THE EXPRESSION OF THE LAPLACE TRANSFORM OF $(P \pm i0)^\lambda$ AS A K-TRANSFORM.

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Abstract - In this Note we obtain the Laplace transform (I,1) of the distributions $(P\pm i0)^\lambda$ ((II,2)) as a K_ν-transform of order $\nu=\frac{n-2}{2}$ (cf. Theorem 4, formula (III,17)).

I. Introduction.

We begin with some definitions. Let $t=(t_0,t_1,\ldots,t_{n-1})$ be a point of $I\!R^n$. We shall write $t_0^2-t_1^2-\ldots-t_{n-1}^2=u$. By Γ_+ we designate the interior of the forward cone: $\Gamma_+=\{t\in I\!R^n/t_0>0,u>0\}$; and by $\overline{\Gamma}_+$ we designate its closure. Similarly, Γ_- designates the domain $\Gamma_-=\{t\in I\!R^n/t_0<0,u>0\}$, and $\overline{\Gamma}_-$ designates its closure. We put $z=(z_0,z_1,\ldots,z_{n-1})\in \mathbb{C}^n$, where $z_\nu=x_\nu+iy_\nu, \nu=0,1,\ldots,n-1$; $<t,z>=t_0z_0+t_1z_1+\ldots+t_{n-1}z_{n-1}$ and $dt=dt_0dt_1\ldots dt_{n-1}$. The tube T_- is defined by $T_-=\{z\in\mathbb{C}^n/y\in V_-\}$ where $V_-=\{y\in I\!R^n/y_0<0,y_0^2-y_1^2-\ldots-y_{n-1}^2>0\}$.

The Laplace transform of $\phi(t)$ is

$$f(z)=\mathcal{L}\{\phi\}=\int_{I\!R^n} e^{-i<t,z>}\phi(t)dt\ . \qquad (\mathrm{I},1)$$

Let $F(\lambda)$ be a function of the scalar variable λ, and let $\phi(t)$ be a function endowed with the following properties:

(a) $\phi(t)=F(u)$,

(b) $\operatorname{supp}\phi(t) \subset \overline{\Gamma}_+$,

(c) $e^{<t,y>}\phi(t) \in L$ if $y \in V_-$.

We call $\mathcal{R}$ the family of functions $\phi(t)$ which satisfies (a), (b) and (c). Similarly, we call $\mathcal{A}$ the family of functions which satisfies conditions

(a') $\phi(t) = F(u)$,

(b') $\operatorname{supp}\phi(t) \in \overline{\Gamma}_-$,

(c') $e^{<t,y>}\phi(t) \in L_1$ if $y \in V_+$.

We can state the following Theorem ([1], p.53, form. (I,2;1)).

Theorem 1.

Hypothesis.

(α) $\phi(t) \in \mathcal{R}$,

(β) $z \in T_-$.

Thesis.

$$f(z) = \mathcal{L}\{\phi\} = \frac{(2\pi)^{\frac{n-2}{2}}}{\left(z_1^2 + \ldots + z_{n-1}^2 - z_0^2\right)^{\frac{n-2}{4}}} \cdot \int_0^\infty F(\lambda)\lambda^{\frac{n-2}{4}} K_{\frac{n-2}{2}} \left\{\left[\lambda\left(z_1^2 + \ldots + z_{n-1}^2 - z_0^2\right)\right]^{1/2}\right\} d\lambda \ . \qquad (\mathrm{I}, 2)$$

Here $K_\nu(z)$ designates the modified Bessel function of the third kind ([2], vol.II, p.427).

II. The distribution $(P \pm i0)^\lambda$.

Let $t = (t_1, t_2, \ldots, t_n)$ be a point of the n- dimensional Euclidean space $I\!R^n$. Consider a non-degenerate quadratic form

$$P = P(t) = t_1^2 + \ldots + t_p^2 - t_{p+1}^2 - \ldots - t_{p+q}^2 \ , \qquad (\mathrm{II}, 1)$$

where $n = p + q$. The distribution $(P \pm i0)^\lambda$ is defined by

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{P \pm i\varepsilon|t|^2\}^\lambda \ , \qquad (\mathrm{II}, 2)$$

where $\varepsilon > 0$, $|t|^2 = t_1^2 + \ldots + t_n^2$, $\lambda \in \mathbb{C}$.

