STOCHASTIC DIFFERENTIAL GEOMETRY AND RANDOM DIFFEOMORPHIMS AS INTEGRALS OF THE NAVIER-STOKES EQUATIONS

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Summary: In this article we integrate in closed form, the Navier- Stokes equation for an incompressible fluid on a compact manifold which is isometrically immersed in Euclidean space. We carry out this integration through the application of the methods of Stochastic Differential Geometry, i.e. the theory of diffusion processes on smooth manifolds. Thus we start by defining the invariant infinitesimal generators of diffusion processes of differential forms on smooth compact manifolds, in terms of the laplacians (on differential forms) associated with the Riemann-Cartan-Weyl (RCW) connections. These geometries have a torsion tensor which reduces to a trace 1-form, whose conjugate vector field is the drift of the diffusion of scalar fields. We construct the diffusion processes of differential forms associated with these laplacians by using the property that the solution flow of the stochastic differential equation corresponding to the scalar diffusion is -under Holder regularity conditions- a (random) diffeomorphism of the manifold. We apply these constructions to give a new characterization of the Navier-Stokes equation for the velocity one-form of an incompressible fluid as a non-linear diffusion process determined by a RCW connection. We prove this equation to be equivalent to a linear diffusion equation for the vorticity and the Poisson-de Rham equation for the velocity with the vorticity as a source. We give the invariant random stochastic differential equations for the position (as a Lagrangian representation) of the fluid particles and thus obtain a random diffeomorphism which is a solution of the Navier-Stokes equation. We solve the Cauchy problem for the heat equation for the vorticity two-form and the Dirichlet problem for the Poisson-de Rham equation for the velocity one-form. We discuss the regularity, existance and uniqueness of the solutions. We discuss the relation between our setting with the description of turbulence as random motion of dislocations.

Keywords: incompressible fluid, diffusion processes on smooth manifolds, Riemann-Cartan-Weyl connections, vorticity, Ito formula for differential forms, stochastic differential equations, heat equation, Poisson-de Rham equation.

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1 Introduction.

The purpose of this article is to integrate the Navier-Stokes equation for an incompressible fluid on a compact orientable smooth manifold which is isometrically embedded in Euclidean space. The method of integration we shall apply for this purpose stems from Stochastic Differential Geometry, i.e. the theory of Brownian processes in smooth manifolds developed in the pioneering works by Ito [15], Elworthy and Eells [13], P. Malliavin [11] and further elaborated by Elworthy [12], Ikeda and Watanabe [14], P. Meyer [40], and Rogers and Williams [37].

The analysis of the velocimetry signal of a turbulent fluid shows that its velocity is a random variable, even though that the dynamics is ruled by the deterministic Navier-Stokes equation [8]. The concept of a turbulent fluid as a stochastic process was first proposed by Reynolds [16], who decomposed the velocity into mean velocity plus fluctuations. The Reynolds approach is currently used in most numerical simulations of turbulent fluids in spite of the fact that it leads to unsurmountable non-closure problems of the transport equations.; see Lumley [18], Mollo and Christiansen [38]. Furthermore, the Reynolds decomposition is non invariant alike as the usual decomposition into drift and white noise perturbation in the non-invariant theory of diffusion processes. Other treatments of stochasticity in turbulence were advanced from the point of view of Feynman path integrals, as initiated by Monin and Yaglom [17]. From the point of view of diffusion processes, invariant measures for modifications of the Navier-Stokes equations on euclidean domains, has been constructed by Vishik and Fursikov [36] and Cruzeiro and Albeverio [42]. (It is important to remark that the existance of an invariant measure for NS as a *classical* dynamical system is the starting point of the classical dynamical systems approach to turbulence; see Ruelle [48].) Contemporary investigations develop the relations between randomness and the many-scale structure of turbulence which stems from the Kolmogorov theory as presented by Fritsch[8] and Lesieur [3], and apply the renormalization group method; see Orzsag [19].

A completely new line of research followed from the understanding of the fundamental importance of the vorticity in the self-organization of turbulent fluids, which was assessed by numerical simulations by Lesieur [3,4], and theoretically by Majda [39] and Chorin [1]. It was found that the Navier-Stokes equations for an incompressible fluids on Euclidean domains is equivalent to a linear diffusion equation for the vorticity which becomes a source for the velocity through the Poisson equation. This observation was the starting point for the random vortex method in Computational Fluid Mechanics largely due to Chorin [1,2,6]. This conception lead to apply methods of statistical mechanics (as originally proposed by Onsager [20]) to study the complex topology of vortex dynamics and to relate this to polymer dynamics [1]. In the random vortex method a random lagrangian representation for the position of the incompressible fluid particles was proposed. Consequently, the Navier-Stokes linear ('heat') equation for the vorticity was integrated only for two dimensional fluids, while the general case was numerically integrated by discretization of the this "heat" equation; see Chorin [1,2]. The difficulty for the exact integration in the general case apparently stems from the fact that while in dimension 2 the vorticity 2-form can be identified with a density and then the integration of the Navier-Stokes equation for the vorticity follows from the application of the well known Ito formula for scalar fields, in the case of higher dimension this identification is no longer valid and an Ito formula for 2-forms is required to carry out the integration. This formula became only recently available in the works by Elworthy [27] and Kunita [24], in the context of the theory of random flows on smooth manifolds.

The importance of a Stochastic Differential Geometry treatment of the Navier-Stokes equation on a smooth n-manifold M stems from several fundamental facts which are keenly interwoven. For a start, it provides an intrinsic geometrical characterization of diffusion processes of differential forms which follows from the characterizations of the laplacians associated to non-Riemannian geometries with torsion of the trace type, as the infinitesimal generators of the diffusions. In particular, this will allow to obtain a new way of writing the Navier-Stokes equation for an incompressible fluid in terms of these laplacians acting on differential one-forms (velocities) and two-forms (vorticities). Furthermore, these diffusion processes of differential forms, are constructed starting from the scalar diffusion process which under Holder continuity regularity conditions yields a time-dependant random diffeomorphism of M which will represent the Lagrangian trajectories for the fluid particles position. This diffeomorphic property will allow us to use the Ito formula for differential forms (following the presentation due to Elworthy) as the key instrument for the integration of the Navier-Stokes equation (NS for short, in the following) for an incompressible fluid.

