative topological algebra A. Let M be a symmetric A-bimodule; even in the topological case, it is a fact that the tensor algebra $T^*_A(M)$ localizes, i.e. if $S \subset A$ is a multiplicative set, then $T^*_{A_S}(M_S) = (T^*_A(M))_S$, because

$$(M\otimes_A ... \otimes_A M)\otimes_A A_S = M\otimes_A ... \otimes_A M_S = M_S\otimes_A ... \otimes_A M_S = M_S\otimes_{A_S} ... \otimes_{A_S} M_S$$

Then

$$(T^*_A(M))_S = \left(\bigoplus_{n\geq 0} M^{\otimes_A n}\right) \otimes_A A_S = \bigoplus_{n\geq 0}\left(M_S^{\otimes_{A_S} n}\right) = T^*_{A_S}(M_S)$$

This isomorphism induces an isomorphism between the quotients, so $(\Lambda^*_A(M))_S \cong \Lambda^*_{A_S}(M_S)$. In particular, for $M = \Omega^1(A)$ we have $\Omega^1(A)_S = \Omega^1(A_S)$ and so $\Omega^*(A)_S = \Omega^*(A_S)$.

Denoting by C the continuous dual coalgebra A' and by $\Lambda^*_C(M')$ the graded cosymmetric coalgebra, the fact that $\Lambda^n_C(M') = (\Lambda^n_A(M))'$ is deduced by checking that $(\Lambda^n_A(M))'$ satisfies the corresponding universal property:

$$Hom_{gcCoalg}(X, \Lambda^*_A(M)') = Hom_{gcAlg}(\Lambda^*_A(M), X') =$$

$$= Hom_A(M, X') = Com_C(X, M') = Hom_{gcCoalg}(X, \Lambda^*_C(M'))$$

Where $gcCoalg$ = graded cocommutative coalgebras and $gcAlg$ = graded commutative algebras. Taking $M = \Omega^1(A)$, we already know that $\Omega^1_C = (\Omega^1(A))'$. Concerning localization, we have that

$$\begin{aligned} (\Omega^*_C)_{[S]} = & \ (\Omega^*(A))'_{[S]} = (\Omega^*(A)_S)' = \\ = & \ (\Omega^*(A_S)) = \\ = & \ \Omega^*_{C_{[S]}} \end{aligned}$$

The main example of this situation that will be considered in this work is the following:

Let X be a compact smooth manifold, $A = C^\infty(X)$ and $C = A' = \mathcal{D}(X)$. Choose an open covering $X = \cup_{i=1}^n U_i$ consisting of sets U_i homeomorphic to open balls and let $\{\phi_i\}_{i=1,\ldots,n}$ be a partition of unity subordinated to the covering $\{U_i\}_{i=1,\ldots,n}$. Consider $S_i = \{f \in C^\infty(X) \ / \ f(x) \neq 0, \ \forall x \in U_i\}$, then A_{Si} is the topological algebra $C^\infty(U_i)$ (see [15]). The canonical morphism $A \to \oplus_{i=1}^n A_{Si}$ is in this case not only a monomorphism but also a section, with left inverse $A_{Si} \to A$ given by $f \mapsto f.\phi_i$. Consider the following exact diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \bigoplus_{i=1}^{n} A_{Si} & \longrightarrow & \bigoplus_{i<j} (A_{Si})_{S_j} & \longrightarrow & \cdots \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & \bigoplus_{i=1}^{n} A_{Si} & \longrightarrow & \bigoplus_{i<j} A_{S_i.S_j} & \longrightarrow & \cdots \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & C^\infty(X) & \longrightarrow & \bigoplus_{i=1}^{n} C^\infty(U_i) & \longrightarrow & \bigoplus_{i<j} C^\infty(U_i \cap U_j) & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \longrightarrow & (..(A_{S_1})_{S_2})...)_{S_n} & \longrightarrow & 0 \\
 & & \| & & \\
\cdots & \longrightarrow & A_{S_1.\ldots.S_n} & \longrightarrow & 0 \\
 & & \| & & \\
\cdots & \longrightarrow & C^\infty(U_1 \cap \ldots \cap U_n) & \longrightarrow & 0
\end{array}$$

which splits by an A-linear homotopy s, then, for any A-module M the same exact sequence replacing A_{Si} by M_{Si} holds. In particular, taking $M = \Omega^1(A)$, we obtain a (split) exact sequence

$$0 \longrightarrow \Omega^1(X) \longrightarrow \bigoplus_{i=1}^{n} \Omega^1(U_i) \longrightarrow \bigoplus_{i<j} \Omega^1(U_i \cap U_j) \longrightarrow \cdots$$

$$\cdots \longrightarrow \Omega^1(U_1 \cap \ldots \cap U_n) \longrightarrow 0 \qquad (*)$$

where $\Omega^1(U_i)$ denotes $\Omega^1(C^\infty(U_i)) = \Omega^1(C^\infty(X)_{Si}) = (\Omega^1(C^\infty(X))_{Si}$. The $C^\infty(U_i)$-module $\Omega^1(U_i)$ is free with basis $\{dx_i^1, ..., dx_i^n\}$ where x_i^j is the j-th coordinate with respect to the local chart associated to U_i.

In this local case, all definitions of Ω^1 (sections of the cotangent bundle, I/I^2, universal object for derivations) are coincident (to prove it, consider the universal property of each one of this objects). Moreover, they all localize, so the exact sequence $(*)$ gives that all different definitions of $\Omega^1(X)$ coincide.

As $\Omega^n(A_S) = \Lambda^n_{A_S}\Omega^1(A_S) = \Lambda^n_{A_S}\Omega^1(A)_S = (\Lambda^n_A\Omega^1(A))_S$, the n-th. component of Ω^* also localizes.

Following Connes' computations [4], a $\mathbb{C}$-split projective resolution $\{E_i\}_{i\in\mathbb{N}}$ of $C^\infty(X)$ as $C^\infty(X)\widetilde{\otimes}C^\infty(X) = C^\infty(X \times X)$-module is obtained by taking E_i = pull-back of $\Omega^i(X)$ over the second projection. Tensoring this resolution by $A \otimes_{A^e} -$, the complex calculating Hochschild homology (its topological version) has $\Omega^i(X)$ in degree i and the differential is null, so $HH_n(A) = \Omega^n(A)$. This resolution being $\mathbb{C}$-split, the dual complex of $\mathcal{D}(X \times X)$-injective comodules is also exact and $\mathbb{C}$-split. The complex obtained by cotensoring with C over C^e calculates then $Hoch^*(C)$ and has Ω^i_C in degree i because if P is a finitely generated projective A^e-module then P' is a C^e-injective finitely cogenerated comodule and $(A \otimes_{A^e} P)' = C\square_{C^e}P'$. Then $C\square_{C^e}E'_n = (\Omega^n(A))' = \Omega^n_C$. We conclude that in this case $Hoch^n(C) = \Omega^n_C$.

Also, if M is a $\mathcal{D}(U_i)$-bicomodule (for some i, $1 \leq i \leq n$) we can compare the cohomology $Hoch^*(\mathcal{D}(U_i), M)$ with $Hoch^*(\mathcal{D}(X), M)_{[Si]}$. In a more general setting, we have:

Proposition 3.1 *Given a continuous map of nuclear Fréchet (resp. DF) coalgebras $f : D \to C$ such that*

- $D\square_C D \cong D$, *and*

- $Hoch^i(D\widetilde{\otimes}D, C) = 0 \quad \forall i > 0$,

we have $Hoch^i(M, C) = Hoch^i(M, D)$ *for all* $i \in \mathbb{N}_0$ *and arbitrary Fréchet (resp. DF) D-bicomodule M.*

As a corollary, $Hoch^*(M, \mathcal{D}(U_i)) = Hoch^*(M, \mathcal{D}(X))$ for all $\mathcal{D}(U_i)$-bicomodule M, taking $C = \mathcal{D}(X)$ and $D = \mathcal{D}(U_i)$.

When $D = C_{[S]}$ for a multiplicative subset $S \subset Z(C')$, the first condition of the above proposition is verified because localization is an "idempotent" functor, while the second condition is satisfied whenever one can prove that colocalization is exact.

Proof: It is a consequence of Lemma 1.3, taking $K_p(M) := Hoch^p(M, C)$ because

$$\begin{aligned} Hoch^0(M, C) = \;& C\square_{C^e}M = C\square_{C^e}D^e\square_{D^e}M \cong \\ \cong \;& (D\square_C C\square_C D)\square_{D^e}M \cong (D\square_C D)\square_{D^e}M \cong \\ \cong \;& D\square^e_D M = Hoch^0(M, D) \end{aligned}$$

Also $Hoch^p(D\widetilde{\otimes}M\widetilde{\otimes}D, C) = 0$ because $D\widetilde{\otimes}D$ is C-coflat. The condition concerning long exact sequences is also satisfied.

4 Smooth coalgebras

Given a k-algebra A, there are several ways of defining the smoothness of A. One of them says that A is smooth if and only if the second cohomology Harrison group is trivial for every symmetric A-bimodule M, or equivalently, every extension

$$0 \to M \to B \to A$$

of commutative algebras with M a square zero ideal of B, splits by an algebra morphism. Hochschild - Kostant - Rosenberg's well known theorem says that if A is finitely generated commutative smooth algebra over a perfect field, then its Hochschild homology groups are isomorphic to $\Omega^n(A)$. In fact, both conditions are equivalent ([3], [13]).

Given a coalgebra C, we have shown an example where the Hochschild cohomology groups associated to the coalgebra, $Hoch^n(C)$ are respectively isomorphic to the n-th component of the exterior coalgebra on Ω^1_C. We want to characterize coalgebras satisfying this property, in an analogous way to the Hochschild - Kostant - Rosenberg theorem.

This section is devoted to the definition and properties of smooth coalgebras. We begin by recalling some results and definitions from [6].

Definition 4.1 *Given a k-coalgebra C, an* **extension** *of C is a coalgebra D such that C is a subcoalgebra of D. The exact sequence*

$$0 \longrightarrow C \longrightarrow D \stackrel{p}{\longrightarrow} D/C \longrightarrow 0$$

endows D/C with a structure of D-bicomodule. A structure of (non-counital) coalgebra on D/C is given by $\Delta_{D/C}(\overline{d}) := (p \otimes p)\Delta_D(d)$ $(d \in D)$.

Moreover, if $D = C \wedge C$ (i.e. $\Delta(D) \subseteq C \otimes D + D \otimes C$) we have that $\Delta_{D/C} = 0$ and also $\overline{D} := D/C$ is a C-bicomodule. In this case, given a k-linear map $\psi : D \to C$ extending id_C, the following diagrams commute:

$$\begin{array}{ccc} D & \stackrel{\Delta}{\longrightarrow} & D \otimes D \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p\otimes\psi} \\ \overline{D} & \stackrel{\rho^+}{\longrightarrow} & \overline{D} \otimes C \end{array} \qquad\qquad \begin{array}{ccc} d & \longmapsto & d_1 \otimes d_2 \\ \downarrow & & \downarrow \\ \overline{d} & \longmapsto & \overline{d_1} \otimes d_2 \end{array}$$

and analogously for $\rho^- : \overline{D} \to C \otimes \overline{D}$.

