EQUIVALENCE OF NORMS IN ONE-SIDED H^p SPACES

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Abstract - One-sided versions of maximal functions for suitable defined distributions are considered. Weighted norm equivalences of these maximal functions for weights in the Sawyer's A_q^+ classes are obtained.

1. Notations and Definitions

Let f be a locally integrable function on $I\!R$. The one-sided Hardy-Littlewood maximal functions are defined as

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)|dy \ ,$$

and

$$M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|dy \ .$$

It is well known, see [S], that if $1 < q < \infty$ and $w(x) \geq 0$, a necessary and sufficient condition for

$$\int_{-\infty}^{\infty} M^+f(x)^q w(x)dx \leq C(w,q) \int_{-\infty}^{\infty} |f(x)|^q w(x)dx$$

to hold is that there exists a constant $C < \infty$ such that if $-\infty < a < b < c < \infty$, then

$$\left(\int_a^b w(x)dx\right)^{1/q} \left(\int_b^c w(x)^{-q'/q}dx\right)^{1/q'} \leq C(c-a) \ . \tag{1.1}$$

1991 Mathematics Subject Classification. Primary 42B30. Secondary 42B25.

Keywords and phrases. H^p -spaces, maximal functions, weights.

We say that a non-negative and measurable function $w(x)$ is a weight in the class A_q^+ if (1.1) holds. Similar results are valid for the maximal function $M^- f$.

In the limit case of $q = 1$ the condition $M^- w(x) \leq Cw(x)$ a.e. is necessary and sufficient for the weak type 1-1. These weights are said to belong to the class A_1^+. For more properties of the classes A_q^+ and A_q^- see [S] and [M].

If $w \in A_q^+$, $q \geq 1$, the weight $w(x)$ can be equal to zero on a set of positive measure, however there exists $x_{-\infty}$, $-\infty \leq x_{-\infty} < \infty$ such that $w(x) = 0$ a.e. if $x \leq x_{-\infty}$ and $w(I) > 0$ if I is a bounded interval contained in $[x_{-\infty}, \infty)$.

Let $\mathcal{D}$ the space of the C^∞ functions with compact support defined on $I\!R$. By $\mathcal{D}(c, +\infty)$ we denote the space of C^∞ functions with bounded support contained in $(c, +\infty)$. The dual spaces of $\mathcal{D}$ and $\mathcal{D}(c, +\infty)$ shall be denoted by $\mathcal{D}'$ and $\mathcal{D}'(c, +\infty)$. Its elements are called distributions.

Let $\mathcal{S}$ be the space of C^∞ functions on $I\!R$ with rapidly decreasing derivatives of all orders. The topology of $\mathcal{S}$ is given for the family of norms

$$\|\varphi\|_{\alpha,\beta} = \sup_{x \in I\!R} \left[(1 + |x|)^\beta \sum_{k=0}^{\alpha} |D^k \varphi(x)| \right] ,$$

where α is a non-negative integer and β a non-negative real number. For any $c \in I\!R$ let $\mathcal{S}_c$ be the closed subspace of $\mathcal{S}$ of all rapidly decreasing functions with support contained in $[c, +\infty)$. The dual spaces of $\mathcal{S}$ and $\mathcal{S}_c$ are denoted by $\mathcal{S}'$ and $\mathcal{S}_c'$ respectively. The elements of $\mathcal{S}'$ and $\mathcal{S}_c'$ are called tempered distributions. If $-\infty \leq d < \infty$ we define $\mathcal{S}_d^+$ as the union

$$\mathcal{S}_d^+ = \bigcup_{d<c} \mathcal{S}_c .$$

By $(\mathcal{S}_d^+)'$ we denote the set of linear functionals of $\mathcal{S}_d^+$ that restricted to $\mathcal{S}_c$ belong to $\mathcal{S}_c'$, for any number c greater than d.

Let γ be a non-negative integer. We say that φ belongs to the class Φ_γ^+ if $\varphi \in \mathcal{D}$, $\text{supp}(\varphi) \subseteq I$, where I is a bounded interval of the form $[-a, 0]$, $a > 0$, and $|I|^{\gamma+1} \|D^\gamma \varphi\|_\infty \leq 1$.