2 Riemann-Cartan-Weyl Geometry of Diffusions

In this section M denotes a smooth compact orientable n-dimensional manifold (without boundary) provided with an affine connection described by a covariant derivative operator ∇ which we assume to be compatible with a given metric g on M, i.e. $\nabla g = 0$. Given a coordinate chart (x^α) $(\alpha = 1, \ldots, n)$ of M, a system of functions on M (the Christoffel symbols of ∇) are defined by $\nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\gamma} = \Gamma(x)^\alpha_{\beta\gamma} \frac{\partial}{\partial x^\alpha}$. The Christoffel coefficients of ∇ can be decomposed as in

[10,46]:

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + K^{\alpha}_{\beta\gamma}. \tag{1}$$

The first term in (1) stands for the metric Christoffel coefficients of the Levi-Civita connection ∇^g associated to g, i.e. $\left\{ {\alpha \atop \beta\gamma} \right\} = \frac{1}{2}(\frac{\partial}{\partial x^{\beta}} g_{\nu\gamma} + \frac{\partial}{\partial x^{\gamma}} g_{\beta\nu} - \frac{\partial}{\partial x^{\nu}} g_{\beta\gamma}) g^{\alpha\nu}$, and

$$K^{\alpha}_{\beta\gamma} = T^{\alpha}_{\beta\gamma} + S^{\alpha}_{\beta\gamma} + S^{\alpha}_{\gamma\beta}, \tag{2}$$

is the cotorsion tensor, with $S^{\alpha}_{\beta\gamma} = g^{\alpha\nu} g_{\beta\kappa} T^{\kappa}_{\nu\gamma}$, and $T^{\alpha}_{\beta\gamma} = 1/2(\Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta})$ the skew-symmetric torsion tensor. We are interested in (one-half) the Laplacian operator associated to ∇, i.e. the operator acting on smooth functions on M defined as

$$H(\nabla) := 1/2\nabla^2 = 1/2 g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta}. \tag{3}$$

A straightforward computation shows that that $H(\nabla)$ only depends in the trace of the torsion tensor and g:

$$H(\nabla) = 1/2\triangle_g + \hat{Q}, \tag{4}$$

with $Q := Q_{\beta} dx^{\beta} = T^{\nu}_{\nu\beta} dx^{\beta}$ the trace-torsion one-form and where $\hat{Q}$ is the vector field associated to Q via g: $\hat{Q}(f) = g(Q, df)$, for any smooth function f defined on M. Finally, $\triangle_g$ is the Laplace-Beltrami operator of g: $\triangle_g f = \mathrm{div_g}\ \mathrm{grad} f$, $f \in C^{\infty}(M)$, with $\mathrm{div_g}$ the Riemannian divergence.

Consider the family of zero-th order differential operators acting on smooth k-forms, i.e. differential forms of degree k ($k = 0, \ldots, n$) defined on M:

$$H_k(g, Q) := 1/2\triangle_k + L_{\hat{Q}}, \tag{5}$$

In (5) we have the de Hodge operator acting on k-forms:

$$\triangle_k = (d - \delta)^2 = -(d\delta + \delta d), \tag{6}$$

with d and δ the exterior differential and codifferential operators respectively, i.e. δ is the adjoint operator of d defined through the pairing of k-forms on M: $<< \omega_1, \omega_2 >>:= \int g(\omega_1, \omega_2) vol_g$, for arbitrary k-forms ω_1, ω_2, where $vol_g(x) = det(g(x))^{\frac{1}{2}} dx$ is the volume density. The last identity in (6) follows from the fact that $d^2 = 0$ so that $\delta^2 = 0$. Furthermore, the second term in (5) denotes the Lie-derivative with respect to the vectorfield $\hat{Q}$. Recall that the Lie-derivative is independant of the metric:for any smooth vectorfield X on M

$$L_X = i_X d + d i_X,, \tag{7}$$

where i_X is the interior product with respect to X: for arbitrary vectorfields $X_1, \ldots, X_{k-1}$ and ϕ a k-form defined on M, we have $(i_X \phi)(X_1, \ldots, X_{k-1}) = \phi(X, X_1, \ldots, X_{k-1})$. Then, for f a scalar field, $i_X f = 0$ and

$$L_X f = (i_X d + d i_X) f = i_X df = g(\tilde{X}, df) = X(f). \tag{8}$$

where $\tilde{X}$ denotes the 1-form associated to a vectorfield X on M via g. We shall need later the following identities between operators acting on smooth k-forms, which follow easily from algebraic manipulation of the definitions:

$$d\triangle_k = \triangle_{k+1} d, \; k = 0, \ldots, n, \tag{9}$$

and

$$\delta\triangle_k = \triangle_{k-1}\delta, \; k = 1, \ldots, n, \tag{10}$$

and finally, for any vectorfield X on M we have that $dL_X = L_X d$ and therefore

$$dH_k(g, Q) = H_{k+1}(g, Q)d, \; k = 0, \ldots, n. \tag{11}$$

Let $R : (TM \oplus TM) \oplus TM \to TM$ be the (metric) curvature tensor defined by: $(\nabla^g)^2 Y(v_1, v_2) = (\nabla^g)^2 Y(v_2, v_1) + R(v_1, v_2)Y(x)$. From the Weitzenbock formula [14] we have

$$\triangle_1 \phi(v) = \text{trace } (\nabla^g)^2 \phi(-,-)(v) - Ric_x(v, \hat{\phi}_x),$$

for $v \in T_x M$ and $Ric_x(v_1, v_2) = \text{trace } < R(-, v_1)v_2, - >_x$. Since $\triangle_0 = (\nabla^g)^2 = \triangle_g$, we see that from the family defined in (5) we retrieve for scalar fields ($k = 0$) the operator $H(\nabla)$ defined in (4).

Proposition 1: Assume that g is non-degenerate. There is a one-to-one mapping

$$\nabla \rightsquigarrow H_k(g, Q) = 1/2\triangle_k + L_{\hat{Q}}$$

between the space of g-compatible affine connections ∇ with Christoffel coefficients of the form

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{2}{(n-1)} \left\{ \delta^{\alpha}_{\beta} \, Q_{\gamma} \; - \; g_{\beta\gamma} \, Q^{\alpha} \right\} \tag{12}$$

and the space of elliptic second order differential operators on k-forms ($k = 0, \ldots, n$) with zero potential term.