If $f : D \to C \otimes C$ is defined by $f := (\psi \otimes \psi)\Delta - \Delta\psi$, then $f(x) = 0$ for $x \in C$, so, there exists $\overline{f} : D/C \to C \otimes C$ such that $\overline{f} \circ p = f$.

Lemma 4.2 *Given an extension D of C and f as above, $\overline{f}$ is a 2-cocycle in the complex $Hom(\overline{D}, C^{\otimes *})$.*

Proof: As p is surjective, it is enough to see that $\delta^2(\overline{f}) \circ p = 0$:

$$\begin{aligned} \delta^2(\overline{f}) \circ p = \; & (id \otimes \overline{f})\rho^- p - (\Delta \otimes id)\overline{f}p - (\overline{f} \otimes id)\rho^+ p + (id \otimes \Delta)\overline{f}p = \\ = \; & (id \otimes \overline{f})(id \otimes p)\Delta - (\Delta \otimes id)f - (\overline{f} \otimes id)(p \otimes id)\Delta + (id \otimes \Delta)f = \\ = \; & (id \otimes f)\Delta - (\Delta \otimes id)f - (f \otimes id)\Delta + (id \otimes \Delta)f \end{aligned}$$

This last expression is zero, as one verifies using the definition of f and the fact that $D = C \wedge C$.

The proof of the following result was given in [6]:

Lemma 4.3 ([6], Lemma 7) *Given k-linear maps $\psi_1, \psi_2 : D \to C$ such that $\psi_i|_C = id_C$ $(i = 1, 2)$, let $\overline{f}_1$ and $\overline{f}_2$ be defined as above. Then there exists $h \in Hom(\overline{D}, C)$ such that $\overline{f}_1 - \overline{f}_2 = \delta^1(h)$.*

The above Lemmas show that there is a 2-cocycle associated to each extension of coalgebras. Doi proved ([6], Theorem 4) that, in the above situation, $[\overline{f}] = 0$ in $H^2(\overline{D}, C)$ if and only if there exists $\psi : D \to C$ such that $\psi|_C = id_C$ and ψ is a coalgebra morphism.

As a consequence, given a coalgebra C and a C-bicomodule M, the equivalence classes of extensions are in 1-1 correspondence with the elements of $H^2(M, C)$. As we are interested in cocommutative coalgebras, we will consider only cosymmetric bicomodules and cocommutative extensions. This class of extensions is in 1-1 correspondence with a subgroup of $H^2(M, C)$. Given a 2-cocycle $[f] \in H^2(M, C)$ with M cosymmetric, we say that f is symmetric if $f(m) = \sigma_{12}(f(m))$. We notice that if g is an arbitrary 2-cocycle, then $\widehat{g} := \sigma_{12}(g)$ is also a 2-cocycle. Assuming $1/2 \in k$, the space of 2-cocycles decomposes into the direct sum of subspaces corresponding to the eigenvalues 1 and -1 of σ_{12}. The boundary of a 1-cocycle is always symmetric, so the decomposition of cocycles gives a decomposition of $H^2(M, C) = H^2(M, C)^{sym} \oplus H^2(M, C)^{antisym}$. It is clear that symmetric 2-cocycles correspond to cocommutative extensions and viceversa, then smoothness of C is defined as follows:

Definition 4.4 *Given a cocommutative k-coalgebra (k a field), we say that C is* **smooth** *if and only if $H^2(M, C)^{sym} = 0$ for every cosymmetric C-bicomodule M.*

Remark: The subspace $H^2(M, C)^{sym}$ is analogous to the second Harrison cohomology group for commutative algebras and symmetric bimodules (which is a direct summand of the second Hochschild cohomology group). A similar Harrison-type theory might be defined for cocommutative coalgebras, but we are not going to consider other degrees because in practice, every property proved in this work depends only on extension properties.

Next, we give an equivalent description of smoothness:

Proposition 4.5 *The following facts are equivalent:*

1. *C is k-smooth.*
2. *Given an extension of cocommutative k-coalgebras*

$$0 \to D \to E \to M \to 0$$

with $E = D \wedge D$ (and so M is a D-cosymmetric bicomodule), and a k-coalgebra morphism $\psi : D \to C$, the following diagram may be completed *with a morphism ϕ of k-coalgebras:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle\psi}\big\downarrow & \swarrow{\scriptstyle\phi} & & & & & \\ & & C & & & & & & \end{array}$$

Proof: 2. $\Rightarrow$ 1.)

Let M be a cosymmetric C-bicomodule and consider an extension $0 \to C \to E \to M \to 0$ with E cocommutative and the solid arrows diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & \swarrow{\scriptstyle\phi} & & & & & \\ & & C & & & & & & \end{array}$$

as M is a C-bicomodule, it is also a cosymmetric E-bicomodule; the exactness of the sequence is equivalent to $E = C \wedge C$. By the hypothesis, the sequence splits by a coalgebra morphism ϕ, then, as $H^2(M, C)^{sym}$ classifies commutative extensions, $H^2(M, C)^{sym} = 0$.

1. $\Rightarrow$ 2.)

Given an extension $0 \to D \to E \to M \to 0$ where M is a cosymmetric D-bicomodule, E and D are cocommutative and $\psi : D \to C$ a coalgebra morphism, consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \stackrel{i}{\longrightarrow} & E & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\bar{\psi}} & & & & \\ 0 & \longrightarrow & C & \longrightarrow & P & \longrightarrow & P/C & \longrightarrow & 0 \end{array}$$

where $P = (C \oplus E)/\sim$ and $\sim$ is defined by $(0, d) \sim (\psi(d), 0)$ for all $d \in D$. Notice that P is a cocommutative coalgebra which contains C (we write c for the class of $(c, 0)$ in P and similarly for the elements of E), with coproduct given by:

$$\Delta_P : P \to P \otimes P$$

$$\begin{cases} \Delta_P(c) = \Delta_C(c) \\ \Delta_P(e) = \Delta_E(e) \end{cases}$$

Suppose that given $c \in C$ and $e \in E$, there exists $d \in D$ such that $c = \psi(d)$ and $e = i(d)$, then (using Sweedler's notation)

$$\Delta_C(c) = \Delta_C(\psi(d)) = (\psi \otimes \psi)\Delta_D(d) = \psi(d_1) \otimes \psi(d_2)$$

and

$$\Delta_E(e) = \Delta_E(i(d)) = (i \otimes i)\Delta_D(d) = i(d_1) \otimes i(d_2)$$

But $\psi(d_j) = i(d_j)$ in P, so Δ_P is well defined (and it is coassociative and counitary). Clearly, the map $C \to P$ $\left(c \mapsto \overline{(c,0)}\right)$ is a monomorphism.

We want to see that $P = C \wedge C$, or equivalently that P/C is a C-bicomodule. Taking into account that $E = D \wedge D$ and that $P/C = (0 \oplus E)/\!\sim\, = E/\!\sim$, then, for $e \in E$, $e \sim 0$ if and only if there exists $d \in D$ such that $e = i(d)$, then $\Delta_{P/C}(\overline{(0,e)}) = \Delta_M(e) = 0$. In fact $P/C \cong M$, the C-structure on P/C is induced by the D-structure on M and the morphism ψ.

Now by hypothesis C is smooth. As these extensions are classified by $H^2(C, P/C)^{sym}$, which is null, it follows that there exists a coalgebra morphism $\sigma : P \to C$ splitting the exact sequence $0 \to C \to P \to M \to 0$. Defining $\phi := \sigma \circ \overline{\psi}$ we obtained the desired morphism.

As in the case of algebras, the smoothness of the coalgebra C has consequences on the structure of the comodule of differentials, as the following proposition shows:

Proposition 4.6 *If C is a k-smooth cocommutative coalgebra, then Ω^1_C is an injective C-comodule.*

Proof: Given a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{\ i\ } & M \\ & & {\scriptstyle f}\big\downarrow & \swarrow {\scriptstyle \overline{f}} & \\ & & \Omega^1_C & & \end{array}$$

where M and N are C comodules, and i, f are morphisms of comodules (i is a monomorphism), we need a morphism $\overline{f}$ extending f.

By the universal property of Ω^1_C, f corresponds to a coderivation $\nabla_f : N \to C$. Let us consider $C \oplus N$ as cocommutative k-coalgebra, whose structure is given by:

$$\Delta : C \oplus N \longrightarrow (C \oplus N) \otimes (C \oplus N) \cong (C \otimes C) \oplus (C \otimes N) \oplus (N \otimes C) \oplus (N \otimes N)$$

$$(c, n) \longmapsto (\Delta_C(c), \rho^-(n), \rho^+(n), 0)$$

and similarly for $C \oplus M$ (notice that cosymmetry of N implies cocommutativity of $C \oplus N$). Then we have:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C \oplus N & \xrightarrow{(id_C, i)} & C \oplus M & \longrightarrow & M/i(N) & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle id_C \oplus \nabla_f} & \swarrow {\scriptstyle \gamma} & & & & & \\ & & C & & & & & & \end{array}$$

The dotted arrow γ exists because (id_C, i) is a coalgebra morphism and $C \oplus M = (C \oplus N) \wedge (C \oplus N)$.

As the above diagram commutes, it necessarily holds that $\gamma = id_C \oplus D_\gamma$, where $D_\gamma : M \to C$ is a coderivation, extending ∇f. This coderivation corresponds to a C-colinear morphism $f_\gamma : M \to \Omega^1_C$, such that $f_\gamma \circ i = f$, because $D_\gamma \circ i = \nabla_f$.

As expected, the smoothness property remains true locally:

Proposition 4.7 *Let C be a cocommutative smooth k-coalgebra, then $C_{[S]}$ is smooth for any multiplicatively closed subset S of C'.*

Proof: Let us consider an extension $0 \to D \to E \to M \to 0$ with D and E cocommutative (and hence M cosymmetric), and a coalgebra morphism $\nu : D \to C_{[S]}$, composing with the canonical map $C_{[S]} \to C$, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \xrightarrow{i} & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle \nu} & & \swarrow & & & & \\ & & C_{[S]} & {\scriptstyle u'} & & & & & \\ & & \downarrow {\scriptstyle \pi} & \swarrow & & & & & \\ & & C & & & & & & \end{array}$$

As C is smooth, the big triangle may be completed with u', such that $u' \circ i = \pi \circ \nu$. Given $s \in S$, $s \circ u' \circ i = s \circ \pi \circ \nu$ is an invertible element in D', as $s \circ \pi$ is invertible in $(C_{[S]})'$, so, there exists $\tilde{t} \in D'$ such that $i^*(s \circ u').\tilde{t} = 1$. As i^* is an epimorphism, there exists $t \in E'$ such that $i^*(t) = \tilde{t}$ and then, for some $m \in M'$ we have $t.(s \circ u') = 1 + m$ and this element is invertible because m is nilpotent (in fact $m^2 = 0$). As a consequence $s \circ u'$ is invertible for all $s \in S$ and so u' factors through $C_{[S]}$.