Let f belong to $(\mathcal{S}_d^+)'$. We define the maximal function $f_{+,\gamma}^*$ of f as

$$f_{+,\gamma}^*(x) = \sup_{\varphi \in \Phi_\gamma^+} | < f, \varphi(x - \cdot\,) > | ,$$

for $x > d$. Given a weight $w(x)$ in the class A_q^+ and $0 < p < \infty$, we say that $f \in \mathcal{D}'(x_{-\infty +\infty})$ belongs to $H^p_{+,\gamma}(w)$ if

$$\left(\int_{x_{-\infty}}^{\infty} f^*_{+,\gamma}(x)^p w(x) dx \right)^{1/p} = \|f\|_{H^p_{+,\gamma}(w)} < \infty \, .$$

It is easy to see that this definition of the space $H^p_{+,\gamma}(w)$ coincides with the definition given in [RS1].

If $\phi \in \mathcal{S}$ and $\operatorname{supp}(\phi) \subset (-\infty, 0]$, $f \in (\mathcal{S}^+_{x_{-\infty}})'$ and $x > x_{-\infty}$, then

$$f * \phi_t(x) = < f, \phi_t(x - \cdot\,) > ,$$

where $\phi_t(x) = t^{-1}\phi(x/t)$ is well defined. We shall consider three other maximal functions:

$$M_0(f, \phi, x) = \sup_{t>0} |f * \phi_t(x)| \, ,$$

$$M_1^+(f, \phi, x) = \sup_{0 \le y - x < t} |f * \phi_t(y)| \, , \quad \text{and}$$

$$N_\lambda^+(f, \phi, x) = \sup_{y \ge x, t > 0} \frac{|f * \phi_t(y)|}{\left(1 + \frac{y-x}{t}\right)^\lambda} \, ,$$

where $\lambda > 0$ and $x > x_{-\infty}$. We shall say that $f \in (\mathcal{S}^+_{x_{-\infty}})'$ belongs to $H^p_{+,\phi}(w)$ if

$$\left(\int_{x_{-\infty}}^{\infty} M_1^+(f, \phi, x)^p w(x) dx \right)^{1/p} = \|f\|_{H^p_{+,\phi}(w)} < \infty \, ,$$

again, $0 < p < \infty$ and $w \in A_q^+$.

2. Statement of the main results

The purpose of this paper is to show that, under suitable condition on p, γ, q and ϕ, the spaces $H^p_{+,\gamma}(w)$ and $H^p_{+,\phi}(w)$ are the same and that the norms on these spaces are equivalent. The method we shall apply in order to obtain our results relies on Calderon's formula. In this connection see [CT] and [StT].

The main results of this paper are stated in the following theorems.

Theorem A. *Let w belong to A_q^+ and $0 < p < \infty$. If β and γ are positive integers satisfying $p(\beta+1) > q$ and $p(\gamma+1) > q$, then*

$$H^p_{+,\beta}(w) = H^p_{+,\gamma}(w) \ . \tag{2.1}$$

Moreover, for any distribution $f \in \mathcal{D}'(x_{-\infty}, +\infty)$

$$c_1 \|f\|_{H^p_{+,\beta}(w)} \le \|f\|_{H^p_{+,\gamma}(w)} \le c_2 \|f\|_{H^p_{+,\beta}(w)} \tag{2.2}$$

hold with constants c_1 and c_2 not depending on f.

Theorem B. *Let w belong to A_q^+, $0 < p < \infty$, $\phi \in \mathcal{S}$ with support contained in $(-\infty, 0]$ and $\int \phi \neq 0$. Then, if $p(\gamma+1) > q$*

$$H^p_{+,\phi}(w) = H^p_{+,\gamma}(w) \ .$$

Moreover, if f belong to $(\mathcal{S}^+_{x_{-\infty}})'$, then

$$c_1 \|f\|_{H^p_{+,\gamma}(w)} \le \|f\|_{H^p_{+,\phi}(w)} \le c_2 \|f\|_{H^p_{+,\gamma}(w)}$$

hold with constants c_1 and c_2 not depending on f.