The connections defined in (12) are called Riemann-Cartan-Weyl (RCW for short) connections [10,21,22,26]. The naming after Weyl of the trace-torsion is motivated by the fact that these geometries can be introduced through scale transformations which extend the Weyl transformations in the first ever conceived gauge theory; see Rapoport [10 b,c].

3 Riemann-Cartan-Weyl Diffusions of Differential forms

In this and the next section we shall extend the correspondance of Proposition 1 to a correspondance between RCW connections and diffusion processes of k-forms ($k = 0, \ldots, n$) having $H_k(g, Q)$ as infinitesimal generators (i.g. for short, in the following).Thus, naturally we shall call these processes as RCW diffusion processes.

In the following we shall further assume that $Q = Q(\tau, x)$ is a time-dependant 1-form. The stochastic flow associated to the diffusion generated by $H_0(g, Q)$ has for sample paths the continuous curves $\tau \mapsto x_\tau \in M$ satisfying the Ito invariant non-degenerate s.d.e. (stochastic differential equation)

$$dx(\tau) = X(x(\tau))dW(\tau) + \hat{Q}(\tau, x(\tau))d\tau. \tag{13}$$

In this expression, $X = (X^\alpha_\beta(x))$ satisfies $X^\alpha_\nu X^\beta_\nu = g^{\alpha\beta}$, and $\{W(\tau), \tau \geq 0\}$ is a standard Wiener process on R^n. Here τ denotes the time-evolution parameter of the diffusion (in a relativistic setting it should not be confused with the time variable), and for simplicity we shall assume always that $\tau \geq 0$. Consider the canonical Wiener space Ω of continuous maps $\omega : R \to R^n, \omega(0) = 0$, with the canonical realization of the Wiener process $W(\tau)(\omega) = \omega(\tau)$. The (stochastic) flow of the s.d.e. (13) is a mapping

$$F_\tau : M \times \Omega \to M, \quad \tau \geq 0, \tag{14}$$

such that for each $\omega \in \Omega$, the mapping $F_.(. \, , \omega) : [0, \infty) \times M \to M$, is continuous and such that $\{F_\tau(x) : \tau \geq 0\}$ is a solution of equation (13) with $F_0(x) = x$, for any $x \in M$.

Let us assume in the following that the components X^α_β, $\hat{Q}^\alpha$, $\alpha, \beta = 1, \ldots, n$ of the vectorfields X and $\hat{Q}$ on M in (13) are predictable functions which further belong to $C^{m,\epsilon}_b$ ($0 \leq \epsilon \leq 1$, m a non-negative integer), the space of Holder bounded continuous functions of degree $m \geq 1$ and exponent ϵ, and also that $\hat{Q}^\alpha(\tau) \in L^1(R)$, for any $\alpha = 1, \ldots, n$. With these regularity conditions, if we further assume that $x(\tau)$ is a semimartingale on a probability space $(\Omega, \mathcal{F}, P)$, then it follows from Kunita [24] that the flow of (13) has a modification (which with abuse of notation we denote as)

$$F_\tau(\omega) : M \to M, \quad F_\tau(\omega)(x) = F_\tau(x, \omega), \tag{15}$$

which is a diffeomorphism of class C^m, almost surely for $\tau \geq 0$ and $\omega \in \Omega$.

Remarks 1: In the differential geometric approach -pioneered by V. Arnold- for integrating NS on a smooth manifold as a perturbation (due to the diffusion term we shall present below) of the geodesic flow in the group of volume preserving diffeomorphisms of M (as the solution of the Euler equation), it was proved that under the above regularity conditions on the initial velocity, the solution flow of NS defines a diffeomorphism in M of class C^m; see Ebin and Marsden [9]. The difference of this classical approach with the one presented here, is to integrate NS through a time-dependant *random* diffeomorphism associated with a RCW connection. As wellknown, these regularity conditions are basic in the usual functional analytical treatment of NS pioneered by Leray [45] (see also Temam [7]), and they are further related to the multifractal structure of turbulence [41]. This diffeomorphism property of random flows is fundamental for the construction of their ergodic theory (provided an invariant measure for the processes exists), and in particular, of quantum mechanics amd non-linear non-equilibrium thermodynamics [10,21,22].

Let us describe the first derivative flow of (13), i.e. the stochastic process $\{v(\tau) := T_{x_0}F_\tau(v(0)) \in T_{F_\tau(x_0)}M, v(0) \in T_{x_0}M\}$; here T_zM denotes the tangent space to M at z and $T_{x_0}F_\tau$ is the linear derivative of F_τ at x_0. The process $\{v_\tau, \tau \geq 0\}$ can be described [27] as the solution of the invariant Ito s.d.e. on TM:

$$dv(\tau) = \nabla^g \hat{Q}(v(\tau), \tau)d\tau + \nabla^g X(v(\tau))dW(\tau) \tag{16}$$

If we take U to be an open neighborhood in R^n so that $TU = U \times R^n$, then $v(\tau) = (x(\tau), \tilde{v}(\tau))$ is described by the system given by integrating (13) and the covariant Ito s.d.e.

$$d\tilde{v}(\tau)(x(\tau)) = \nabla^g X(x(\tau))(\tilde{v}(\tau))dW(\tau) + \nabla^g \hat{Q}(\tau, x(\tau))(\tilde{v}(\tau))d\tau, \tag{17}$$

with initial condition $\tilde{v}(0) = v_0$. Thus, $\{v(\tau) = (x(\tau), \tilde{v}(\tau)), \tau \geq 0\}$ defines a random flow on TM.

