Weighted eigenfunctions and Gauss curvature of conical revolution surfaces

E. LAMI DOZO and B. TOUKOUROU

Abstract

We give description of Gauss curvatures in revolution surfaces with conical singularities at the extreme opposite points thanks to positive eigenfunctions of an eigenvalue problem in dimension one with a prescribed singular weight.

1 Introduction

Given a revolution surface

$$S = \{(\alpha(v)\cos u, \alpha(v)\sin u, \beta(v) \,|; 0 < u < 2\pi,\ a < v < b\} \tag{1}$$

where $\alpha(v) > 0, \alpha, \beta$ regular functions and supposing the generating curve $\gamma = (\alpha(v), 0, \beta(v))$ parametrized by arc-length, that is

$$\alpha'^2 + \beta'^2 = 1 \text{ in }]a, b[$$

Then the Gauss curvature K of S is given by

$$K = \frac{-\alpha''(v)}{\alpha(v)},\ v \in]a, b[\tag{2}$$

[DC, p. 162].

If α, β are regular up to $[a, b]$ and

$$\begin{cases} \alpha(a) = \alpha(b) = 0,\ 0 < \alpha'(a) \leq 1,\ -1 \leq \alpha'(b) < 0 \\ \beta(a) < \beta(b) \end{cases} \tag{3}$$

the surface S will have conical singularities at $P_a = (0, 0, \beta(a))$ and $P_b = (0, 0, \beta(b))$ with angles θ_a and θ_b determined by

$$(\cos\theta_a, \sin\theta_a) = (\beta'(a), \alpha'(a)), (\cos\theta_b, \sin\theta_b) = (-\beta'(b), -\alpha'(b))$$

where θ_a, θ_b are in $]0, \pi[$.

We describe the family of K's looking for solutions of the following boundary problem

$$\begin{cases} \alpha'' + \lambda g(v)\alpha = 0 \text{ in }]a,b[\\ \alpha(a) = \alpha(b) = 0 \\ \alpha(v) > 0 \text{ in }]a,b[\end{cases} \tag{4}$$

which are in $C^2(]a,b[) \cap C^1([a,b])$ and satisfy $\alpha'(a) > 0 > \alpha'(b)$. We do so because for a given weight $g(v)$, in the half-line $\{tg; t > 0\}$ there will be at most one $K = \lambda g$. Functions g will be allowed to have singularities at a and b like simple poles (if they were analytic) by considering natural examples. The main result on (4) for such a g is supplementary to those on the subject found in [DF, M-M].

In [T], M. Troyanov fixes a Riemann surface with a metric ds_0^2 having prescribed conical singularities at a prescribed finite numbers of points and gives rather complete results on the Gauss curvatures on metrics ds^2 conformal to ds_0^2, (i.e. $ds^2 = e^{2f} ds_0^2$). Here we describe curvatures associated to warped singular metrics $\alpha^2(v)du^2 + dv^2$ on $]a,b[\times S^1$ which are not conformally equivalent.

In §2 we give a pointwise necessary condition on K, additional to the integral-ones given in [E-T] and examples motivating the conditions on g which appear in the result on (4) in §3. Finally, reconstructing surfaces S having the same curvature λ_g associated to $\{s\alpha_g; 0 < 1 \leq 1\}$ where $||\alpha_g'||_\alpha = \Lambda$ is indicated.

2 Necessary condition, examples

The area element of S is $dA = \alpha(v)\, du\, dv$. A curvature K given by (2) satisfies [cf E-T],

$$\int_S K dA = 2\pi(\alpha'(a) - \alpha'(b)) > 0 \tag{5}$$

and

$$\int_S K' dA = -2\pi(\alpha'(a) + \alpha'(b))(\alpha'(a) - \alpha'(b)) \tag{6}$$

(5) implies that K is positive somewhere. A pointwise necessary condition, independent of α, is given by Barta's inequality [B]

$$\sup_{]a,b[} K \geq \left(\frac{\pi}{b-a}\right)^2 \geq \inf_{]a,b} K \tag{7}$$

For $\varphi_1 = \sin \dfrac{\pi v}{b-a}$, we have

$$\int_S [K - \left(\frac{\pi}{b-a}\right)^2] \varphi_1\, dA = 0 \tag{8}$$

integrating $K\alpha\varphi_1 = -\alpha''$ by parts. If we have one equality in (7), we deduce from (8) that $K \equiv \left(\frac{\pi}{b-a}\right)^2$ and $\alpha = \varphi_1$ is a solution which is unique modulo a normalization. We remark that $\left(\frac{\pi}{b-a}\right)^2$ is the only positive constant curvature. This uniqueness of K in $\{tK; t > 0\}$ will also hold for non constant K's.

The following examples are characteristic of the type of curvatures we will prescribe.