Theorem C. *Let w belong to A_q^+, $0 < p < \infty$, $\phi \in \mathcal{S}$ with support contained in $(-\infty, 0]$ and $\int \phi \neq 0$. If $p(\gamma+1) > q$ and $f \in H^p_{+,\gamma}(w)$ then*

$$c_1 \|f\|^p_{H^p_{+,\gamma}(w)} \le \int_{x_{-\infty}}^{\infty} M_0(f, \phi, x)^p w(x) dx \le c_2 \|f\|^p_{H^p_{+,\gamma}(w)} \tag{2.3}$$

hold with constants c_1 and c_2 not depending on f.

3. Proof of the results

The main ingredient in the proof of the theorems A, B and C is the following one sided version of Calderon's formula, see for instance [CT].

Lemma (3.1) *Let $\phi \in \mathcal{S}$, $\phi \not\equiv 0$ and* $\operatorname{supp}(\phi) \subset (-\infty, 0]$. *Given a non-negative integer n, there exists $\eta \in \mathcal{D}$ with support contained in $(-\infty, 0]$ and $D^k\hat{\eta}(0) = 0$ for every k, $0 \le k \le n$, such that*

$$\int_0^\infty \hat{\phi}(sx)\hat{\eta}(sx)\frac{ds}{s} = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} . \tag{3.2}$$

Proof. The Fourier transform $\hat{\eta}$ of the function η we are looking for has a zero of order greater than or equal to $n+1$ at the origen. Thus, it can be written as $\hat{\eta}(x) = (2\pi i x)^{n+1}\hat{\rho}(x)$ with $\rho \in \mathcal{S}$ and $\operatorname{supp}(\rho) \subset (-\infty, 0]$. Then, the integral in (3.2) is equal to

$$2\pi i \int_0^\infty x\widehat{D^n\phi}(sx)\hat{\rho}(sx)ds = 2\pi i \int_0^\infty x\hat{\varphi}(sx)\hat{\rho}(sx)ds \ , \tag{3.3}$$

where $\varphi = D^n\phi$. By change of variables, it follows that (3.3) is equal to

$$2\pi i \int_0^\infty \hat{\varphi}(s)\hat{\rho}(s)ds \ , \quad \text{if} \quad x > 0 \ , \quad \text{and} \tag{3.4}$$

$$-2\pi i \int_{-\infty}^0 \hat{\varphi}(s)\hat{\rho}(s)ds \ , \quad \text{if} \quad x < 0 \ . \tag{3.5}$$

Since the supports of φ and ρ are contained in $(-\infty, 0]$, the functions $\hat{\varphi}$ and $\hat{\rho}$ are boundary values of analytic functions on the upper half-plane and rapidly decreasing for $|z|$ tending to infinity. Then, by Cauchy's Theorem, we have

$$\int_{-\infty}^\infty \hat{\varphi}(s)\hat{\rho}(s)ds = 0 \ .$$

This implies that (3.4) and (3.5) have the same value. Therefore, in order to prove the lemma we should find ρ satisfying

$$2\pi i \int_0^\infty \hat{\varphi}(z)\hat{\rho}(s)ds = 1 \ . \tag{3.6}$$

By Cauchy's Theorem, we have

$$\begin{aligned} 2\pi i \int_0^\infty \hat{\varphi}(s)\hat{\rho}(s)ds &= -2\pi \int_0^\infty \hat{\varphi}(is)\hat{\rho}(is)ds \\ &= -2\pi \int_0^\infty \left(\int_{-\infty}^0 e^{2\pi sx}\varphi(x)dx\right)\left(\int_{-\infty}^0 e^{2\pi sy}\rho(y)dy\right) ds \ . \end{aligned}$$

It is easy to show that the triple integral above is absolutely convergent and equal to

$$
(3.7) \qquad -\int_{-\infty}^{0}\left(\int_{-\infty}^{0}\frac{1}{-y-x}\varphi(x)dx\right)\rho(y)dy = -\int_{-\infty}^{0}H(\varphi)(-y)\rho(y)dy
$$

where $H(\varphi)$ stands for the Hilbert transform of φ. If (3.7) were equal to zero for every $\rho \in \mathcal{D}$ with support contained in $(-\infty, 0]$, then $H(\varphi)(y)$ would be equal to zero for $y > 0$. It is well known that this implies that φ, and therefore ϕ, are identically equal to zero, which is not the case. Thus, there exists ρ satisfying (3.6) as we wanted to show. ∎

The following lemma is a consequence of Lemma (3.1).