Theorem 1 : For any differential 1-form ϕ of class $C^{1,2}(R \times M)$ (i.e. in a local coordinate system $\phi = a_\alpha(\tau)dx^\alpha$, with $a_\alpha(\tau, .) \in C^2(M)$ and $a_\alpha(., x) \in C^1(R)$) we have the Ito formula (Corollary 3E1 in [27]):

$$\begin{aligned} \phi(v_\tau) &= \phi(v_0) + \int_0^\tau \nabla^g \phi(X(x)dW_s)(v_s) + \int_0^\tau \phi(\nabla^g X(v_s)dW_s) \\ &+ \int_0^\tau [\frac{\partial}{\partial s} + H_1(g, Q)]\phi(v_s)ds + \int_0^\tau \text{trace } d\phi(X(x_s)-, \nabla^g X(v_s))(-)ds \end{aligned} \tag{18}$$

In the last term in (18) the trace is taken in the argument $-$ of the bilinear form and further we have the mappings

$$\nabla^g Y : TM \to TM; \nabla^g \phi : TM \to T^*M.$$

Remarks 2 : From (18) we conclude that the i.g. of the 'velocity' stochastic process is not $\partial_\tau + H_1(g, Q)$, due to the last term in (18). This term vanishes identically in the case we shall present in the following section.

4 Riemann-Cartan-Weyl Gradient Diffusions

Suppose that there is an isometric immersion of an n-dimensional compact manifold M into a Euclidean space R^m:$f : M \rightarrow R^m, f(x) = (f^1(x), \ldots, f^m(x))$. For example, $M = S^n, T^n$, the n-dimensional sphere or torus respectively, and f is the standard inclusion into R^{n+1}. Suppose further that $X(x) : R^m \rightarrow T_xM$, is the orthogonal projection of R^m onto T_xM the tangent space at x to M, considered as a subset of R^m. Then, if $e_1, \ldots, e_m$ denotes the standard basis of R^m, we have

$$X = X^i e_i, \text{ with } X^i = \text{grad } f^i, i = 1, \ldots, m. \tag{19}$$

The second fundamental form [25] is a bilinear symmetric map

$$\alpha_x : T_xM \times T_xM \rightarrow \nu_xM, x \in M, \tag{20}$$

with $\nu_xM = (T_xM)^\perp$ the space of normal vectors at x to M. We then have the associated mapping

$$A_x : T_xM \times \nu_xM \rightarrow T_xM, < A_x(u, \zeta), v >_{R^m} = < \alpha_x(u, v), \zeta >_{R^m}, \tag{21}$$

for all $\zeta \in \nu_xM$, $u, v \in T_xM$. Let $Y(x)$ be the orthogonal projection onto ν_xM

$$Y(x) = e - X(x)(e), x \in M, e \in R^m. \tag{22}$$

Then:

$$\nabla^g X(v)(e) = A_x(v, Y(x)e), v \in T_xM, x \in M. \tag{23}$$

For any $x \in M$, if we take $e_1, \ldots, e_m$ to be an orthonormal base for R^m such that $e_1, \ldots, e_m \in T_xM$, then for any $v \in T_xM$,we have

$$\text{either } \nabla^g X(v)e_i = 0, \text{ or } X(x)e_i = 0. \tag{24}$$

We are interested in the RCW *gradient* diffusion processes on compact manifolds isometrically immersed in Euclidean space, given by (13) with X given by (19). We shall now give the Ito formula for 1-forms.

Theorem 2 : Let $f : M \rightarrow R^m$ be an isometric immersion. For any differential form ϕ of degree 1 in $C^{1,2}(R \times M)$, the Ito formula is

$$
\begin{aligned}
\phi(v_\tau) &= \phi(v_0) + \int_0^\tau \nabla^g \phi(X(x_s)dW_s)v_s \\
&+ \int_0^\tau \phi(A_x(v_s, Y(x_s)dW_s) + \int_0^\tau [\frac{\partial}{\partial s} + H_1(g,Q)]\phi(v_s)ds, \qquad (25)
\end{aligned}
$$

i.e. $\partial_\tau + H_1(g,Q)$, is the i.g. (with domain the differential 1-forms belonging to $C^{1,2}(R\times M)$) of $\{v_\tau : \tau \geq 0\}$.

Proof: It follows immediately from the facts that the last term in the r.h.s. of (18) vanishes due to (24), while the third term in the r.h.s. of (18) coincides with the third term in (25) due to (23).

Consider the value Φ_x of a k-form at $x\in M$ as a linear map: $\Phi_x : \Lambda^k T_x M \rightarrow R$. In general, if E is a vector space and $A : E \rightarrow E$ is a linear map, we have the induced maps

$$
\Lambda^k A : \Lambda^k E \rightarrow \Lambda^k E,\ \Lambda^k(v^1 \wedge \ldots \wedge v^k) := Av^1 \wedge \ldots \wedge Av^k;
$$

and

$$
\begin{aligned}
&(d\Lambda^k)A : \Lambda^k E \rightarrow \Lambda^k E,\ (d\Lambda^k)A(v^1 \wedge \ldots \wedge v^k) \\
&:= \sum_{j=1}^k v^1 \wedge \ldots \wedge v^{j-1} \wedge Av^j \wedge v^{j+1} \wedge \ldots \wedge v^k.
\end{aligned}
$$

For $k = 1, (d\Lambda)A = \Lambda A$. The Ito formula for k-forms, $1 \leq k \leq n$, is due to Elworthy (Prop. 4B [27]).

Theorem 3 : Let M be isometrically immersed in R^m. Let $V_0 \in \Lambda^k T_{x_0} M$. Set $V_\tau = \Lambda^k(TF_\tau)(V_0)$ Then for any differential form ϕ of degree k in $C^{1,2}(R\times M)$, $(1 \leq k \leq n)$,

$$
\begin{aligned}
\phi(V_\tau) &= \phi(V_0) + \int_0^\tau \nabla^g \phi(X(x_s)dWs)(V_s) \\
&+ \int_0^\tau \phi((d\Lambda)^k A_{x_s}(-, Y(x_s)dW_s)(V_s)) + \int_0^\tau [\frac{\partial}{\partial s} + H_k(g,\hat{Q})]\phi(V_s)ds \qquad (26)
\end{aligned}
$$

i.e., $\partial_\tau + H_k(g,\hat{Q})$ is the i.g. (with domain of definition the differential forms of degree k in $C^{1,2}(R\times M)$) of $\{V_\tau : \tau \geq 0\}$.