The distributions $(P \pm i0)^\lambda$ are an important contribution of Gelfand ([3], p.274). The distributions $(P \pm i0)^\lambda$ are analytic in λ everywhere except at $\lambda = -\frac{n}{2} - k$, $k = 0, 1, \ldots$, where they have simple poles ([3], p.275). But, we can obtain the expression of $(P \pm i0)^\lambda$, when $\lambda = -\frac{n}{2} - k$, $k = 0, 1, \ldots$; this is, the $Pf(P \pm i0)^{-\frac{n}{2}-k}$, Pf indicates the Finite Part ([4]). Our formula permits to give a sense, in a natural way, to some of the so called "infinities" of quantum electrodynamics. (See, for ex. [5], p.95, form. (IV,8;1)).

It is useful to state an equivalent definition of the distributions $(P \pm i0)^\lambda$.

In this definition appears the distributions

$$P_+^\lambda = \begin{cases} P^\lambda & \text{if } P \geq 0, \\ 0 & \text{if } P \leq 0; \end{cases} \tag{II,3}$$

$$P_-^\lambda = \begin{cases} 0 & \text{if } P \geq 0, \\ (-P)^\lambda & \text{if } P \leq 0; \end{cases} \tag{II,4}$$

([3], Ch.III, p.276).

We can prove, without difficulty, that the following formula is valid ([3], Ch.III, p.276)

$$(P \pm i0)^\lambda = P_+^\lambda + e^{\pm i\pi\lambda} P_-^\lambda \ . \tag{II,5}$$

From this formula we conclude immediately that

$$(P + i0)^\lambda = (P - i0)^\lambda = P^\lambda \ . \tag{II,6}$$

when $\lambda = k =$ positive integer.

III. The expression of the Laplace transform of $(P \pm i0)^\lambda$ as a K-transform.

We put, by definition,

$$(P_+^\lambda, t_1 > 0) = \begin{cases} P^\lambda & \text{if } P \geq 0, t_1 > 0 \\ 0 & \text{if } P \leq 0; \end{cases} \tag{III,1}$$

and

$$(P_-^\lambda, t_1 > 0) = \begin{cases} 0 & \text{if } P > 0, \\ (-P)^\lambda & \text{if } P \leq 0, t_1 > 0. \end{cases} \tag{III,2}$$

Therefore, from (II,5), (III,1) and (III,2), we have

$$\left\{(P \pm i0)^{\lambda}, t_1 > 0\right\} = (P_+^{\lambda}, t_1 > 0) + e^{\pm i\pi\lambda}(P_-^{\lambda}, t_1 > 0)\ . \qquad \text{(III, 3)}$$

Analogously, we define

$$(P_+^{\lambda}, t_1 < 0) = \begin{cases} P^{\lambda} & \text{if } P \geq 0, t_1 < 0, \\ 0 & \text{if } P < 0; \end{cases} \qquad \text{(III, 4)}$$

and

$$(P_-^{\lambda}, t_1 < 0) = \begin{cases} 0 & \text{if } P > 0 \\ (-P)^{\lambda} & \text{if } P \leq 0, t_1 < 0. \end{cases} \qquad \text{(III, 5)}$$

From (III,4), (III,5) and (II,5), we obtain

$$\{(P \pm i0)^{\lambda}, t_1 < 0\} = (P_+^{\lambda}, t_1 < 0) + e^{\pm i\pi\lambda}(P_-^{\lambda}, t_1 < 0)\ . \qquad \text{(III, 6)}$$

From (III,3) and (III,6), we obtain

$$\begin{aligned}(P \pm i0)^{\lambda} = \Big\{ & \left[(P_+^{\lambda}, t_1 > 0) + (P_+^{\lambda}, t_1 < 0)\right] \\ & + e^{\pm i\pi\lambda}\left[(P_-^{\lambda}, t_1 > 0) + (P_-^{\lambda}, t_1 < 0)\right] \Big\}\ .\end{aligned} \qquad \text{(III, 7)}$$

Equivalently, we can write,

$$\begin{aligned}(P \pm i0)^{\lambda} = \Big\{ & \left[(P_+^{\lambda}, t_1 > 0) + e^{\pm i\pi\lambda}(P_-^{\lambda}, t_1 > 0)\right] \\ & + \left[(P_+^{\lambda}, t_1 < 0) + e^{\pm i\pi\lambda}(P_-^{\lambda}, t_1 < 0)\right] \Big\}\ .\end{aligned} \qquad \text{(III, 8)}$$