Example 1. (Small circle). The curve

$$\gamma_s = (\sin v - \sin\delta, 0, -\cos v + \cos\delta),\ v \in [\delta, \pi - \delta]$$

where $0 < \delta < \frac{\pi}{2}$, describes a circular arc of length $\pi - 2\delta$ parametrized by arc-length. The corresponding surface S_s has conical singularities with same angle at $(0, 0, 0)$ and $(0, 0, 2\cos\delta)$. The curvature

$$K_s = \frac{\sin v}{\sin v - \sin\delta}, \quad]\delta, \pi - \delta[$$

satisfies $K_p(v) > 0$ and has simple poles at δ and $\pi - \delta$.

Example 2. (Big circle)

$$\gamma_b = (\sin v + \sin\delta, 0, \cos\delta - \cos v),\ v \in [-\delta, \pi + \delta]$$

describes the complementary circular arc of length to γ_s. The surfaces S_b has singularities at the same points than S_s with complementary angles to those of S_s. The Gauss curvature of S_b is

$$K_b = \frac{\sin v}{\sin v + \sin\delta},\ v \in [-\delta, \pi + S[.$$

K_p changes sign at $v = 0$ and $v = \pi$ and has also simple poles at $-\delta$ and $\pi + \delta$.

3 A sufficient condition

Taking into account the examples in §2 we introduce a condition on g to obtain a positive eigenfunction α of (4) in the Sobolev space $H_0^1(]a, b[)$. We proced as in [M-M,DF], consequently we only detail the differences in our proof.

Theorem 3.1 *Let $g \in C(]a, b[)$ be such that*

$$d_a = \lim_{v \to a^+} (v - a) g(v),\ d_b = \lim_{v \to b^-} (b - v) g(v) \tag{9}$$

exist. If g is positive at one point, then there is a unique positive λ such that (4) has a solution $\alpha \in H_0^1(]a,b[)$. Moreover $\alpha \in C^1([a,b])$ and if $d_\alpha d_b \neq 0$ we have $\alpha'(a) > 0 > \alpha'(b)$.

Proof. We may suppose $[a,b] = [0,L]$. Let $\varphi \in C_c^1(]0,L[)$ and $\psi \in C_0^1([a,b])$, i.e. $\psi \in C^1([a,b])$, $\psi(0) = \psi(L) = 0$. From

$$\int_0^L g\psi\varphi\,dv = \int_0^{L/2} vg(v)\left(\frac{1}{v}\int_0^v \psi'(t)\,dt\right)\varphi(v)\,dv + \int_0^{L/2}(L-v)g(v)\left(\frac{1}{L-v}\int_{L-v}^L \psi'(t)\,dt\right)\varphi(v)\,dv. \quad (10)$$

and from Hardy's inequality : $||\frac{1}{v}\int_0^v w(t)\,dt||_{L^2(\mathbb{R}_+)} \leq 2||w||_{L^2(\mathbb{R}_+)}$ applied to ψ extended by 0 out of [0,L] and to $\tilde{\varphi}(v) = \psi(L-v)$ on [0,L] also extended by 0 to $\mathbb{R}_+$, we deduce

$$|\int_0^L g\psi\varphi dv| \leq ||vg(v)\varphi(v)||_{L^2}2||\psi'||_{L^2} + ||(L-v)g(v)\varphi(v)||_{L^2}2||\psi'||_{L^2}$$

From (10) we obtain

$$|\int_0^L g\psi\varphi\,dv| \leq M||\varphi||_{L^2}||\psi'||_{L^2},\ \varphi \in C_c^1(]0,L[),\ \psi \in C_0^1([0,L]) \quad (11)$$

where $M = 2(||vg(v)||_\infty + ||(L-v)g(v)||_\infty)$.

As in [M-M,DF] the map $\varphi \to T_\varphi : H_0^1(]0,L[) \to H_0^1(]0,L[)$ defined by

$$\int_0^L (T_\varphi)'\psi'\,dv = \int_0^L g\varphi\psi\,dv,\ \psi \in H_0^1(]0,L[)$$

is then linear, compact and symmetric for the scalar product $\int_0^L \varphi'\psi'\,dv$ in $H_0^1(]0,l[)$. The hypothesis $g(v_0) > 0$ for some $v_0 \in]0,L[$ gives that the eigenvalues $\lambda \geq 0$ of (4) form a sequence $0 < \lambda_k < \lambda_{k+1}$, $k = 1,2,\ldots$ with $\lim_{k\to\infty} \lambda_k = \infty$. The first one λ_1 called principal eigenvalue is simple and is the only λ_k with a positive eigenfunction $\varphi_1(v) \underset{\neq}{>} 0$ on $]0,L[$. Besides $\varphi_1(v) > 0$ on $]0,L[$ and $\varphi_1 \in C_0([0,L])$.