5 Local Coalgebras

In this section (where C will always denote a cocommutative k-coalgebra) we study local coalgebras, i.e. coalgebras obtained after "localization by maximal ideals".

Definition 5.1 *Given C, a* **coideal** *D is a C-comodule provided of an epimorphism $\phi : C \to D$ of (cosymmetric) C-comodules.*

Remark: In this situation, $Ker(\phi)$ is a subcoalgebra of C, and as C-comodule, D is isomorphic to $C/Ker(\phi)$. We also remark that this definition is dual to the definition of an ideal in ring theory. There, ideals are subobjects of rings such that their quotients are also rings; here, a coideal is a quotient such that the Kernel of the projection is a (sub)coalgebra. Our definition of coideal differs from Sweedler's, notwithstanding we prefer ours because of the reason exposed above.

With this definition the notions of maximal and prime coideal make sense.

Definition 5.2 *A coideal D of C is called* **maximal** *if $Ker(\phi)$ is a one dimensional subcoalgebra (necessarily isomorphic to the base field k).*

If $(D, \phi : C \to D)$ is a maximal coideal and $f : k \to Ker(\phi)$ is the corresponding isomorphism of coalgebras, then $e = f(1)$ is a group-like element in C. We shall keep in mind in what follows this correspondence between "points" of the coalgebra, its maximal coideals and group-like elements of C. As an example, taking $C = k[x]^0$ with k algebraically closed, $k[x]^0 = \oplus_{\lambda \in k} k[s].e^{\lambda s}$. It is easy to see that the only group-like elements of C are the exponentials $e^{\lambda s}$, then C has 'as many points as' elements of k.

When k is algebraically closed, maximal coideals always exist. This follows because it is sufficient to find a group-like element of a finite dimensional subcoalgebra $\widetilde{C} \subset C$, and such an element always exists because there always exists algebra morphisms $\widetilde{C}^* \to k$.

Now, given a maximal coideal D in C, clearly $C' - D'$ is a multiplicative subset of C', then we are able to construct $C_{[C'-D']}$, which is a topological coalgebra. We shall denote it by C_D.

We next define the notion of prime coideals of a coalgebra:

Definition 5.3 *A coideal D with kernel K_D of a coalgebra C is* **prime** *if the restriction map*

$$C^* \cong Com_C(C, C) \to Com_C(K_D, K_D)$$

is such that for all $f \in C^*$, $f|K_D : K_D \to K_D$ is null or an epimorphism.

Note that every maximal coideal is prime.

Definition 5.4 *A coalgebra C will be called a* **local coalgebra** *if it has a unique group-like element.*

Remark: given a maximal coideal D in C, the exact sequence

$$0 \to k \to C \to D \to 0$$

splits by means of the coalgebra morphism $\epsilon : C \to k$, so localization is exact for this sequence, then $0 \to k \to C_D \to D_D \to 0$ is exact ($k_D = k$ because D' acts by isomorphisms on k), so D_D is a maximal coideal in C_D.

There is a natural question arising at this point: how are coideals of C related to coideals of $C_{[S]}$?

Let us look it from one side first:

Giving a coideal of $C_{[S]}$ is the same thing as giving a subcoalgebra of $C_{[S]}$ (the kernel of the map from $C_{[S]}$ to the coideal). If we have a subcoalgebra of $C_{[S]}$, we obtain a subcoalgebra of C as follows:

Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \xrightarrow{i} & C_{[S]} & \xrightarrow{p} & C_{[S]}/E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi \circ i} & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
0 & \longrightarrow & \pi \circ i(E) & \hookrightarrow & C & \xrightarrow{\widetilde{p}} & X & \longrightarrow & 0
\end{array}$$

where X is the push-out of $C_{[S]}/E$ and C over $C_{[S]}$: $X = \frac{C \oplus C_{[S]}/E}{\langle (\pi(z),0)-(0,p(z)) \ : \ z \in C_{[S]} \rangle}$. A direct computation shows that $Ker(\widetilde{p}) = Im(\pi \circ i)$, so the diagram is commutative and the rows are exact. We will denote $Im(\pi \circ i)$ by $Im(E)$.

On the other hand, given a subcoalgebra of C: $\widetilde{C} \hookrightarrow C$, we obtain a subcomodule $\widetilde{C}_{[S]}$ of $C_{[S]}$ which is a subcoalgebra because $\widetilde{C}_{[S]} = \widetilde{C}_{[i^*(S)]}$, and one can ask when these constructions are inverse.

Starting with a subcoalgebra E of $C_{[S]}$, let us see that $Im(E)_{[S]} = E$.

As E is a $C_{[S]}$-comodule, $E = E_{[S]}$. The inclusion $\widetilde{\imath} : Im(E) \to C$ induces a map $\widetilde{\imath}_{[S]} : Im(E)_{[S]} \to C_{[S]}$; we claim that $Im(\widetilde{\imath}_{[S]}) \subseteq E = E_{[S]}$, in fact, given $\{x_t\}_{t \in S} \in Im(E)_{[S]}$, we know that $x_1 \in \pi(i\{e_t\}_{t \in S}))$ and $t.x_{ts} = x_s$, forall $s, t \in S$. Then $\widetilde{i}(\{x_t\}_{t \in S}) = \{t^{-1}.\widetilde{i}(x_1)\}_{t \in S} = \{t^{-1}.x_1\}_{t \in S}$. But $x_t = t^{-1}.x_1 = t^{-1}.e_1 =$

$t^{-1}.(t.e_t) = e_t$. Then we have the following commutative diagram:

$$\begin{array}{ccc} Im(E)_{[S]} & \xrightarrow{\widetilde{\imath}_{[S]}} & C_{[S]} \\ & \searrow^{\widetilde{\imath}_{[S]}} & \uparrow i \\ & & E = E_{[S]} \end{array}$$

As the localization of a monomorphism is again a monomorphism, $\widetilde{\imath}_{[S]}$ is injective.

Concerning surjectivity, if $\{e_t\}_{t\in S} \in E$, then $e_1 \in Im(E)$. We may take the element $\{y_t\}_{t\in S} \in (Im(E))_{[S]}$ defined by $t_1 = e_1$, $y_t = t^{-1}.e_1$, which maps onto $\{e_t\}_{t\in S}$.

Next, we look for a necessary condition of those subcoalgebras $\widetilde{C}$ of C of type $Im(E)$ for some subcoalgebra E of $C_{[S]}$.

Given a subcoalgebra $\widetilde{C}$ of C, it is always true that $Im(\widetilde{C}_{[S]}) \subseteq \widetilde{C}$ because $x \in Im(\widetilde{C}_{[S]}) \;\Leftrightarrow\; x \in Ker(\widetilde{p} : C \to C \coprod_{\widetilde{C}_{[S]}} C_{[S]}/\widetilde{C}) \;\Leftrightarrow\; \overline{(x, \overline{0})} = 0$ in the push-out $\Leftrightarrow\; \exists y \in C_{[S]}$ such that $(x, \overline{0}) = (\pi(y), -p(y)) \;\Leftrightarrow\; \exists y \in C_{[S]}$ such that $(x, 0) = (\pi(y), -y + z)$ for some $z \in \widetilde{C}_{[S]} \;\Leftrightarrow\; \exists y \in \widetilde{C}_{[S]}$ such that $x = \pi(y)$.

Assertion: If $\widetilde{C}$ belongs to the class of subcoalgebras we are considering, and $\widetilde{C}^{\perp}$ denotes $\{f \in C^* \;/\; f|_{\widetilde{C}} = 0\}$, then $\widetilde{C}^{\perp} \cap S = \emptyset$, because if not, $Im(\widetilde{C}_{[S]}) = 0$, and then $Im(\widetilde{C}_{[S]}) \neq \widetilde{C}$.

Lemma 5.5 *With notation as above,* $Im(\widetilde{C}_{[S]}) = 0 \;\Leftrightarrow\; \widetilde{C}_{[S]} = 0$*.*

Proof: $\Leftarrow$) This is evident.

$\Rightarrow$) If $Im(\widetilde{C}_{[S]})$ is zero, then, using the commutativity of the diagram

$$\begin{array}{ccc} Im(\widetilde{C}_{[S]}) & \hookrightarrow & \widetilde{C} \\ \uparrow {\scriptstyle \pi \circ i_{\widetilde{C}_{[S]} \to C_{[S]}}} & & \uparrow \pi \\ \widetilde{C}_{[S]} & = & \widetilde{C}_{[S]} \end{array}$$

we obtain that the map π is null, and $\widetilde{C}_{[S]} = 0$ because $\epsilon_{\widetilde{C}_{[S]}} = \epsilon_C|_{\widetilde{C}_{[S]}} \circ \pi = 0$.

Lemma 5.6 *If* $\widetilde{C}^{\perp} \cap S = \emptyset$ *then* $\widetilde{C}_{[S]} = 0$*.*

Proof: The dual algebra $\widetilde{C}'$ is isomorphic to $C'/\widetilde{C}^{\perp}$. If $i : \widetilde{C} \to C$ denotes the inclusion, then

$$i^*(S) = \{S|_{\widetilde{C}} : \; s \in S\}$$

Supposing $\widetilde{C}^{\perp} \cap S \neq \emptyset$, there exists $f \in S$ such that $f|_{\widetilde{C}} = 0$, then $0 \in i^*(S)$, so $\widetilde{C}_{[S]} = \widetilde{C}_{[i^*(S)]} = 0$.

Proposition 5.7 *With the notation as above, given a maximal coideal D of C (corresponding to a subcoalgebra $\widetilde{C} = k.x$ of C), the coalgebra $C_D = C_{[C'-D']}$ is a local coalgebra (i.e., it has a unique group-like element).*

Proof: Consider a subcoalgebra K of C_D corresponding to a maximal coideal; we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & C_D & \longrightarrow & C_D/K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \pi} & & & & \\ 0 & \longrightarrow & Im(K) & \longrightarrow & C & & & & \end{array}$$

As $(Im(K))^{\perp}$ is a subset of $D' = (k.x)^{\perp}$, $Im(K) = k.x$, because they have the same dimension.

As a consequence, if C_D has a maximal coideal then it is unique. Also, if $s \in S = C' - D' = \{f \in C' \; / \; f(x) \neq 0\}$, then $s.x = (1 \otimes s)\Delta(x) = s(x).x$. Then we can define the element $\{\frac{1}{s(x)}.x\}_{s \in S} \in \prod_{s \in S}(k.x)^{(s)}$. Denoting by λ_s the element $\frac{1}{s(x)}.x$, we have, for $t \in S$,

$$t.\lambda_{st} = \frac{1}{(st)(x)}t(x).x = \frac{1}{s(x)t(x)}t(x).x = \lambda_s$$

so, $\{\lambda_s\}_{s \in S} \in k.x_{[S]} = k.\pi(x)$. This proves that there is at least one maximal coideal (notice that the argument assuring the existence of maximal coideals was stated only in the algebraic context).

We finish this section with a description of injective comodules over local coalgebras. In fact we will prove that given an injective comodule M finitely cogenerated over a local coalgebra C, is C-free.

We need a previous result, which is the analogue to Nakayama's Lemma for algebras.