Lemma (3.8) *Let ψ belong to $\mathcal{S}$ with $D^k\hat{\psi}(0) = 0$ for every k, $0 \le k \le m$. If ϕ and η are as in Lemma (3.1), then*

$$
\lim_{\substack{a\to 0\\ b\to\infty}}\left\|\hat{\psi}(x)\int_a^b\hat{\phi}(sx)\hat{\eta}(sx)\frac{ds}{s} - \hat{\psi}(x)\right\|_{m,h} = 0
$$

for every $h \ge 0$. Moreover, there exists a constant C such that

$$
\left\|\hat{\psi}(x)\int_a^b\hat{\phi}(sx)\hat{\eta}(sx)\frac{ds}{s} - \hat{\psi}(x)\right\|_{m+1,h} \le C
$$

holds for every $0 < a < 1 < b < \infty$.

<u>*Proof*</u>. By Lemma (3.1), we have

$$
\hat{\psi}(x)-\hat{\psi}(x)\int_a^b\hat{\phi}(sx)\hat{\eta}(sx)\frac{ds}{s} = \hat{\psi}(x)\left(\int_0^a+\int_b^\infty\right)\hat{\phi}(sx)\hat{\eta}(sx)\frac{ds}{s} = A(a,x)+B(b,x).
$$

The hypotheses on ψ and η imply that $\hat{\psi}(x) = x^{m+1}\hat{\rho}(x)$ and $\hat{\eta}(x) = x\hat{\mu}(x)$, where ρ and μ belong to $\mathcal{S}$ with their supports contained in $(-\infty, 0]$. Let us estimate $A(a,x)$. Since

$$
A(a,x) = \hat{\rho}(x)x^{m+2}\int_0^a(\hat{\phi}\hat{\mu})(sx)ds\ ,
$$

the derivative $D_x^\alpha A(a,x)$, for $0 \le \alpha \le m+1$, is equal to a finite sum of terms of the form constants times

$$D^{\alpha_1}\hat\rho(x)x^{m+2-\alpha_2}\int_0^a D^{\alpha_3}(\hat\phi\hat\mu)(sx)s^{\alpha_3}ds \ , \tag{3.9}$$

where $\alpha = \alpha_1+\alpha_2+\alpha_3$. We observe that all the terms of the form (3.9) tend to zero for x tending to zero and are bounded by a constant times

$$|D^{\alpha_1}\hat\rho(x)|(1+|x|)^{m+2}a \ .$$

Therefore,

$$(1+|x|)^h\sum_{\alpha=0}^{m+1}|D^\alpha A(a,x)| \le C\cdot a \ ,$$

where the constant c does not depends on a. This implies that $\|A(a,x)\|_{m+1,h}$ is bounded for $0<a<1$ and that it goes to zero for a tending to zero.

Procceding as before, we have

$$B(b,x) = \hat\rho(x)x^{m+2}\int_b^\infty(\hat\phi\hat\mu)(sx)ds \ .$$

Thus, its derivative of order α, $0 \le \alpha \le m+1$, turn out to be a sum of constants times terms of the form

$$D^{\alpha_1}\hat\rho(x)x^{m+1-\alpha_2-\alpha_3}\int_b^\infty D^{\alpha_3}(\hat\phi\hat\mu)(sx)^{\alpha_3}dsx \ , \tag{3.10}$$

where $\alpha = \alpha_1+\alpha_2+\alpha_3$. The integral above is equal to

$$\begin{aligned} &\int_{bx}^\infty D^{\alpha_3}(\hat\phi\hat\mu)(s)s^{\alpha_3}ds && \text{if } \ x>0\ , \quad \text{and} \\ -&\int_{-\infty}^{bx} D^{\alpha_3}(\hat\phi\hat\mu)(s)s^{\alpha_3}ds && \text{if } \ x<0\ . \end{aligned} \tag{3.11}$$