Remarks 3 : Therefore, starting from the flow $\{F_\tau : \tau \geq 0\}$ of the s.d.e. (13) with i.g. given by $\partial_\tau + H_0(g,Q)$, we obtained that the derived velocity process $\{v(\tau) : \tau \geq 0\}$ given by (16) (or (13) and (17)) has $H_1(g,Q)$ as i.g.; finally, if we consider the diffusion processes of differential forms of degree $k \geq 1$, we get that

$\partial_\tau + H_k(g, Q)$ is the i.g. of the process $\{\Lambda^k v(\tau) : \tau \geq 0\}$, on the Grassmannian bundle $\Lambda^k TM$, $(k = 0, \ldots, n)$. In particular, $\partial_\tau + H_2(g, Q)$ is the i.g. of the stochastic process $\{v(\tau) \wedge v(\tau) : \tau \geq 0\}$ on $TM \wedge TM$.

5 The Navier-Stokes Equation and Riemann -Cartan-Weyl Gradient Diffusions

In the sequel, M is a compact (possibly with smooth boundary ∂M) n-manifold with a Riemannian metric g. We provide M with a 1-form whose Hodge decomposition is

$$Q(x) = df(x) + u(x), \quad \delta u = -div(\hat{u}) = 0, \tag{27}$$

where f is a scalar field and u is a coclosed 1-form, weakly orthogonal to df, i.e. $\int g(df, u)vol_g = 0$. We shall assume that $u(x, 0) = u(x)$ is the initial velocity 1-form of an incompressible viscous fluid on M, and that we further have a 1-form $Q(x, \tau) = Q_\alpha(x, \tau)dx^\alpha$ whose Hodge decomposition is:

$$Q(x, \tau) = df(x, \tau) + u(x, \tau),$$

with $\delta u_\tau(x) = \delta u(x, \tau) = 0$ (incompressibility condition), and

$$\int g(df_\tau, u_\tau)vol(g) = 0,$$

which satisfies the evolution equation on $M \times R$ (Eulerian representation of the fluid):

$$\frac{\partial Q_\alpha}{\partial \tau} + \nabla^g_{\hat{u}} Q_\alpha = -Q_\beta \nabla^g_\alpha u^\beta + \nu \triangle_1 Q_\alpha, \tag{28}$$

Here ν is the kinematical viscosity. In the above notations and in the following, all covariant operators act in the M variables only. In the formulation of Fluid Mechanics in Euclidean domains, $Q(x, \tau)$ receives the name of (Buttke) "magnetization variable" [1].

Remarks 4: We recall that to take the Hodge decomposition of the velocity of a viscous fluid is a basic procedure in Fluid Mechanics [1,6,7,9]. We shall see below that Q and in particular u are related to a natural RCW geometry of the incompressible fluid. In the formulation of Quantum Mechanics and of non-linear non-equilibrium thermodynamics stemming from RCW diffusions, we have a Hodge decomposition of the trace-torsion associated to a stationary state; see

[10,21,22,26]. This decomposition allows to associate with the divergenceless term of the trace-torsion a probability current which characterizes the time-invariance symmetry breaking of the diffusion process, and is central to the construction of the ergodic theory of these flows.

Equation (28) is the gauge-invariant form of the NS for the velocity 1-form $u(x,\tau)$. Indeed, if we substitute the Hodge decomposition of $Q(x,\tau) = Q_\tau(x)$ into (28) we obtain,

$$\frac{\partial u}{\partial \tau} + \nabla^g_{\hat{u}_\tau} u_\tau = \nu \triangle_1 u_\tau - d(\frac{\partial f}{\partial \tau} + \nabla^g_{\hat{u}_\tau} f + \frac{1}{2}|u_\tau|^2 - \nu \triangle_g f). \tag{29}$$

Consider the operator P of projection of 1-forms into co-closed 1-forms: $P\omega = \alpha$ for any one-form ω whose Hodge decomposition is $\omega = df + \alpha$, with $\delta\alpha = 0$. From (10) we get that

$$P\triangle_1 u_\tau = \triangle_1 u_\tau, \tag{30}$$

and further applying P to (29) we finally get the well known covariant NS (with no exterior forces; the gradient of the pressure term disappears by projecting with P [1,9])

$$\frac{\partial u}{\partial \tau} + P[\nabla^g_{\hat{u}_\tau} u_\tau] - \nu \triangle_1 u_\tau = 0. \tag{31}$$

Conversely, starting with equation (29) which is equivalent to NS we obtain (28). Note that Q_τ and u_τ differ by a differential of a function for all times. Multiplication of (29) by $I - P$ (I the identity operator) yields an equation for the evolution of f which is only arbitrary for $\tau = 0$. Now we note that the non-linearity of NS originates from applying P to the term

$$\nabla^g_{\hat{u}_\tau} u_\tau = i_{\hat{u}_\tau} du_\tau,$$

which taking in account (7) can still be written as

$$L_{\hat{u}_\tau} u_\tau - d i_{\hat{u}_\tau} u_\tau = L_{\hat{u}_\tau} u_\tau - d(|u_\tau|^2). \tag{32}$$

Applying P to (32), we see that the kinetic energy term there disappears and the non-linear term in NS can be written as

$$P[\nabla^g_{\hat{u}_\tau} u_\tau] = P[\text{Ł}_{\hat{u}_\tau} u_\tau]. \tag{33}$$

Therefore, from (5) and (33), NS takes the final concise form

$$\frac{\partial u}{\partial \tau} = P H_1(2\nu g, \frac{-1}{2\nu} u_\tau) u_\tau. \tag{34}$$

This is a new way of writing NS, and through it we have found that NS for the velocity of an incompressible fluid is a a non-linear diffusion process determined by a RCW connection. This is a characterization of NS hitherto unknown (cf. [1-9]). This RCW connection which determines NS has $2\nu g$ for the metric, and the time-dependant trace-torsion of this connection is $-u/(2\nu)$. Then, the drift of this process does not depend explicitly on ν, as it coincides with the vectorfield associated via g to $-u_\tau$, i.e.$-\hat{u}_\tau$. Thus, we have a static metric which depends on the kinematical viscosity, and the trace-torsion, initially unnoticed appeared through a dynamical field given by $-u/(2\nu)$ which in the limit $\nu \to 0$ in which the Euler equations substitutes NS, becomes singular while the drift is still well behaved.