Lemma 5.8 *Consider a cocommutative local coalgebra C and a C-comodule M, together with an injection $M \hookrightarrow C^n$ for some $n \in \mathbb{N}$. Then, if the composition:*

$$0 \longrightarrow M \xrightarrow{\rho} C \otimes M \xrightarrow{\pi_K \otimes id} C/K \otimes M$$

is a momorphism for some subcoalgebra $K \subseteq C$, then $M = 0$.

Proof:

First step: suppose that M is a subcomodule of C.

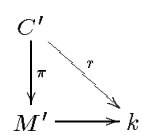
As C is cocommutative, M is a subcoalgebra and hence, if $M \neq 0$, it contains at least an irreducible subcoalgebra, but there is only one, so $k.x \subseteq M$.

This last assertion is clear in the algebraic context (see [16]), for the topological case it worths a proof:

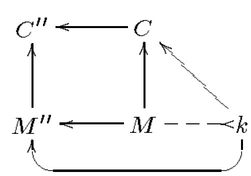
Consider the inclusion $k.x \hookrightarrow C$, by restriction it induces an algebra map $r : C' \to (k.x)' = k$ with Kernel $(k.x)^{\perp}$, which is a maximal ideal of C'. The composition

$$C' \xrightarrow{\pi} M' \longrightarrow M'/\pi((k.x)^{\perp})$$

is clearly a surjection, and $(k.x)^{\perp}$ maps to zero, then it induces a surjection $C'/(k.x)^{\perp} \to M'/\pi((k.x)^{\perp})$. Since $C'/(k.x)^{\perp}$ is a field, it is a monomorphism, then $M'/\pi((k.x)^{\perp})$ is isimorphic to k and this proves that $\pi((k.x)^{\perp})$ is a maximal ideal in M'. Dualizing the diagram



one obtains the commutative diagram



The dashed arrow exists because $M = M'' \cap C$, as x is the image of $1 \in k$, the assertion is proved.

If K is another subcoalgebra of C, by the same argument $k.x \subseteq K$. By hypothesis, the composition $M \to C \otimes M \to C/K \otimes M$ is injective, but the element $x \in M$ is on the kernel of this composition; this is a contradiction unless $M = 0$.

Second step: Let us take $n = \min\{m \in \mathbb{N} \;/\; M \text{ is embeddable in } C^m\}$.

The case $n = 1$ has already been studied, then suppose $n > 1$. Let $\phi : M \to C^n$ denote the embedding and take $N = M \cap \phi^{-1}(C^{n-1} \oplus 0)$.

If $N = 0$, then the composition $M \xrightarrow{\phi} C^n \xrightarrow{\pi_n} C$ is injective, because if $m \in M \cap Ker(\pi_n \circ \phi)$ then $\pi_n(\phi(m)) = 0$, $m \in N = \{0\}$, and this contradicts the minimality of n. Then $N \neq \{0\}$. Now consider a subcoalgebra K of C and the diagram,

$$\begin{array}{ccccc} N & \longrightarrow & C \otimes N & \longrightarrow & C/K \otimes N \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ M & \longrightarrow & C \otimes M & \longrightarrow & C/K \otimes M \end{array}$$

If the bottom row is a monomorphism, then the top row is too, since the columns are monomorphisms. Also $\phi|_N : N \to C^{n-1} \oplus 0$ is injective; by the inductive hypothesis $N = 0$, and this is a contradiction.

When the coalgebra C is algebraic, Lemma 5.8 can be generalized as follows:

Corollary 5.9 *Let M be a C comodule (not necessarily finitely cogenerated) where C is a cocommutative local (algebraic) coalgebra. Then if the composition:*

$$0 \longrightarrow M \xrightarrow{\rho} C \otimes M \xrightarrow{\pi_K \otimes id} C/K \otimes M$$

is a momorphism for some subcoalgebra $K \subseteq C$, then $M = 0$.

Proof: Consider N an arbitrary finite dimensional subcomodule of M, then, if $(\pi \otimes is_M) \circ \rho_M$ is a monomorphism, its restriction to N is so. Since a finite dimensional comodule is finitely cogenerated (consider the structure morphism $N \to C \otimes N \cong C^{dim_k(N)}$), Lemma 5.8 holds for N and then $N = 0$. But M is the union of its finite dimensional subcomodules, then $M = 0$.

Proposition 5.10 *Let C be a local coalgebra and M an injective finitely cogenerated C-comodule. Then M is a free C-comodule.*

Proof: Let us suppose $M \neq 0$ and let n be as in the proof of the above Lemma. Consider the diagram,

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \longrightarrow & C^n \\ & & \uparrow & & \uparrow \\ 0 & \longrightarrow & (k.x)^n \cap M & \longrightarrow & (kx)^n \end{array}$$

1. $M \cap (kx)^n \neq \{0\}$:

Identifying $M \otimes C$ as a subspace of $C^n \otimes C \cong (C \otimes C)^n$ and then considering the composition

$$M \to C \otimes M \to C/k.x \otimes M$$

we see that the image of an element $m \in M$ is zero if $m = (m_1, ..., m_n)$ is an element of the type $(\epsilon(m_1)x, ..., \epsilon(m_n)x)$, so if $M \cap (k.x)^n = 0$ this composition is injective and by Nakayama's Lemma $M = 0$ (a contradiction). Then $M \cap (k.x)^n$ is a proper subspace of $(k.x)^n$. Let $\{m_1, ..., m_s\}$ denote a base of it.

2. Consider the map $\epsilon^n : C^n \to (k.x)^n$ ($(c_1, ..., c_n) \mapsto (\epsilon(c_1), ..., \epsilon(c_n))$ and the map $\phi : M \to C^s = C \otimes (k.m_1 \oplus ... \oplus k.m_s)$ given by

$$\phi(m) = m_{-1} \otimes \epsilon^n(m_0)$$

Notice that $\epsilon^n(M) \subseteq (k.x)^n \cap M = k.m_1 \oplus ... \oplus k.m_s$. Since ϕ is a morphism of C-comodules, then $Ker(\phi)$ is a subcomodule of M. We want to show that it is zero. But looking again at the composition

$$\begin{array}{ccccc} Ker(\phi) & \longrightarrow & C \otimes Ker(\phi) & \longrightarrow & C/k.x \otimes Ker(\phi) \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & C \otimes M & \longrightarrow & C/k.x \otimes M \end{array}$$

we observe that $\phi|_{(k.x)^n} = id_{(k.x)^n}$ and hence $Ker(\phi) \cap (k.x)^n = 0$ then $Ker(\phi) = 0$ again by Nakayama's Lemma.

3. Next we want to show that the cokernel of ϕ is zero. Let us write K_ϕ the cokernel of ϕ and consider the short exact sequence

$$0 \to M \to C^s \to K_\phi \to 0$$

As M is injective, the sequence splits, and $C^s \cong M \oplus K_\phi$. But $K_\phi \cap (k.m_1 \oplus \dots \oplus k.m_s) = 0$ because $(k.m_1 \oplus \dots k.m_s) = (k.x)^n \cap M$ and $K_\phi \cap M = 0$. This proves that $K_\phi = 0$ because $(k.m_1 \oplus \ldots \oplus k.m_s)$ is the kernel of the composition $C^s \to C \otimes C^s \to C/k.x \otimes C^s$ which induces by restriction the composition $K_\phi \to C \otimes K_\phi \to C/k.x \otimes K_\phi$.

We conclude this section with a Lemma that will be necessary later, but which is also interesting on its own.

Lemma 5.11 *Let C and D be two cocommutative k-coalgebras with $k = \overline{k}$. Then*

1. *If C and D are local, then $C \otimes D$ is also local.*
2. *if C and D are smooth, then $C \otimes D$ is also a smooth coalgebra.*

Proof: 1. Concerning the 'local' part of the Lemma, we remark that $C \otimes D$ is the product in the category of cocommutative k coalgebras, the 'projections' being $p_C = 1 \otimes \epsilon : C \otimes D \to C$ and $p_D = \epsilon \otimes 1 : C \otimes D \to D$. If $\phi : E \to C \otimes D$ is a coalgebra morphism and E is cocommutative, then ϕ determines two coalgebra morphisms $\phi_C = p_C \circ \phi$, $\phi_D = p_D \circ \phi$, and $\phi(x) = \phi_C(x) \otimes \phi_D(x)$ for all $x \in E$. If e is a group-like element of $C \otimes D$ then it corresponds to a coalgebra morphism $k \to C \otimes D$, and as a consequence of the above argument e must be of the type $e = e_C \otimes e_D$ with e_C (resp. e_D) a group-like element of C (resp. D). But if C and D have unique group-like elements, then a group-like element of $C \otimes D$ is also unique.

2. We now focus our attention in the 'smooth' part. Let

$$0 \to C \otimes D \to E \to M \to 0$$

be an extension of cocommutative coalgebras with $E = (C \otimes D) \wedge_E (C \otimes D)$. We must produce a coalgebra splitting. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \longrightarrow & E \oplus_{p_D} D & \longrightarrow & M_D & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle p_D} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C \otimes D & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_C} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & E \oplus_{p_C} C & \longrightarrow & M_C & \longrightarrow & 0
\end{array}$$

where M_C and M_D are the respective Cokernels (which are cosymmetric). By diagram chasing, one see that the induced maps $M \to M_C$ and $M \to M_D$ are both surjective, so $\Delta_{M_C} = 0 = \Delta_{M_D}$, or in other words, $E \oplus_{p_C} C = C \wedge_{E \oplus_{p_C} C} C$ and similarly for $E \oplus_{p_D} D$ (notice that both are cocommutative coalgebras). Now by smoothness of C and D there are coalgebra splittings of both extensions; let us denote them by $s : E \oplus_{p_C} C \to C$ and $t : E \oplus_{p_D} D \to D$. Composing with the canonical projections $E \to E \oplus_{p_C} C$ and $E \to E \oplus_{p_D} D$ we obtain two coalgebra morphisms $S : E \to C$ and $T : E \to D$, the coalgebra morphism $(S \otimes T) \circ \Delta_E : E \to C \otimes D$ is the desired splitting for this extension.

6 Return to the comodule of Kähler differentials

In this section we shall prove some facts about the comodule of Kähler differentials of a cocommutative coalgebra C, and further results for cases local and smooth.

Given a cocommutative coalgebra C and a coideal D, consider the usual exact sequence $0 \to K \to C \to D \to 0$ (K is a subcoalgebra) and let γ be the composition map $D \xrightarrow{\rho^+} D \otimes C \xrightarrow{id \otimes \pi} D \otimes D$.

Lemma 6.1 *Given a subcoalgebra K of C, then $Ker(\gamma) \cong (K \wedge K)/K$.*

Proof: We recall from [16] that $K \wedge K = \{x \in C \ / \ \Delta(x) \in C \otimes K + K \otimes C\} = Ker((\pi_K \otimes \pi_K) \circ \Delta)$ and we have the inclusion $K \to K \wedge K$. Also $K \wedge K = Ker(\pi \circ \gamma)$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & Ker(\pi \circ \gamma) & \longrightarrow & Ker(\gamma) & \longrightarrow & 0 \\ & & \| & & \| & & \uparrow & & \\ 0 & \longrightarrow & K & \longrightarrow & K \wedge K & \longrightarrow & (K \wedge K)/K & \longrightarrow & 0 \end{array}$$

The map $(K \wedge K)/K \to Ker(\gamma)$ exists by direct inspection, and by the five lemma it is an isomorphism.