Since $D^{\alpha_3}(\hat\phi\hat\mu)(s)s^{\alpha_3}$ is the boundary value of an analytic function on the upper half-plane, by Cauchy's Theorem, we get that the limit of $D^\alpha B(b,x)$ for x tending to zero exists if $0 \le \alpha \le m+1$. Moreover it follows from (3.10) that

$$(1+|x|)^h\sum_{\alpha=0}^{m+1}|D^\alpha B(b,x)| \le C_m\|\hat\rho\|_{m+1,h+m+1}\|\hat\phi\hat\mu\|_{m+1,m+3} \ .$$

If we restrict α to be less than or equal to m, the integrals in (3.11) are bounded by

$$\|\hat{\phi}\hat{\mu}\|_{m,m+2} \int_{b|x|}^{\infty} \frac{ds}{s^2} = \frac{1}{|x|b} \|\hat{\phi}\hat{\mu}\|_{m,m+2} \ .$$

Then,

$$(1+|x|)^h \sum_{\alpha=0}^{m} |D^\alpha B(b,x)| \leq C_m \|\hat{\rho}\|_{m,h+m} \|\hat{\phi}\hat{\mu}\|_{m,m+2} b^{-1} \ .$$

This shows that

$$\lim_{b\to\infty} \|B(b,x)\|_{m,h} = 0 \ ,$$

for every $h \geq 0$. ∎

The following theorem and Lemma (3.18) will be used in the proof of Lemma (3.21).

Theorem (3.12) *Let $\psi \in \mathcal{S}$, $\operatorname{supp}(\psi) \subset (-\infty, 0]$ and $D^k\hat{\psi}(0) = 0$ for every k, $0 \leq k \leq m$, and assume that ϕ and η are as in Lemma (3.1). If $f \in \mathcal{S}'_c$ and there exist a non-negative integer ℓ and $0 \leq \alpha < 1$ such that*

$$| < f, \rho > | \leq K \|\rho\|_{\ell, m+\alpha} \ ,$$

for every $\rho \in \mathcal{S}_c$, then

$$f * \psi_t(x) = \int_0^\infty \left(\int (f * \phi_s)(y)(\psi_t * \eta_s)(x-y)dy \right) \frac{ds}{s} \tag{3.13}$$

holds for every $x > c$.

<u>*Proof*</u>. By Lemma (3.8), the function

$$\hat{\psi}(x) \int_a^b \hat{\phi}(sx)\hat{\eta}(sx)\frac{ds}{s} - \hat{\psi}(x) \tag{3.14}$$

converges to zero in the norm $\| \ \|_{m,h}$ and its norm $\| \ \|_{m+1,h}$ is bounded uniformly for $0 < a < 1 < b < \infty$, for every $h \geq 0$. The inverse Fourier transform of (3.14) is equal to

$$\int_a^b \psi * \phi_s * \eta_s(x) \frac{ds}{s} - \psi(x) = \psi_{a,b}(x) \ .$$

Then, taking $h=\ell+2$ we have that

$$\|\psi_{a,b}\|_{\ell,m+1} \leq C \ ,$$

holds for every $0<a<1<b<\infty$, and

$$\lim_{\substack{a\to 0\\ b\to\infty}} \|\psi_{a,b}\|_{\ell,m} = 0 \ .$$

This implies that for every k, $0\leq k\leq \ell$,

$$(1+|x|)^{m+\alpha}|D^k\psi_{a,b}(x)| \leq \|\psi_{a,b}\|_{\ell,m+1}(1+|x|)^{\alpha-1} \ , \quad \text{and} \tag{3.15}$$

$$(1+|x|)^{m+\alpha}|D^k\psi_{a,b}(x)| \leq \|\psi_{a,b}\|_{\ell,m}(1+|x|)^{\alpha} \ . \tag{3.16}$$