Let us introduce the vorticity two-form

$$\Omega_\tau = du_\tau. \tag{35}$$

Note that also $\Omega_\tau = dQ_\tau$. Now, if we know Ω_τ for any $\tau \geq 0$, we can obtain u_τ (or still Q_τ) by inverting the definition (35). Namely, applying δ to (35) and taking in account (6) we obtain the Poisson-de Rham equation (would g be hyperbolic, it is the Maxwell-de Rham equation [10a])

$$\triangle_1 u_\tau = -\delta\Omega_\tau. \tag{36}$$

and an identical equation for Q_τ. (Note that if we know Q_τ we can reconstruct f_τ by solving $-\delta Q_\tau = div(\hat{Q}_\tau) = \triangle_g f_\tau$, for any τ.) From the Weitzenbock formula we can write (36) showing the coupling of the Ricci metric curvature to the velocity $u = u_\alpha(x,\tau)dx^\alpha$:

$$(\nabla^g)^2 u_\tau - R_{\alpha\beta}u_\tau^\beta dx^\alpha = -\delta\Omega_\tau. \tag{37}$$

with $R_\alpha^\beta(g) = R_{\mu\alpha}{}^{\mu\beta}(g)$, the Ricci (metric) curvature tensor. Thus, the vorticity Ω_τ is a source for the velocity one-form u_τ, for all τ; in the case that M is a compact euclidean domain, (36) is integrated to give the Biot-Savart law of Fluid Mechanics [1,39].

Now, apply d to (34) and further (Hodge) decompose $L_{-\hat{u}_\tau}u_\tau = \alpha_\tau + dp_\tau$ (with p_τ the pressure at time τ); in account that

$$dPL_{-\hat{u}_\tau}u_\tau = d\alpha_\tau = d(\alpha_\tau + dp_\tau) = dL_{-\hat{u}_\tau}u_\tau = L_{-\hat{u}_\tau}du_\tau = L_{-\hat{u}_\tau}\Omega_\tau,$$

and that from (9) we have that $d\triangle_1 u_\tau = \triangle_2\Omega_\tau$, we therefore obtain the linear evolution equation

$$\frac{\partial\Omega_\tau}{\partial\tau} = H_2(2\nu g, \frac{-1}{2\nu}u_\tau)\Omega_\tau. \tag{38}$$

Theorem 4 : Given a compact orientable Riemannian manifold with metric g, the Navier-Stokes equation (34) for an incompressible fluid with velocity one-form $u = u(\tau, x)$ such that $\delta u_\tau = 0$, assuming sufficiently regular conditions, is equivalent to a linear diffusion process for the vorticity given by (38) with u_τ satisfying the Poisson-de Rham equation (36). The RCW connection on M generating this process is determined by the metric $2\nu g$ and a trace-torsion 1-form given by $-u/2\nu$.

6 Integration of the Navier-Stokes equation for the vorticity:

In the following we assume additional conditions on M, namely that it is isometrically immersed in an Euclidean space.

Let u denote a solution of (34) and consider the flow $\{F_\tau : \tau \geq 0\})$ of the s.d.e. whose i.g. is $\frac{\partial}{\partial \tau} + H_0(2\nu g, \frac{-1}{2\nu} u)$; from (13) we know that this is the flow defined by integrating the non-autonomous Ito s.d.e.

$$dx(\tau) = [2\nu]^{\frac{1}{2}} X(x(\tau)) dW(\tau) - \hat{u}(\tau, x(\tau)) d\tau, x(0) = x, 0 \leq \tau. \tag{39}$$

We shall assume in the following that X and $\hat{u}_\tau$ have the regularity conditions stated above so that the random flow of (39) is a diffeomorphism of M of class C^m.

Theorem 5: Equation (39) is a random Lagrangian representation for the fluid particles positions, i.e. $x(\tau)$ is the random position of the particles of the incompressible fluid whose velocity obeys (34).

Proof: In the case that M is 2-dimensional we note that 2-forms are pseudoscalars so that Ω_τ is a density on M for every τ; then, on (38) we can replace $H_2(2\nu g, -u/2\nu)$ by $H_0(2\nu g, -u/2\nu)$ which thus becomes the scalar backward Fokker-Planck equation for the probability density of the random process (39). Therefore, in dimension 2 our assertion follows from the Ito formula for scalar fields. In the general case $n \geq 3$ the proof requires the Ito formula for 2-forms. Indeed, consider the derived velocity flow $\{v(\tau) = T_{x_0} F_\tau(v_0) = (x(\tau), \tilde{v}(\tau)) : \tilde{v}(\tau) \in T_{x(\tau)}M, \tau \geq 0\}$ on TM; this process is given by (39) and the process with initial velocity $\tilde{v}(0) = v_0 \in T_{x_0}M$:

$$d\tilde{v}(\tau) = [2\nu]^{\frac{1}{2}} \nabla^g X(x(\tau))(\tilde{v}(\tau)) dW(\tau) - \nabla^g \hat{u}(\tau, x(\tau))(\tilde{v}(\tau)) d\tau. \tag{40}$$

for any $0 \leq \tau$. From the Ito formula we know that $\frac{\partial}{\partial \tau} + H_1(2\nu g, -\frac{1}{2\nu} u_\tau)$ is the backward i.g. of $\{v(\tau), \tau \geq 0\}$. From the Ito formula for 2-forms we conclude that $\frac{\partial}{\partial \tau} + H_2(2\nu g, -\frac{1}{2\nu} u_\tau)$ is the backward i.g. of the stochastic process $\{v(\tau) \wedge v(\tau), \tau \geq$

0} on $TM \wedge TM$. This concludes with the proof of our assertion in the general case.

Remark 5: Note that the drift of $\{\tilde{v}_\tau : \tau \geq 0\}$ is minus the deformation tensor of the fluid.

6.1 Cauchy Problem for the Vorticity

Let us solve the Cauchy problem for $\Omega(\tau, x)$ of class C^m in $R \times M$ satisfying (38) with initial condition $\Omega_0(x)$.