Appealing to the linearity of the Laplace transform, we have

$$\begin{aligned}\mathcal{L}\left[(P \pm i0)^{\lambda}\right] = \mathcal{L}\Big\{ & \left[(P_+^{\lambda}, t_1 > 0) + e^{\pm i\pi\lambda}(P_-^{\lambda}, t_1 > 0)\right] \\ & + \mathcal{L}\left[(P_+^{\lambda}, t_1 < 0) + e^{\pm i\pi\lambda}(P_-^{\lambda}, t_1 < 0)\right] \Big\}.\end{aligned} \qquad \text{(III, 9)}$$

Remembering the definition (III,1), the function $(P_+^{\lambda}, t_1 > 0)$ is a retarded-function, then we can apply the formula (I,2); that is,

$$\begin{aligned}\mathcal{L}[(P_+^{\lambda}, t_1 > 0)] = & \frac{(2\pi)^{\frac{n-2}{2}}}{(z_1^2 + \ldots + z_{n-1}^2 - z_0^2)^{\frac{n-2}{4}}} \\ & \cdot \int_0^{\infty} (P_+^{\lambda}, t_1 > 0) t^{\frac{n-2}{4}} K_{\frac{n-2}{2}} \left\{\left[t\left(z_1^2 + \ldots + z_{n-1}^2 - z_0^2\right)\right]^{1/2}\right\} dt.\end{aligned} \qquad \text{(III, 10)}$$

Analogously, we obtain

$$\mathcal{L}[(P_-^\lambda, t_1 > 0)] = \frac{(2\pi)^{\frac{n-2}{2}}}{(z_1^2 + \ldots + z_{n-1}^2 - z_0^2)^{\frac{n-2}{4}}} \cdot \int_0^\infty (P_-^\lambda, t_1 > 0) t^{\frac{n-2}{2}} K_{\frac{n-2}{2}} \left\{ \left[t \left(z_1^2 + \ldots + z_{n-1}^2 - z_0^2 \right) \right]^{1/2} \right\} dt. \tag{III, 11}$$

Similarly, we have

$$\mathcal{L}[(P_+^\lambda, t_1 < 0)] = \frac{(2\pi)^{\frac{n-2}{2}}}{(z_1^2 + \ldots + z_{n-1}^2 - z_0^2)^{\frac{n-2}{4}}} \cdot \int_0^\infty (P_+^\lambda, t_1 < 0) t^{\frac{n-2}{4}} K_{\frac{n-2}{2}} \left\{ \left[t \left(z_1^2 + \ldots + z_{n-1}^2 - z_0^2 \right) \right]^{1/2} \right\} dt \tag{III, 12}$$

and

$$\mathcal{L}[(P_-^\lambda, t_1 < 0)] = \frac{(2\pi)^{\frac{n-2}{2}}}{(z_1^2 + \ldots + z_{n-1}^2 - z_0^2)^{\frac{n-2}{4}}} \cdot \int_0^\infty (P_-^\lambda, t_1 < 0) t^{\frac{n-2}{2}} K_{\frac{n-2}{2}} \left\{ \left[t \left(z_1^2 + \ldots + z_{n-1}^2 - z_0^2 \right) \right]^{1/2} \right\} dt. \tag{III, 13}$$

From (III,9), (III,10), (III,11), (III,12) and (III,13), we arrive at

$$\begin{aligned} \mathcal{L}[(P \pm i0)^\lambda] = {} & \frac{(2\pi)^{\frac{n-2}{2}}}{(z_1^2 + \ldots + z_{n-1}^2 - z_0^2)^{\frac{n-2}{4}}} \\ & \cdot \left\{ \int_0^\infty \left[(P_+^\lambda, t_1 > 0) + e^{\pm i\pi\lambda} (P_-^\lambda, t_1 > 0) \right] \right. \\ & \cdot t^{\frac{n-2}{4}} K_{\frac{n-2}{2}} \left\{ \left[t \left(z_1^2 + \ldots + z_{n-1}^2 - z_0^2 \right) \right]^{1/2} \right\} dt \\ & + \int_0^\infty \left[(P_+^\lambda, t_1 < 0) + e^{\pm i\pi\lambda} (P_-^\lambda, t_1 < 0) \right] \\ & \left. \cdot t^{\frac{n-2}{4}} K_{\frac{n-2}{2}} \left\{ \left[t \left(z_1^2 + \ldots + z_{n-1}^2 - z_0^2 \right) \right]^{1/2} \right\} dt \right\}. \end{aligned} \tag{III, 14}$$