Remark: In the dual situation, $K \wedge K$ corresponds to I^2, namely, if $K = I^{\perp}$ then $K \wedge K = (I^2)^{\perp}$ (see for example [16].

The comodules of Kähler differentials of both coalgebras are related by the following proposition:

Proposition 6.2 *1. There is an exact sequence of C-comodules*

$$i \ 0 \longrightarrow \Omega^1_K \xrightarrow{i} \Omega^1_C \square_C K \xrightarrow{\delta} Ker(\gamma)$$

2. *If D is a maximal coideal, so that $K = k.x$, the monomorphism $\Omega^1_C \square_C k \rightarrow Ker(\gamma)$ is an isomorphism.*

Proof:

1. The map i is defined as the composition of the inclusion of $\Omega^1_K = Sym(K \otimes K/\Delta(K)) \subseteq Sym(C \otimes C/\Delta(C)) = \Omega^1_C$ with the structure map giving the isomorphism $\Omega^1_C \cong \Omega^1_C \square_C C$. We notice that if an element z belongs to $\Omega^1_K \subseteq \Omega^1_C$, then $\rho(z) \in \Omega^1_C \otimes K$, then the image of the above composition is included in $\Omega^1_C \square_C K$.

The map δ is defined by $\delta = p_K \circ (d \otimes \epsilon) : \Omega^1_C \square_C K \rightarrow Ker(\gamma)$ where $d : \Omega^1_C \rightarrow C$ is the universal coderivation, as defined in section 1 and $p_K : C \rightarrow C/K$ is the canonical projection. The image of $d \otimes \epsilon$ is contained in $K \wedge K$ because the domain is $\Omega^1 \square_C K$ and not $\Omega^1_C \square_C C$.

It is then sufficient to prove that, for each K-comodule T, the following sequence is exact:

$$0 \rightarrow Com_K(T, \Omega^1_K) \rightarrow Com_K(T, \Omega^1_C \square_C K) \rightarrow Com_K(T, Ker(\gamma))$$

By the universal property of Ω^1_K, the first term is $Coder_k(T, K)$, the second one is (by adjunction) isomorphic to $Com_C(h_K(K, T), \Omega^1_C)$ which is isomorphic to $Coder_k(T, C)$. The third one may be considered as embedded into $Com_C(T, D)$, so we get

$$0 \rightarrow Coder_k(T, K) \rightarrow Coder_k(T, C) \rightarrow Com_C(T, Ker(\gamma)) \subseteq Com_C(T, D)$$

Observe that $h_K(K, T)$ exists for all T, because K is a quasi-finite C-comodule (for a definition of h see for example [6]).

The exactness of the above sequence follows by a direct computation.

2. From 1., when D is maximal we have a monomorphism $\Omega^1_C \square_C k \rightarrow Ker(\gamma)$ (notice that $\Omega^1_k = 0$ because $Coder_k(T, k) = 0$ for all k-vector spaces T). Denoting by e the group-like element corresponding to D, then $K = k.e$ and $Ker(\gamma)$ is a k-vector space, with the structure defined as follows:
Given $x \in Ker(\gamma)$, $\rho^+(x) \in Ker(\gamma) \otimes C$ (as $Ker(\gamma)$ is a C-comodule). Moreover, $\rho^+(x) \in Ker(\gamma) \otimes k.e$, as the following diagram shows:

$$\begin{array}{ccccccccc}
 & & Ker(\gamma) & \hookrightarrow & D & \xrightarrow{\gamma} & D \otimes D & & \\
 & & \downarrow{\scriptstyle \rho^+|_{Ker(\gamma)}} & & \downarrow{\scriptstyle \rho^+} & & \| & & \\
0 & \longrightarrow & D \otimes k.e & \longrightarrow & D \otimes C & \longrightarrow & D \otimes D & \longrightarrow & 0
\end{array}$$

The inverse of the map $\Omega^1_C \square_C k \to Ker(\gamma)$ can be constructed as follows:

Since

$$Com_k(Ker(\gamma), \Omega^1_C \square_C k) = Com_C(h_k(k, Ker(\gamma)), \Omega^1_C) = Com_C(Ker(\gamma), \Omega^1_C)$$

and the last one equals $Coder_k(Ker(\gamma), C)$, we can define instead a coderivation from $Ker(\gamma)$ into C. The map $f : Ker(\gamma) \to C$ defined by $f(\overline{x}) = x - \epsilon(x).e$ is well-defined and it is a coderivation (this last statement is easily verified). The k-linear morphism $\widetilde{f} : Ker(\gamma) \to \Omega^1_C \square_C k$ obtained from f is the desired inverse. In order to see that, we shall make more explicit the formula for $\widetilde{f}$:

The colinear map $\widetilde{f}$ corresponding to f is defined as $\widetilde{f}(\overline{x}) := (1 \otimes f)\rho^-(\overline{x}) =$

$$= [x_1 \otimes f(\overline{x_2})] \otimes x_3 = [x_1 \otimes (x_2 - \epsilon(x_2).e)] \otimes x_3 =$$

$$= -[x_1 \otimes \epsilon(x_2)e] \otimes x_3 = -[x_1 \otimes e] \otimes x_2 =$$

$$= -[x_1 \otimes e] \otimes \epsilon(x_2)e = -[x \otimes e] \otimes e$$

The composition $\delta \circ \widetilde{f}$ is easy. Let us consider $\overline{x} \in (k.e \wedge k.e)/k.e$, then

$$\delta(\widetilde{f})(\overline{x}) = \delta(-[x \otimes e] \otimes e) = -\overline{\epsilon(x).e - x} = \overline{x}$$

For the other composition we will use the equation

$$[z \otimes w_1] \otimes w_2 \otimes e = [z_1 \otimes w] \otimes z_2 \otimes e = [z \otimes w] \otimes e \otimes e$$

which is verified for all elements $[z \otimes w] \otimes e \in \Omega^1_C \square_C k.e$. Applying $d \otimes id \otimes \epsilon$ we get the formula

$$(\epsilon(z)w_1 - z\epsilon(w_1)) \otimes w_2 = (\epsilon(z_1)w - z_1\epsilon(w)) \otimes z_2 = \epsilon(z)w \otimes e - \epsilon(w)z \otimes e$$

or equivalently

$$\epsilon(z)\Delta(w) - z \otimes w = w \otimes z - \epsilon(w)\Delta(z) = \epsilon(z)w \otimes e - \epsilon(w)z \otimes e$$

With this formulae in hand we can check the other composition:

$$\widetilde{f}\delta([z \otimes w] \otimes e) = \widetilde{f}(\overline{\epsilon(z)w - \epsilon(w)z}) =$$

$$= -[\epsilon(z)w \otimes e] \otimes e + [\epsilon(w)z \otimes e] \otimes e$$

Now, using the equality $\epsilon(z)w \otimes e - \epsilon(w)z \otimes e = \epsilon(z)\Delta(w) - z \otimes w$, one has that

$$[\epsilon(z)w \otimes e] \otimes e - [\epsilon(w)z \otimes e] \otimes e = [z \otimes w] \otimes e$$

When K is a smooth subcoalgebra of C, the previous proposition can be made more precise:

Proposition 6.3 *In the situation of proposition 6.2, if K is smooth, the sequence*

$$0 \to \Omega^1_K \to \Omega^1_C \square_C K \to Ker(\gamma) \to 0$$

is split exact.

Proof: We only have to prove that the map $\Omega^1_C \square_C K \to Ker(\gamma)$ is split surjective, but this is equivalent to the fact that the induced maps

$$(*)\ Com_K(T, \Omega^1_C \square_C K) \to Com_K(T, Ker(\gamma))$$

are surjective for every K-comodule T. The domain is isomorphic to $Com_K(h_K(K,T), \Omega^1_C) \cong Coder_C(T, C)$.

First remark: given a coderivation $\delta : T \to C$ (where the K-comodule T is a C-comodule by the inclusion $K \to C$), one has that $Im(\delta) \subseteq Ker(\pi \circ \gamma)$ (and there is an exact sequence $0 \to K \to Ker(\pi \circ \gamma) \to Ker(\gamma) \to 0$)
Second remark: the map $\pi \circ \delta : T \to Ker(\gamma)$ is K-colinear.
Third remark: if K is a smooth coalgebra, we look at the exact sequence

$$0 \longrightarrow K \underset{i}{\overset{r}{\rightleftarrows}} C \underset{\pi}{\overset{s}{\rightleftarrows}} D \longrightarrow 0$$

and think of D as K-bicomodule with null coalgebra structure. Then the sequence splits, C is isomorphic (as a vector space) to $D \oplus K$ and r is a coalgebra morphism. If $r' : C \to K$ is another coalgebra morphism, then $r' \circ s : D \to K$ is a coderivation.

Given $f : T \to Ker(\gamma)$ a morphism of K-comodules, we construct the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \dashrightarrow & E & \xrightarrow{p_2} & T & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & K & \longrightarrow & Ker(\gamma \circ \pi) & \xrightarrow[\pi]{} & Ker(\gamma) & \longrightarrow & 0
\end{array}$$

where E is the pull-back of f along π, that is, $E = \{(c,t) \in C \oplus T \ / \ f(t) = \pi(c)$ and $\gamma\pi(c) = 0\}$. The diagram may be completed with the dashed arrow after noticing that K is the kernel of the projection on the second factor p_2. Also, as K is smooth, and E is provided of a coalgebra structure, $E \cong K \oplus T$. Then we obtain a coderivation $\delta_f : T \to K$, and hence the map $(*)$ is an epimorphism. In particular, choosing $T = Ker(\gamma)$ and $f = id$, we get the splitting of the sequence.

Corollary 6.4 *In the situation of the proposition, if C is smooth then $Ker(\gamma)$ is a K-injective comodule.*

Proof: As we have a retraction of $\Omega^1_C \square_C K \to Ker(\gamma)$ it is enough to see that $\Omega^1_C \square_C K$ is K-injective, but as C is smooth, Ω^1_C is C-injective and so $\Omega^1_C \square_C K$ is K-injective.

Remark that the proof above implies that if Ω^1_C is finitely cogenerated as C-comodule then $Ker(\gamma)$ is finitely cogenerated as K-comodule.

If K corresponds to a maximal ideal then K is local, the above map is an isomorphism and $Ker(\gamma)$ is a free K-comodule ($Ker(\gamma) \cong K^n$ for some $n \in \mathbb{N}$).

Looking for a moment at the dual situation (an ideal I of an algebra A and the quotient algebra A/I) it is clear after the corollary that the object corresponding to $Ker(\gamma)$ is I/I^2.