The type of singularity of $K(v) \equiv \lambda_1 g(v)$ at $v = 0$ implies for $\alpha = \varphi_1$ that $\lim_{v\to 0^+} v\alpha''(v) = -\lim K(v)\alpha(v) = 0$, so $v\alpha''(v) \in C([0,\frac{L}{2}]$ and $va''(v) = h'(v)$ on

$]0,L[$, where $h(v)=v\alpha'(v)-\alpha(v)$. Hence $\lim_{v\to 0^+}\dfrac{h(v)}{v}=\lim_{v\to 0^+}\alpha'(v)-\dfrac{\alpha(v)}{v}=0$ also, so $\alpha'(0^+)\equiv\alpha'(0)$ exists. Analogously $\alpha'(L^-)\equiv\alpha'(L)$ exists and $\alpha\in C_0^1([a,b])$. Finally $\lambda=\lambda_1$, $\alpha=\varphi_1$ is our solution.

If $d_a\equiv\lim_{v\to 0^+} vg(v)\neq 0$, we have a $\delta>0$ such that $g(v)>0$ on $]0,\delta[$ and $vg(v)$ is continuous and bounded on $]0,\delta[$ and $\alpha''+\lambda g\alpha=0$ on $]0,\delta[$ with $-\alpha$ having a as maximum value attained at 0. These four properties and a well adapted maximum principle for (9) [P-W, Th. 4, p. 7] insure $\alpha'(0)>0$. Also $\alpha'(L)<0$ follows. Q. E. D.

Remark 3.2 The existence of λ and α holds if $(v-a)g(v)(b-v)$ is bounded in $]a,b[$. Conditions (9) with $d_ad_b\neq 0$ are meaningful for $g(v)$ unbounded. If $g(v)$ is bounded (so $d_a=d_b=0$), from $g=g^+-g^-$ and $-\alpha''-\lambda g^+\alpha=\lambda g^-\alpha$ we have $\alpha'(a)>0>\alpha'(b)$. [DF, Th. 1.17].

4 Building S

Given g fulfilling the hypothesis of the preceding theorem, there is only one $\alpha=\alpha_g\in C^2(]a,b[)\cap C^1([a,b])$ such that $||\alpha_g'||_\infty=1$, $\alpha(v)>0$.

If $g(v)>0$ on $]a,b[$, then $K(v)=\lambda g(v)>0$ and $-\alpha''(v)=K(v)\alpha(v)>0$ also. Hence α is concave on $[a,b]$ and

$$\lim_{v\to a^+}K(v)\alpha(v)=\lim_{v\to a^+}K(v)(v-a)\frac{a(v)}{v-a}=\lambda d_a\alpha'(a)$$

implies $\alpha\in C^2([a,b])$ and α' strictly decreasing in $[a,b]$ with $||\alpha'||_\infty=1$. If $0<\alpha'(a)<1$ we deduce $\alpha'(b)=-1$. Defining

$$\beta(x)=\int_a^v(1-\alpha'(t)^2)^{1/2}\,dt, \tag{12}$$

the surface generated by $(\alpha,0,\beta)$ will have a conical singularity at $(0,0,0)$ with angle $\theta_a\in]0,\frac{\pi}{2}[$ and of angle $\theta_b=\frac{\pi}{2}$ at $(0,0,\beta(b))$ *i.e.* no singularity. If we consider $\rho\alpha$, $0<\rho<1$, (12) gives β_ρ and we obtain a family of surfaces S_ρ with conical singularities with the same curvature $K(v)=\lambda g(v)$. If $\alpha'(a)=-\alpha'(b)=1$, S will have no singularities, however S_ρ will do have.

Two examples illustrating this case are $g\equiv 1$ on $[c,b]=[0,L]$, then $\lambda=\left(\frac{\pi}{L}\right)^2$, $K=\left(\frac{\pi}{L}\right)^2$ and $\alpha_1=\frac{L}{\pi}\sin\frac{\pi}{L}v$ satisfies $\alpha_1'(0)=-\alpha'(L)=1$. The surface S is a sphere of radius $\frac{L}{\pi}$. The other example is $g(v) = [v(L-v)]^{-1}$ on $]0, L[$, then $\lambda = 2$ and $\alpha_g = \frac{1}{L} v(L-v)$.

Finally, if g changes sign a finite number of times (hence K, as in the "big circle") that is if g has a finite number of zeros in $[a, b[$ and at each zero v_0, we have $g'(v_0) \neq 0$, then $\alpha_g'' = \lambda g \alpha_g$ has the same zeros, so $|\alpha_g'(v)| = 1$ has a finite number of solutions. For the associated partition of $]a, b[$, α will be successively convex then concave or vice-versa on the contiguous subintervals. A convenient choice of the sign in $\pm(1 - \alpha'(t)^2)^{1/2}$ at each subinterval and (12) define β and we obtain $S = S_g$ of class C^1.

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Enrique LAMI DOZO
CONICET - Univ. de Buenos Aires and
Univ. Libre de Bruxelles.

Bouraïma TOUKOUROU
Univ. Nationale du Bénin

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