Given $\varepsilon>0$, by (3.15) there exists N such that

$$(1+|x|)^{m+\alpha}|D^k\psi_{a,b}(x)| < \varepsilon \tag{3.17}$$

holds for $|x|\geq N$ uniformly in $0<a<1<b<\infty$. If $|x|<N$, (3.16) shows that (3.17) holds true if a is small and b is large. Thus, we have shown that

$$\lim_{\substack{a\to 0\\ b\to\infty}} \|\psi_{a,b}\|_{\ell,m+\alpha} = 0 \ .$$

This implies that

$$< f,\psi >= \lim_{\substack{a\to 0\\ b\to\infty}} < f, \int_a^b \psi*\phi_s*\eta_s(x)\frac{ds}{s} > \ ,$$

where right hand side can be easily shown to be equal to

$$\lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b < f,\psi*\phi_s*\eta_s > \frac{ds}{s} \ .$$

In particular taking $\psi_t(x-\cdot)$ instead of $\psi(\cdot)$ we obtain (3.13) as we wanted to show. ∎

Lemma (3.18) *Let* $\psi\in\mathcal{S}$, $\operatorname{supp}(\psi)\subset(-\infty,0]$ *and* $\int\psi=0$. *If* $\eta\in\mathcal{D}$ *with* $\operatorname{supp}(\eta)\subset(-\infty,0]$ *and* $\int x^k\eta(x)dx=0$ *for every* k, $0\leq k\leq n-1$, *then*

$$|\psi_t*\eta_s(x)| \leq \begin{cases} 0 & \textit{if } x>0 \ \text{ and} \\ C(n,\eta)\|\psi\|_{n,n+4}\frac{1}{s}\frac{\min(t/s,(s/t)^r)}{\left(1+\frac{|x|}{s}\right)^{n+1-r}} & \textit{if } x\leq 0 \end{cases} \tag{3.19}$$

holds for every r , $0 \le r \le n+1$.

Proof. Since, for $t > 0$ and $s > 0$,

$$\psi_t * \eta_s(x) = \frac{1}{s}\left(\psi_{t/s} * \eta\right)(x/s)$$

holds, it is enough to consider the case $s = 1$. Let us first assume that $0 < t \le 1$. By hypothesis $\int \psi = 0$, thus

$$\begin{aligned}\psi_t * \eta(x) &= \int \psi(y)[\eta(x-ty) - \eta(x)]dy \\ &= \left(\int_{|y|\le 1} + \sum_{k=0}^{\infty} \int_{2^k < |y| \le 2^{k+1}}\right) \psi(y)[\eta(x-ty) - \eta(x)]dy \ .\end{aligned}$$

Using the mean value theorem, we have

$$|\eta(x-ty) - \eta(x)| \le t|y||\eta'(x - t\theta y)| \le \frac{t|y| \, \|\eta\|_{1,n+1}}{(1 + |x - t\theta y|)^{n+1}} \ .$$

If $|x| \ge 2^{k+2}$ and $|y| \le 2^{k+1}$, we have $|x - t\theta y| \ge \frac{|x|}{2}$ and therefore, the k-th term of the series is bounded by

$$t\, 2^{n+1} \frac{\|\eta\|_{1,n+1}}{(1+|x|)^{n+1}} \int_{2^k < |y| \le 2^{k+1}} |y| \ |\psi(y)| dy \ .$$

If $|x| < 2^{k+2}$, the k-th term is bounded by

$$\begin{aligned}t\|\eta\|_{1,n+1} \int_{2^k < |y| \le 2^{k+1}} |y||\psi(y)|dy &\le t \frac{2^{(k+3)(n+1)}}{(1+|x|)^{n+1}} \|\eta\|_{1,n+1} \int_{2^k < |y| \le 2^{k+1}} |y| \, |\psi(y)|dy \\ &\le \frac{t\, 2^{3(n+1)}}{(1+|x|)^{n+1}} \|\eta\|_{1,n+1} \int_{2^k < |y| \le 2^{k+1}} |y|^{n+2} |\psi(y)|dy \ .\end{aligned}$$