The statement "$Ker(\gamma) \cong K^n$" is much less clear that in the case of algebras, because we cannot speak of generators or linear combinations, so it becomes necessary to express it in terms of morphisms. If $Ker(\gamma) \cong K^n$, then we can take duals, obtaining $Ker(\gamma)^* \cong (K^n)^*$. But $(K^n)^* \cong (K^*)^n$, and the meaning of $Ker(\gamma)^* \cong (K^*)^n$ is clear.

Let us suppose for a while that $n = 1$. Denote by $u : Ker(\gamma) \to K$ the isomorphism and by f the element $u^*(\epsilon)$ (where u^* is the isomorphism $u^* : K^* \to Ker(\gamma)^*$ obtained as the transpose of u), $f \in Ker(\gamma)^* = (K \wedge K/K)^* \cong D^*/(D^*)^2$ (remark that D^* is an ideal of C^*).

Also $Ker(\gamma)^*$ is embedded into $Ker(\gamma \circ \pi)^*$ (thinking of f as $\pi \circ f \in Ker(\gamma \circ \pi)^*$), and we look at $f \circ \pi$ as an element of $(K \wedge K)^*$, vanishing over K.

Consider the short sequence

$$0 \longrightarrow K \longrightarrow C \xrightarrow{(f\circ\pi).} C \longrightarrow 0$$

The composition is clearly zero as K is a subcoalgebra and $K = Ker(\pi)$. Consider also the short sequence

$$(*) \quad 0 \longrightarrow Ker((f \circ \pi).) \longrightarrow C \xrightarrow{(f\circ\pi).} C \longrightarrow 0$$

In the case $(*)$ splits (notice that in the algebraic case this is not an additional assumption and in the topological case the map $\Delta : C \hookrightarrow C^e$ always splits by means of $\epsilon \otimes 1$), by dualization we obtain the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longleftarrow & K^* & \longleftarrow & C^* & \overset{(f\circ\pi).^*}{\longleftarrow} & C^* & \longleftarrow & 0 \\ & & \downarrow & & \| & & \| & & \\ 0 & \longleftarrow & Ker((f\circ\pi).)^* & \longleftarrow & C^* & \overset{(f\circ\pi).}{\longleftarrow} & C^* & \longleftarrow & 0 \end{array}$$

The right horizontal map is a monomorphism (this is known to be true as in this case $D^*/(D^*)^2$ is a C^*/D^*-module of rank one, generated by the class of $(f \circ \pi)$). By the same reasons, we know that both sequences are exact. Then we have that $K = Ker((f \circ \pi).)$ and that the map $(f \circ \pi). : C \to C$ is an epimorphism, and this gives the exactness of the above sequence. In this case, we have just obtained a "small" injective resolution of K as C-comodule.

Now we return to the general case, giving an analogue of the Koszul complex for algebras. The following theorem will then allow us to construct a resolution of the coalgebra C as C^e-comodule which leads to the proof of the coalgebra version of Hochschild - Kostant - Rosenberg theorem (Theorem 8.1):

Theorem 6.5 *Given a cocommutative smooth coalgebra C which is also local (hence every subcoalgebra is local), let us consider $f_1, ..., f_n \in C^* = Com_C(C, C)$ and $K = \cap_{i=1}^n Ker(f_i.)$. If $\cup_{n\in\mathbb{N}}(\Lambda^n K) = C$ then the following statements are equivalent:*

1. *$(K \wedge K)/K$ is a free K-comodule, finitely cogenerated (i.e., $(K \wedge K)/K \cong \oplus_{i=1}^n K.f_i$) and $\bigoplus_{n\in\mathbb{N}} \Lambda^{n+1}K/\Lambda^n K$ is isomorphic to the K-free cocommutative coalgebra on $(K \wedge K)/K$, that is $\bigoplus_{n\in\mathbb{N}} \Lambda^{n+1}K/\Lambda^n K \cong K \otimes sh(k^n)$, where $sh(k^n)$ is the graded dual coalgebra of the symmetric algebra $S(k^n)$.*

2. *The sequence $0 \to K \to C \to \oplus_{i=1}^n Cf_i \to \oplus_{i<j} Cf_i \wedge f_j \to ......Cf_1 \wedge ... \wedge f_n \to 0$ is exact.*

Remark: The condition $\cup_{n\in\mathbb{N}}\Lambda^n K = C$ corresponds in the algebra case to having an ideal $I \subset A$ such that $\cap_{n\in\mathbb{N}} I^n = 0$.

The proof of the theorem will be obtained in section 8, we shall principally spend the next section in the study of the graded coalgebra $\oplus_i(\Lambda^{i+1}K/\Lambda^i K)$

7 The associated graded coalgebra and a structure theorem

In this section we give a description of C in terms of a shuffle coalgebra (Theorem 7.4 and Proposition 7.5). It enables us to finish the proof of Theorem 6.5 by means of an inductive argument.

In case $n = 1$ we have already seen that statement 2. is equivalent to the fact that

$$(2') \quad f. : C \to C \text{ is an epimorphism}$$

In this situation 1. clearly implies both. As an example, we shall see now how (2.') implies (2.) for $n = 2$.

Example: Suppose that $K = Ker(f_1.) \cap Ker(f_2.)$ and that $f_1. : C \to C$ and $f_2.|_{Ker(f_1)} : Ker(f_1) \to Ker(f_1)$ are epimorphisms, then consider the sequence

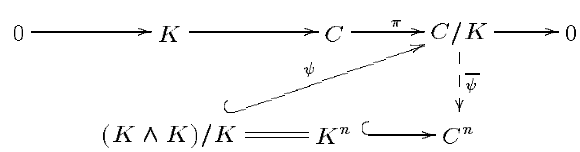
$$0 \longrightarrow K \hookrightarrow C \xrightarrow{\delta_1} Cf_1 \oplus Cf_2 \xrightarrow{\delta_2} C.f_1 \wedge f_2 \longrightarrow 0$$

Suppose that $(c, c') \in Cf_1 \oplus Cf_2$ is such that $0 = \delta_2(c, c') = f_2.c - f_1.c'$, then, as $f_1. : C \to C$ is an epimorphism, there exists $\widetilde{c} \in C$ such that $c = f_1.\widetilde{c}$ and we can write $(c, c') = (f_1.\widetilde{c}, c')$. Considering the element $(c, c') - \delta_1(\widetilde{c}) = (0, c' - f_2.\widetilde{c})$ we have that $0 = \delta_2(0, c' - f_2.\widetilde{c}) = f_1.c' - f_1.f_2.\widetilde{c} = f_1.(c' - f_2.\widetilde{c})$. As $(f_2.)$ is an epimorphism when restricted to $Ker(f_1.)$, there is an element $d \in Ker(f_1.)$ such that $f_2.d = c' - f_2.\widetilde{c}$ and this proves the exactness in $Cf_1 \oplus Cf_2$ (which is the only non trivial one) because if $\delta_2(c, c') = 0$ then

$$\delta_1(\widetilde{c} + d) = (f_1(\widetilde{c} + d), f_2.(\widetilde{c} + d)) =$$
$$= (f_1.\widetilde{c}, f_2.\widetilde{c} + f_2.d) = (c, f_2.\widetilde{c} + c' - f_2.\widetilde{c}) = (c, c')$$

Next we want to describe $\oplus_i \Lambda^{i+1} K / \Lambda^i K$ when $(K \wedge K)/K$ is isomorphic to K^n.

Proposition 7.1 *Let C be a cocommutative coalgebra such that every epimorphic image of C is a finitely cogenerated C-comodule and $(K \wedge K)/K$ as above, then the following diagram*



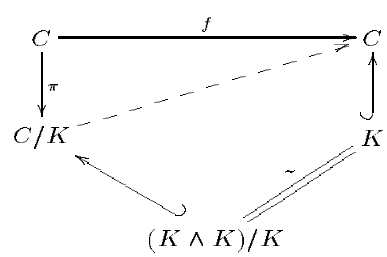
may be completed so as to mantain commutativity, to a morphism of C-comodules $\overline{\psi} : C/K \to C^n$; moreover, $\overline{\psi}$ is a monomorphism.

Proof: The existence of $\overline{\psi}$ is guaranteed by the injectivity of C^n as C-comodule, and $Ker(\overline{\psi})$ is a C-subcomodule of C/K. In order to prove that $Ker(\overline{\psi}) = \{0\}$, consider the diagram

$$\begin{array}{ccccc} Ker(\overline{\psi}) & \xrightarrow{\rho^+} & Ker(\overline{\psi}) \otimes C & \xrightarrow{id\otimes\pi} & Ker(\overline{\psi}) \otimes C/K \\ \downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i\otimes id} \\ C/K & \xrightarrow{\rho^+} & C/K \otimes C & \xrightarrow{id\otimes\pi} & C/K \otimes C/K \end{array}$$

The composition $(id \otimes \pi) \circ \rho^+$ equals γ, and $(i \otimes id_{C/K})$ is a monomorphism as C/K is k-flat. Given $x \in Ker(\overline{\psi})$, if $x \in Ker((id \otimes \pi) \circ \rho^+)$, then $x \in Ker(\gamma) = (K \wedge K)/K$, so $x \in (K \wedge K)/K \cap Ker(\overline{\psi}) = Ker(\psi)$, but ψ is a monomorphism, then $x = 0$. By Lemma 5.8, $Ker(\overline{\psi}) = 0$.

Consider now the case $n = 1$ (i.e. $(K \wedge K)/K \cong K$), then the map $f. : C \to C$ is defined in the following way:



Where $f.$ is defined as the composition of $\pi : C \to C/K$ with the extension of the inclusion $K \to C$, looking at K as a subcomodule of C/K via the isomorphism $K \cong (K \wedge K)/K \subset C/K$. Clearly $Ker(f.) = Ker(\pi) = K$, as the extension provided by the above proposition is injective.

With this definition of $f.$, the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \to & K & \to & K \wedge K & \to & (K \wedge K)/K & \to & 0 \\ & & \| & & \| & & \|{\scriptstyle \sim} & & \\ 0 & \to & K & \to & K \wedge K & \xrightarrow{f.} & K & \to & 0 \end{array}$$

As a consequence $f.: \Lambda^2 K \to K$ is an epimorphism.

We can now prove:

Proposition 7.2 *If $\cup_{n\in\mathbb{N}}\Lambda^n K = C$, then the following statements are equivalent:*

1. *$f.: C \to C$ is an epimorphism*

2. *$gr(f.): \oplus_{n\in\mathbb{N}}\Lambda^{n+1}K/\Lambda^n K \to \oplus_{n\in\mathbb{N}}\Lambda^{n+1}K/\Lambda^n K$ is an isomorphism.*

Proof: One can first observe that $gr(f.): \oplus_{n\in\mathbb{N}}\Lambda^{n+1}K/\Lambda^n K \to \oplus_{n\in\mathbb{N}}\Lambda^{n+1}K/\Lambda^n K$ is always a monomorphism, because given $x \in \Lambda^{n+1}K$, then $f.x \in \Lambda^n K$ ($x \in \Lambda^{n+1}K \Leftrightarrow \pi^n\Delta^n x = 0 \Leftrightarrow f^n.x = o \Rightarrow f.x \in \Lambda^n K$). But $f.x \in \Lambda^{n-1}K$ if and only if $0 = \pi^{n-2}\Delta^{n-2}f.x = (f\otimes 1)\pi^{n-1}\Delta^{n-1}x$, and as $f.: C/K \to C/K$ is a monomorphism, then $\pi^{n-1}\Delta^{n-1}x = 0$, so $x \in \Lambda^n K$.