Equivalently, we can write,

$$\mathcal{L}[(P \pm i0)^{\lambda}] = \frac{(2\pi)^{\frac{n-2}{2}}}{(z_1^2 + \ldots + z_{n-1}^2 - z_0^2)^{\frac{n-2}{4}}}$$
$$\cdot \left\{ \int_0^{\infty} \left[(P_+^{\lambda}, t_1 > 0 + P_+^{\lambda}, t_1 < 0) + e^{\pm i\pi\lambda} (P_-^{\lambda}, t_1 > 0 + P_-^{\lambda}, t_1 < 0) \right] \right.$$
$$\left. \cdot\, t^{\frac{n-2}{4}} K_{\frac{n-2}{2}} \left\{ \left[t \left(z_1^2 + \ldots + z_{n-1}^2 - z_0^2 \right) \right]^{1/2} \right\} dt \right\}. \tag{III, 15}$$

Taking into account the Hypothesis (β) of the Theorem 1, we can write more explicit the formula (III,15), that is,

$$\mathcal{L}[(P \pm i0)^{\lambda}] = \frac{(2\pi)^{\frac{n-2}{2}}}{(z_1^2 + \ldots + z_{n-1}^2 - z_0^2)^{\frac{n-2}{4}}}$$
$$\cdot \left\{ \int_0^{\infty} \left\{ \left[(P_+^{\lambda}, t_1 > 0, y_1 < 0) + (P_+^{\lambda}, t_1 < 0, y_1 > 0) \right] \right. \right. \tag{III, 16}$$
$$\left. + e^{\pm i\pi\lambda} \left[(P_-^{\lambda}, t_1 > 0, y_1 < 0) + (P_-^{\lambda}, t_1 < 0, y_1 > 0) \right] \right\}$$
$$\left. \cdot\, t^{\frac{n-2}{4}} K_{\frac{n-2}{2}} \left\{ \left[t \left(z_1^2 + \ldots + z_{n-1}^2 - z_0^2 \right) \right]^{1/2} \right\} \right\} dt.$$

Finally, we can establish the following

Theorem 2.

Hypothesis

(α) Let $(P \pm i0)^{\lambda}$ be defined by (II,2)

(β) The hypothesis of the Theorem 1 are valid.

Thesis

The following formula is valid

$$\begin{aligned}\mathcal{L}[(P \pm i0)^{\lambda}] = & \frac{(2\pi)^{\frac{n-2}{2}}}{(z_1^2 + \ldots + z_{n-1}^2 - z_0^2)^{\frac{n-2}{4}}} \\ & \cdot \left\{ \int_0^{\infty} \left\{ \left[(P_+^{\lambda}, t_1 > 0, y_1 < 0) + (P_+^{\lambda}, t_1 < 0, y_1 > 0)\right] \right.\right. \\ & + e^{\pm i\pi\lambda} \left[(P_-^{\lambda}, t_1 > 0, y_1 < 0) + (P_-^{\lambda}, t_1 < 0, y_1 > 0)\right] \Big\} \\ & \left. \cdot\, t^{\frac{n-2}{4}} K_{\frac{n-2}{2}} \left\{ \left[t\left(z_1^2 + \ldots + z_{n-1}^2 - z_0^2\right)\right]^{1/2} \right\} dt \right\} .\end{aligned} \tag{III, 17}$$

Putting in the formula (III,17),

$$t = x^2 \ , \tag{III, 18}$$

and

$$z_1^2 + \ldots + z_{n-1}^2 - z_0^2 = y^2 \ , \tag{III, 19}$$

and remembering ([2], vol.2, p.125) the expression of the K_ν-transform of $f(x)$:

$$K_\nu(y) = \int_0^{\infty} f(x) K_\nu(xy)(xy)^{1/2} dx \ , \tag{III, 20}$$

the thesis of the Theorem 2 expresses that the Laplace transform of $(P \pm i0)^{\lambda}$ can be obtained by means of a K_ν-transform of the order $\nu = \frac{n-2}{2}$.

References.

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