For each $\tau \geq 0$ consider the s.d.e. (with $s \in [0, \tau]$):

$$dx_s^\tau = (2\nu)^{\frac{1}{2}} X(x_s^\tau) dW_s - \hat{u}(\tau - s, x_s^\tau) ds. \tag{41}$$

and the derived velocity process $\{v_s^\tau, 0 \leq s \leq \tau\}$. Let $\tilde{\Omega}_\tau(x)$ be a bounded solution of the Cauchy problem; then it follows from the Ito formula (48) (with $k = 2$) that for $s \in [0, \tau)$

$$\begin{aligned} \tilde{\Omega}_{\tau-s}(x_s^\tau)(\Lambda^2 \tilde{v}_s^\tau)) &= \tilde{\Omega}_\tau(x_0^\tau)(\Lambda^2 \tilde{v}_0^\tau) + \text{a local martingale} \\ &+ \int_0^s [-\partial_\tau + H_2(2\nu g, -\frac{1}{2\nu} u_\tau)] \tilde{\Omega}_{\tau-r}(x_r^\tau)(\Lambda^2 \tilde{v}_r^\tau) dr. \end{aligned} \tag{42}$$

Then $M_s = \tilde{\Omega}_{\tau-s}(x_s^\tau)(\tilde{v}_s^\tau \wedge \tilde{v}_s^\tau)$, for $0 \leq s < \tau$ is a local martingale (the last term in (42) vanishes by assumption) which converges to $\Omega_0(x_\tau^\tau))(v_\tau^\tau \wedge v_\tau^\tau)$ as $s \uparrow \tau$, and then as it is bounded, it satisfies $M_s = E_x[\Omega_0(x_\tau^\tau)(\tilde{v}_\tau^\tau \wedge \tilde{v}_\tau^\tau) | \mathcal{F}_s]$, where $\{\mathcal{F}_s, 0 \leq s < \tau\}$ is an increasing filtration adapted to $\{v_\tau^\tau, \tau \geq 0\}$. Set $s = 0$, and let v_τ^τ start in $(x, v(x))$ with $v(x) \in T_x M$, then if there exists a bounded solution of the above Cauchy problem, it must be

$$\begin{aligned} \tilde{\Omega}_\tau(v(x) \wedge v(x)) &= E_x[\Omega_0(x_\tau^\tau)(\Lambda^2 T F_\tau(x)(v(x) \wedge v(x))] \\ &= \int \Omega_0(y)(v(y) \wedge v(y)) p(x, 0, \tau, y) vol_g(y), \end{aligned}$$

where $F_\tau(x)$ denotes the flow of x_τ^τ starting at x, which by taking the bundle projection of TM over M we can finally write as

$$\tilde{\Omega}_\tau(x) = E[\Omega_0(F_\tau(x))] = \int \Omega_0(y) p(x, 0, \tau, y) vol_g(y) \tag{43}$$

where $p(x, 0, \tau, y)$ is the transition density for the Lagrangian representation $\{x_\tau^\tau, \tau \geq 0\}$ of equation (63), i.e. the fundamental solution of the forward Fokker-Planck equation

$$\frac{\partial p}{\partial \tau} = H_0(2\nu g, -\frac{1}{2\nu} u_0)^\dagger(y) p \equiv \nu \triangle_g(y) p + div_g(u_0(y) p) \tag{44}$$

i.e. $p(\tau, x, .) = \delta(x)$ as $\tau \downarrow 0$.

N.B. The solution only requires the initial velocity and initial vorticity of the fluid, and the transition density for the Lagrangian representation starting at time 0.

Theorem 8 If Ω_0 is of class C^m, then $\Omega_\tau(x) = E[\Omega_0(F_\tau(x))]$ is a $C^m(R \times M)$ solution of the Cauchy problem. Conversely, if $\tilde{\Omega}_\tau(x)$ is a solution of the Cauchy problem of class $C^{1,2}(R \times M)$ which is bounded and satisfies further the condition

$$lim_{n\to\infty} E[\tilde{\Omega}_{\tau-\sigma_n}(x)(F_{\sigma_n}(x)) : \sigma_n \leq \tau] = 0$$

where $\sigma_n = inf\{\tau : F_\tau(x, \omega) \notin D_n\}$ and D_n is an increasing sequence of relatively compact sets in M such that $\bigcup D_n = M$, then it coincides with $\Omega_\tau(x) = E[\Omega_0(F_\tau(x)]$.

Proof: The proof of unicity follows as in the case of the scalar heat equation, while from the fact that both Ω_0 and $F_\tau(x)$ are assumed to be of class C^m, then also $\Omega_0(F_\tau(x))$ is of class C^m and we can differentiate with respect to x in $E[\Omega_0(F_\tau(x))]$. The differentiability with respect to τ follows from the fact that since $H_2(2\nu g, -u/2\nu)^m \Omega_0(x)$ is continuous, then we can apply to $\Omega_\tau(x)$ a Taylor-Ito formula; see page 254 [14]).

7 Integration of the Poisson-de Rham equation

In (39) we have that u_τ verifies (36), for every $\tau \geq 0$ which we can rewrite as

$$H_1(g, 0)u_\tau = -\frac{1}{2}\delta\Omega_\tau, \text{ for any } \tau \geq 0. \tag{45}$$

Consider the autonomous s.d.e. generated by $H_0(g, 0) = \frac{1}{2}\triangle_g$:

$$dx^g_s = X(x^g_s)dW_s. \tag{46}$$

We shall solve the Dirichlet problem in an open set U (of a partition of unity) of M given by (45) with the boundary condition $u_\tau \equiv \phi$ on ∂U, with ϕ a given 1-form. Then one can 'glue' the solutions and use the strong Markov property to obtain a global solution. If M has a boundary ∂M we have the no-slip condition with $u_\tau \equiv 0$ on ∂M. Consider the derived velocity process $v^g(s) = (x^g(s), \tilde{v}^g(s))$ on TM, with $\tilde{v}^g(s) \in T_{x^g(s)}M$, whose i.g. is $H_1(g, 0)$:

$$d\tilde{v}^g_s(x^g_s) = \nabla^g X(x^g(s))(\tilde{v}^g(s))dW(s), \tag{47}$$

with initial velocity $\tilde{v}^g(s) = (2\nu)^{\frac{-1}{2}}v_0$. Notice that equations $(46, 47)$ are obtained by taking $u \equiv 0$ in equations $(39, 40)$ and further rescaling by $(2\nu)^{-\frac{1}{2}}$. Then if u_τ

is a solution of (45) for any fixed τ, applying to it the Ito formula we obtain that