Also, as $K^\perp = \langle f\rangle = I$, $I^\perp = \{c \in C \ / \ y(c) = 0, \ \forall y \in I\} = Ker(f.) = K$, and as $I^n =< f^n >= (\Lambda^n K)^\perp$, then $Ker(f^n.) = (I^n)^\perp$ and so (1) implies (2).

Let us now see (2) $\Rightarrow$ (1):

Given $x \in C$, as $\cup_{n\in\mathbb{N}}\Lambda^n K = C$, we can choose n such that $x \in \Lambda^n K$ and $x \notin \Lambda^{n-1}K$. So $\overline{x} \neq 0$ in $\Lambda^n K/\Lambda^{n-1}K$. As $gr(f.)$ is an isomorphism, there exists $\overline{y} \in \Lambda^{n+1}K/\Lambda^n K$ such that $gr(f.)(\overline{y}) = \overline{x}$. Then $f.y - x = z \in \Lambda^{n-1}K$. By inductive hypothesis, there exists $z' \in \Lambda^n K$ such that $z = f.z'$. In order to finish the proof, remark that the case $n = 1$ is easy, as $\Lambda^0 K = 0$.

Remark: By induction, the above proposition provides the equivalence between the following statements:

$$(1)\ f_i.: \cap_{i<j}Ker(f_j.) \to \cap_{i<j}Ker(f_j.) \text{ is an epimorphism}$$

$$(2)\ gr(f_i.): \oplus_{l\geq 0}\Lambda^{l+1}\overline{K}_i/\Lambda^l\overline{K}_i \to \oplus_{l\geq 0}\Lambda^{l+1}\overline{K}_i/\Lambda^l\overline{K}_i \text{ is an isomorphism}$$

where $\overline{K}_i$ is by definition $\cap_{i<j}Ker(f_j.)$.

It also shows that in the 1-dimensional case, we need to prove that $gr(f): \oplus_{n\in\mathbb{N}}\Lambda^{n+1}K/\Lambda^n K \to \oplus_{n\in\mathbb{N}}\Lambda^{n+1}K/\Lambda^n K$ is an isomorphism. It is then sufficient to prove that $gr(f.): \oplus_{n\in\mathbb{N}}\Lambda^{n+1}K/\Lambda^n K \to K$ is an epimorphism. In fact, we shall characterize $\oplus_{n\in\mathbb{N}}\Lambda^{n+1}K/\Lambda^n K$ as a shuffle coalgebra. In the algebra case the corresponding statement concerning $\oplus I^n/I^{n+1}$ follows after noticing that $I = \langle f\rangle$ is not a divisor of 0 because $\{0\}$ is a prime ideal.

However, the existence of chains of coideals will be guaranteed by a stronger result, which is in fact a structure theorem for local smooth coalgebras.

Given a smooth local coalgebra C such that $\cup_n \wedge^n K = C$ (where $K = k.e$), Ω^1_C is C-free of rank $dim_k((K \wedge K)/K)$. We want to prove that if $\phi \in Com_C(C, C) = C^*$, then $\phi = 0$ or ϕ is an epimorphism (i.e. $\{0\}$ is a prime coideal). Notice that in the topological case, we implicitely assume that the equallity $\cup_n \wedge^n K = C$ includes the topology, i.e. that the topology of C is the inductive limit topology of the system $\{\cup_{n \leq m} \wedge^n K\}_{m \in \mathbb{N}}$. An example of lcal smooth coalgebra verifying this condition is the coalgebra of distributions over the real line supported at the origin. In this case the Dirac measure δ_0 is the group-like element, a basis is given by $\{\delta^{(n)}\}_{n \in \mathbb{N}_0}$ and $\wedge^j(\mathbb{C}.\delta_0) = \langle \delta_0, \delta_0', \delta_0^{(j)} \rangle$.

Lemma 7.3 *If C is local and $\Omega^1_C \cong C$, then there exists a coderivation $D : C \to C$ and an element $x \in C^*$ such that $x \circ D = \epsilon$.*

Proof: A coderivation $\widetilde{D}$ is obtained by composition of the isomorphism $C \to \Omega^1_C$ with the universal coderivation $d : \Omega^1_C \to C$. The universal property of Ω^1_C with respect to coderivations implies that $Ker(\widetilde{D})$ does not have non-trivial subcomodules, because if N is a subcomodule of $Ker(\widetilde{D})$, then, denoting by α the composition of the isomorphism $C \cong \Omega^1_C$ with the inclusion $N \subset C$, as $Com_C(N, \Omega^1_C)$ corresponds to $Coder(N, C)$ by means of d, then $\widetilde{D}|_N = d \circ \alpha = 0 \Leftrightarrow \alpha = 0$, so if $N \subseteq Ker(\widetilde{D})$ then $\alpha = 0$ and then $N = 0$.

Denoting by e the group-like element of C, as $k.e$ is a subcomodule of C, the above paragraph shows that $\widetilde{D}(e) \neq 0$. Then there exists $x \in C^*$ such that $x(\widetilde{D}(e)) \neq 0$ (i.e. $x \circ \widetilde{D} \notin (C/K)^*$). As C^* is a local algebra and $(k.e)^\perp \cong (C/K)^* \subseteq C^*$, $x \circ \widetilde{D}$ is a unit in C^*. So take $u \in C^*$ such that $u * (x \circ \widetilde{D}) = \epsilon$ and $D = \widetilde{D} \circ (u.)$. Taking into account that $\epsilon \circ \widetilde{D} = (\epsilon \otimes \epsilon)\Delta\widetilde{D} = (\epsilon \otimes \epsilon \circ \widetilde{D})\rho^- + (\epsilon \circ \widetilde{D} \otimes \epsilon)\rho^+$ then $\epsilon \circ \widetilde{D} = 0$, we have that

$$x.D(c) = D(x.c) + (x \circ D).c = \widetilde{D}(u.x.c) + (x \circ \widetilde{D}).u.c = \widetilde{D}(u.x.c) + c$$

applying ϵ we obtain $x(D(c)) = \epsilon(x.D(c)) = \epsilon(\widetilde{D}(u.x.c)) + \epsilon(c) = \epsilon(c)$. Moreover, we may suppose that $x(e) = 0$, replacing x by $x - x(e).\epsilon$ if necessary.

We remark that as D is a coderivation, the quotient $\widetilde{C} := C/Im(D)$ is also a coalgebra.

Theorem 7.4 *In the conditions of the above Lemma, $C \cong \widetilde{C} \otimes sh(k.x)$ (where $sh(k.x)$ denotes the shuffle coalgebra on the generator x, i.e. the graded dual of the polinomial ring $k[x]$).*

Proof: With the notations of the above proof, given $t \in C^*$ such that $t(e) = 0$, consider the expresion

$$E(D,t)(c) = \sum_{n \geq 0} \frac{D^n}{n!}(t^n.c)$$

We have a well defined map $E(D,t) : C \to C$ because as $C = \cup_n \Lambda^n(k.e)$, then given $c \in C$ there exists $n_0 \in \mathbb{N}$ such that $c \in \Lambda^n(k.e)$ for all $n \geq n_0$ and so the sum is finite as $t^n.c = 0$ for $n \geq n_0$.

Let us take $E : C \to C$ defined by

$$E(c) := E(D,-x)(c) = \sum_{n \geq 0} \frac{(-1)^n D^n(x^n.c)}{n!}$$

It is easy but tedious to see that $Im(E) = Ker(x.)$ (one inclusion is obvious, if $c \in Ker(x.)$ then $c = E(c)$, for the other inclusion one must use a commutation formula for $x.$ and D^n).

Now we define a map

$$\phi : C \to \widetilde{C} \otimes sh(k.x)$$

$$c \mapsto \sum_{n \geq 0} \overline{x^n.c} \otimes x^n$$

(which is a finite sum), whose inverse is given by

$$\psi : \widetilde{C} \otimes sh(k.x) \to C$$

$$\sum_{n \geq 0} \overline{c_n} \otimes x^n \mapsto \sum_{n \geq 0} \frac{D^n}{n!} E(c_n)$$

In order to see that both are coalgebra maps, it is necessary to proceed by cases, but no difficulty arises. When verifying that both compositions are the identity, one should use the following facts:

1. $\sum_{i=0}^{2k} \frac{(-1)^i}{(2k-i)!i!} = 0$
2. $x. \circ D = D \circ x. + \epsilon$
3. $E(c) - c \in Im(D)$
4. $\phi\left(\frac{D^n E(c)}{n!}\right) = \overline{c} \otimes x^n$

5. $c = \sum_{n\geq 0} \frac{D^n E(x^n.c)}{n!}$

Remarks: 1. By means of the above isomorphisms, the map $x.$ acts on $\widetilde{C}\otimes sh(k.x)$ in the following way: it is the identity on the first coordinate and it acts as a coderivation on the second one. As a consequence, it is an epimorphism.
2. The element x of the above result may be chosen as f. In order to do so, it is sufficient to prove that $f \circ D$ is a unit in C^*. But $f \circ D$ is a unit if and only if $f(D(e)) \neq 0$. Now, $f.D(e) = (f \circ D).e + D(f.e)$ as e is group-like. But $f(e) = 0$ then $f.e = 0$, so $f.D(e) = (f \circ D).e = f(D(e))e$. We want to prove then that $f.(D(e)) \neq 0$. This statement is true if and only if $D(e) \neq \lambda.e$ for any $\lambda \in k$. But if $D(e) = \lambda.e$, then $\lambda = \epsilon(D(e)) = 0$, and this is impossible as, for example, $x(D(e)) = 1$.

Proposition 7.5 *In the above proposition, $\widetilde{C}$ is isomorphic to k (in fact, $\widetilde{C}$ is identified with $k.e$).*

Proof: As $Im(D) \subset Ker(\epsilon)$, we have an obvious surjective map $\widetilde{C} = C/Im(D) \to C/Ker(\epsilon) \cong k.e$. Also, there is a map

$$k.e \to \widetilde{C}$$

$$e \mapsto \overline{e}$$

This map is not null because $E(D, -f) : C \to C$ is a coalgebra morphism whose image equals $Ker(f.) = k.e$. As $\phi(e) = \overline{e} \otimes 1 + \overline{f.e} \otimes x + \ldots = \overline{e} \otimes 1 + 0$, the fact that ϕ is an isomorphism implies that $\overline{e} \neq 0$.

The composition $k.e \to \widetilde{C} \to k.e$ is obviously the identity. Consider the other composition:

$$\overline{c} \to \epsilon(c).e \mapsto \overline{\epsilon(c).e} = \overline{E(c)} = \overline{c}$$

where the equality $\overline{\epsilon(c).e} = \overline{E(c)}$ follows using $E(c) = (\epsilon \otimes 1)\Delta(E(c))$.

The above description of C can now be used in order to obtain a description of $gr(C) = \oplus_{n\geq 0}\Lambda^{n+1}K/\Lambda^n K$. We look at the subcomodule $\Lambda^i(k.e)$ in $\widetilde{C} \otimes sh(k.x)$ via the isomorphism ϕ. For example when $i = 2$

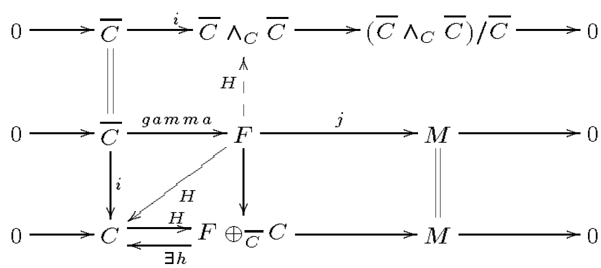
$$(k.e) \wedge (k.e) = Ker(\pi \otimes \pi) = k.e \otimes k.1 \oplus k.e \otimes k.x \cong k.e \oplus (k.e \wedge k.e)/k.e$$

in general $\Lambda^{i+1}K = (k.e \otimes k.1) \oplus (k.e \otimes k.x) \oplus ... \oplus (k.e \otimes k.x^i)$. Then $gr(C) = \oplus_{i\geq 0}(k.e \otimes k.x^i) = k.e \otimes (\oplus_{i\geq 0} k.x^i)$.

We are now able to treat the case $dim_k((K \wedge K)/K) = n = dim_k((K \wedge K)/K)^*$ and prove Theorem 6.5. Denote by $\{f_1, ..., f_n\}$ a basis of $((K \wedge K)/K)^*$. As we have $K^n \cong (K \wedge K)/K \subseteq C/K$, the restriction gives a surjection $K^\perp \cong (C/K)^* \to ((K \wedge K)/K)^*$, so we will consider $\{f_1, \ldots, f_n\}$ as elements of $K^\perp \subset C^*$. In this situation, define $\overline{C}$ as the C-subcoalgebra $\overline{C} = Ker(f_1.)$ and $\overline{K} = Ker(f_1.) \cap K = K$.

Lemma 7.6 *With the notations as above, $\overline{C}$ is a smooth coalgebra.*

Proof: Consider an extension $0 \to \overline{C} \to F \to M \to 0$, where F is a cocommutative coalgebra, M is an F-bicomodule and $F = \overline{C} \wedge_F \overline{C}$ (and then M is a C-bicomodule). We want to prove that the sequence splits. As C is smooth, the last row of the following commutative diagram has a splitting



where $F \oplus_{\overline{C}} C$ is the push-out.

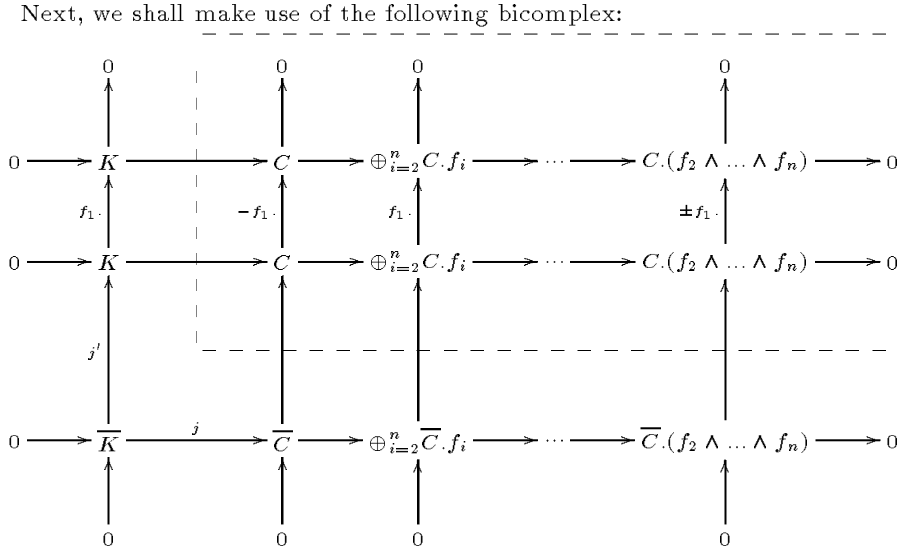
By hypothesis, $F = \{z \in F \;/\; \Delta(z) \in \gamma(\overline{C}) \otimes F + F \otimes \gamma(\overline{C})\}$. We define a map $H : F \to C$ by $H(z) = h((z, 0))$. Given a morphism of coalgebras $f : C \to D$, if $W, V \subseteq C$ then $f(V \wedge_C W) \subseteq f(V) \wedge_D f(W)$, we have that $Im(H) \subseteq \overline{C} \wedge_C \overline{C}$.

Next, we recall that $Im(E) = Im(E(D_1, -f_1)) = Ker(f_1.)$, therefore we a have map $E : \overline{C} \wedge_C \overline{C} \to \overline{C}$ and $E|_{\overline{C}} = id_{\overline{C}}$. As a consequence, the composition $E \circ H$ splits the extension $0 \to \overline{C} \to F \to M \to 0$.

Remark: $\overline{C}$ is a local coalgebra (since it is a subalgebra of C, which is local).

If we consider the coalgebra $\overline{C} = Ker(f_1.)$, then the rank of $\Omega^1_{\overline{C}} = n - 1$ and, by an inductive argument, $gr(\overline{C}) = \oplus_{i \geq 0} \Lambda^{i+1} K / \Lambda^i K \cong k.e \otimes \left(\underset{i \geq 0}{\oplus} k[x_2, ..., x_n]_i \right)$ where $k[x_2, ..., x_n]_i$ is the i^{th}-component of the shuffle coalgebra on $n-2$ generators; in fact $\overline{C} = k.e \otimes sh(k.x_2 \oplus ...k.x_n)$ and $(\overline{K} \wedge \overline{K})/\overline{K} \cong \oplus_{i=2}^n k.f_i$.

Next, we shall make use of the following bicomplex:



By hypothesis, the columns are acyclic (considering the case $n = 1$), and j induces a quasi-isomorphism between the complexes $(0 \to \overline{K} \to 0 \to ...)$ and $(0 \to \overline{C} \to \oplus_{i=2}^{n}\overline{C}.f_i \to ... \to \overline{C}(f_2 \wedge ... \wedge f_n) \to 0)$, which is, in turn, quasi-isomorphic to the total complex of the bicomplex inside the dotted area.

So we obtain that the sequence

$$0 \to \overline{K} \to \overline{C} \to \oplus_{i=1}^{n} C.f_i \to \oplus_{i<j} C(f_i \wedge f_j) \to ... \to C.(f_1 \wedge ... \wedge f_n) \to 0$$

is exact, which proves Theoren 6.5.

8 Proof of the main Theorem

The aim of this section is to calculate $Hoch^n(C)$ for a smooth coalgebra C. Since we make use of simplified resolutions, we consider two kind of situations.

Theorem 8.1 *If C is a cocommutative coalgebra satisfying one of the hypothesis below:*

- *C is a smooth algebraic coalgebra and $ke_i \wedge k.e_i$ is finite dimensional for every group-like element $e_i \in C$.*

- *C is a smooth local topological coalgebra provided of a topology verifying Proposition 1.2,*
  $C = \cup_{n\in\mathbb{N}_0}\wedge^n(k.e)$ (e being the unique group-like element of C) and $k.e\wedge k.e$ is finite dimensional.

then $Hoch^*(C)$ is isomorphic to the exterior coalgebra on Ω^1_C.

Proof: The following argument allows us, when considering an arbitrary cocommutative smooth coalgebra C, to reduce the problem to the local case. It works for "algebraic" coalgebras, but it does not work for topological coalgebras.

Given then an algebraic cocommutative smooth coalgebra C, it is well known ([16], Theorem 8.0.5, p.163) that $C = \oplus_{i\in I}C_i$ where I indexes the set of irreducible subcoalgebras of C. As C is a coalgebra over an algebraically closed field k, each C_i contains at least a group-like element e_i, and it cannot contain two of them due to the irreducibility of C_i. Then each C_i is local.

Let us denote by Λ a set indexing the finite subsets of I and by $\{I_\alpha\}_{\alpha\in\Lambda}$ the set of all finite subsets of I. Then $C = \varinjlim_{\Lambda} \oplus_{i\in I_\alpha}C_i$, and as $Hoch^*$ commutes with direct limits, assuming the theorem for the local case,

$$Hoch^*(C) = \varinjlim_{\Lambda} Hoch^*(\oplus_{i\in I_\alpha}C_i) \cong \varinjlim_{\Lambda} \oplus_{i\in I_\alpha} Hoch^*(C_i) \cong$$

$$\cong \oplus_{i\in I} Hoch^*(C_i) \cong \oplus_{i\in I}\Lambda^*(\Omega^1_{C_i}) \cong$$

$$\cong \Lambda^*(\oplus_{i\in I}\Omega^1_{C_i}) \cong \Lambda^*(\Omega^1_C)$$

where the second equality follows from the Mayer-Vietoris property of $Hoch^*$ (see [10]).

In the local smooth case (for both situations), the existence of a simple resolution allows us to compute in an easy way the cohomology $Hoch^n(C)$. We know that $Hoch^n(C) = Cotor^n_{C^e}(C, C)$ and that C^e is a smooth local coalgebra (see Lemma 5.11). Let $\widetilde{e}$ be the unique group-like element in C^e (in fact $\widetilde{e} = e\otimes e = \Delta(e)$, where e is the unique group-like element of C). We have that $\Delta : C \to C^e$ is a monomorphism of coalgebras and, as C and C^e are smooth, then $C \cong \Delta(C) = \cap_{i=1}^n Ker(f_i.)$, $\cup_j \Lambda^j C = C^e$, for a certain "regular sequence" $\{f_1, ..., f_n\}$ in the sense that

$$0 \to C \to C^e \to \oplus_{i=1}^n C^e.f_i \to \oplus_{i<j} C^e(f_i\wedge f_j) \to ... \to C^e.(f_1\wedge ...\wedge f_n) \to 0$$

is an exact sequence. Then $Cotor^*_{C^e}(C,C) =$

$$= H^*(0 \to C \to \oplus_{i=1}^n C.f_i \to \oplus_{i<j} C(f_i \wedge f_j) \to ... \to C.(f_1 \wedge ... \wedge f_n) \to 0)$$

Using an analogue of the Künneth formula for coalgebras (see [7]) this is the cotensor product of the cohomology of the complexes

$$0 \longrightarrow C \xrightarrow{f_i .} C \longrightarrow 0$$

In other words, $Cotor^*_{C^e}(C,C)$ is a graded coalgebra isomorphic to the exterior coalgebra on $Cotor^1_{C^e}(C,C)$. Since one always has an isomorphism $Cotor^1_{C^e}(C,C) = Hoch^1(C) \cong \Omega^1_C$, the proof is complete.

Conjecture: The reciprocal statement to the Hochschild - Kostant - Rosemberg theorem proved by [3] and [2] suggests that in the coalgebra case the fact of being smooth is equivalent to the above isomorphism.

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