$$M_s := u_\tau(x^g(s))(\tilde{v}^g(s)) + \frac{1}{2}\int_0^s \delta\Omega_\tau(x^g(r))(\tilde{v}^g(r))dr$$

is a local martingale in $[0, \tau_e)$, where τ_e is the first-exit time of U. Then if u_τ and $\delta\Omega_\tau$ are bounded, then for $s < \tau_e$ we get $|M_s| \leq \| u_\tau \|_\infty + \tau_e \| \delta\Omega_\tau \|_\infty$. Assume now that $E_x^B \tau_e < \infty$ for any $x \in U$; then M_s is a uniformly integrable martingale on $[0, \tau_e)$. Further, since $u_\tau = \phi$ on ∂U, it further satisfies

$$lim_{s\uparrow\tau_e} M_s = \phi(x^g(\tau_e))(\tilde{v}^g(\tau_e)) + 1/2\int_0^{\tau_e} \delta\Omega_\tau(x_r^g)(\tilde{v}_r^g)dr, \tag{48}$$

so that

$$M_s \equiv E_x^B[(\phi^g(\tau_e)) + 1/2\int_0^{\tau_e} \delta\Omega(x_r^g)(\tilde{v}_r^g)dr)|\mathcal{F}_s] \tag{49}$$

where $\{\mathcal{F}_s : s \geq 0\}$ is an increasing filtration adapted to $\{x_s^g : s \geq 0\}$. Taking $s = 0$ and $(x^g(0), \tilde{v}^g(0)) = (x, \tilde{v}(x))$, if $\delta\Omega_\tau$ is bounded, we then obtain that a solution of the Dirichlet problem is given by:

$$\begin{aligned} \tilde{u}_\tau(x) &= E_x^B[\phi(x^g(\tau_e) + \int_0^{\tau_e} \frac{1}{2}\delta\Omega_\tau(x^g(s))ds] \\ &= \int(\phi(x_{\tau_e}^g) + 1/2\int_0^{\tau_e} \delta\Omega_\tau(y)ds)p^g(s,x,y)vol_g(y), \end{aligned} \tag{50}$$

where $p^g(s,x,y)$ is the transition density of the s.d.e. (46), i.e.the fundamental solution of the heat equation on M:

$$\partial_\tau p(y) = 1/2\triangle_g p(y) \tag{51}$$

with $p(s,x,-) = \delta_x$ as $s \downarrow 0$.

Remarks 6: If g is uniformly elliptic, and U has a $C^{2,\epsilon}$-boundary, and furthermore $g^{\alpha\beta}$ and $\delta\Omega_\tau$ are Holder-continuous of order ϵ on U and u_τ is uniformly Holder-continuous of order ϵ, then the solution of the Dirichlet problem for *scalars* has a unique solution belonging to $C^{2,\epsilon}(U)$ [31,47].The validity of the extension of this to differential forms, is still an open problem, whose positive solution would guarrantee the uniqueness of the above solution. Futhermore, if this is the case the solution constructed here would be then defined for all times, independantly of the dimension n of M.

Final Observations: The method of integration applied in the previous section is the extension to differential forms of the method of integration of elliptic and parabolic partial differential equations for scalar fields [30-34].

Notice that in the solutions (43) and (50), the local dependance on the curvature is built-in (the curvature is defined by second-order derivatives). This dependance might be exhibited through the scalar curvature term in the Onsager-Machlup lagrangian appearing in the path-integral representation of the fundamental solution of the transition densities of equations (44) and (46) [35,44]. There is further a dependance of the solution on the global geometry and topology of M appearing through the Riemannian spectral invariants of M in the short-time asymptotics of these transition densities [28,29,43]. There is another construction of the solution of NS which exhibits the dependance of the solution on the Ricci curvature, and consists in replacing the velocity process on TM by a generalized Hessian flow for the integration for the vorticity and a Ricci flow (cf. [27]) for the solution of the Poisson-de Rham equation; these alternative constructions -whose details we shall present elsewhere-, allow to integrate NS on a compact manifold, lifting thus the restriction to Euclidean submanifolds we have placed in this article which originated in the Ito formula for one-forms on compact manifolds.

The solution scheme we have presented gives rise to infinite particle random trajectories due to the arbitrariness of the initial point of the Lagrangian paths. To actually integrate NS we choose a finite set of initial points and we take for Ω_0 a linear combination of 2-forms (or area elements in the 2-dimensional case) supported in balls centered in these points, the so-called many vortices solutions; one can choose the original $f_0(x)$ so that Ω_0 is supported in these balls and these localizations persists in time. Thus the role of the potential term in the Buttke magnetization 1-form in the expression (27) is to 'push the vortices to be confined on predetermined finite radii balls; see Chorin [1].

A new approach to NS as a (*random*) dynamical system appears. Given a stationary measure for the *random* diffeomorphic flow of NS given by the stationary flow of equation (39), one can costruct the state space of this flow and further, its *random* Lyapunov spectra. Consequently, assuming ergodicity of this measure, one can conclude that the moment instability of the flow is related to a cohomological property of M, namely the existance of non-trivial harmonic one-forms ϕ, which are preserved by the vectorfield $\hat{u}$ of class C^2, i.e. $L_{\hat{u}}\phi = di_{\hat{u}}\phi = 0$; see page 61 in [27]. We also have the random flow $\{v_\tau \wedge v_\tau : \tau \geq 0\}$ on $TM \wedge TM$ of Theorem 5 which integrates the linear equation for the vorticity. Concerning this last flow, the stability theory of NS (38) requires an invariant measure on a suitable subspace of $TM \wedge TM$ and further, the knowledge of the spectrum of the one-parameter family of *linear* operators depending on ν, $H_2(2\nu g, -\frac{1}{2\nu}u_\tau)$. The latter may play the role of the Schroedinger operators in Ruelle's theory of turbulence, which were introduced by linearising NS for the velocity as the starting point for the discussion of the instability theory; see article in pages $295 - 310$ in Ruelle [48].

Finally, it has been established numerically that turbulent fluids resemble the random motion of dislocations [4]. In the differential geometric gauge theory of crystal dislocations, the torsion tensor is the dislocation tensor [12], and our presentation suggests that this analogy might be established rigorously from the perspective presented here.

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