Then, if $0 < t \le 1$, $|(\psi_t * \eta)(x)|$ is bounded by

$$\begin{aligned}C(n,\eta) \frac{t}{(1+|x|)^{n+1}} \int |\psi(y)|(1+|y|)^{n+2} dy &\le C(n,\eta) \|\psi\|_{0,n+4} \frac{t}{(1+|x|)^{n+1}} \\ &\le C(n,\eta) \|\psi\|_{n,n+4} \frac{t}{(1+|x|)^{n+1-r}} \ .\end{aligned}$$

For the case $1 < t < \infty$, we assume that $\text{supp}(\eta) \subset [-a, 0]$, $a > 0$. Since by hypothesis η has moments equal to zero up to the order $n-1$, it follows that

$$\psi_t * \eta(x) = \frac{1}{t} \int \left[\psi\left(\frac{x-y}{t}\right) - \sum_{k=0}^{n-1} \frac{1}{k!} \left(\frac{-y}{t}\right)^k (D^k\psi)\left(\frac{x}{t}\right) \right] \eta(y) dy .$$

The expression between brackets is bounded by

$$\frac{1}{n!} \left(\frac{|y|}{t}\right)^n \frac{\|\psi\|_{n,n+1}}{\left(1 + \frac{|x-\theta y|}{t}\right)^{n+1}} , \qquad 0 < \theta < 1 . \tag{3.20}$$

If $|x| \geq 2a$, since $|y| \leq a$ this expression is bounded by

$$\frac{2^{n+1}}{n!} \left(\frac{|y|}{t}\right)^n \frac{\|\psi\|_{n,n+1}}{\left(1 + \frac{|x|}{t}\right)^{n+1}} .$$

Thus

$$|\psi_t * \eta(x)| \leq \frac{2^{n+1}}{n!} \frac{\|\psi\|_{n,n+1}}{(t+|x|)^{n+1}} \int |y|^n |\eta(y)| dy$$

whenever $|x| \geq 2a$. If $|x| < 2a$, by (3.20), we have

$$\begin{aligned} |\psi_t * \eta(x)| &\leq \frac{1}{n!} \frac{1}{t^{n+1}} \|\psi\|_{n,n+1} \int |y|^n |\eta(y)| dy \\ &\leq \frac{1}{n!} \left(\frac{1+2a}{t+|x|}\right)^{n+1} \|\psi\|_{n,n+1} \int |y|^n |\eta(y)| dy . \end{aligned}$$

Thus for $t > 1$ and any x we have obtained

$$|\psi_t * \eta(x)| \leq C(n,\eta) \frac{\|\psi\|_{n,n+1}}{(t+|x|)^{n+1}} .$$

Since for $1 < t$, $t^r(1+|x|)^{n+1-r} \leq (t+|x|)^r (t+|x|)^{n+1-r} = (t+|x|)^{n+1}$, it follows that

$$|\psi_t * \eta(x)| \leq C(n,\eta) \frac{\|\psi\|_{n,n+4}}{(1+|x|)^{n+1-r}} t^{-r} .$$

Therefore, collecting our estimates we get that (3.19) holds for $x \leq 0$. Since the supports of ψ and η are both contained in $(-\infty, 0]$ it follows that $\psi_t * \eta(x) = 0$ if $x > 0$. ∎

The next lemma is in the core of the proofs of Theorems A, B and C.

Lemma (3.21) *Let $f \in \mathcal{S}'_c$, ℓ, m non-negative integers and $0 \le \alpha < 1$ such that*

$$| < f, \varphi > | \le K \|\varphi\|_{\ell, m+\alpha} ,$$

holds for every $\varphi \in \mathcal{S}_c$. Let ψ, ϕ and η with their supports contained in $(-\infty, 0]$, ψ and ϕ in $\mathcal{S}$ and η in $\mathcal{D}$. We assume that $D^k \hat{\psi}(0) = 0$ for $0 \le k \le m$, and that there exists a non-negative integer n such that $D^h \eta(0) = 0$ holds for $0 \le h \le n-1$. Then, we have

$$|f * \psi_t(x)| \le C(\varepsilon, n, \eta) \|\psi\|_{n,n+4} N^+_\lambda(f, \phi, x)^{\frac{n-r}{n-r+1}} M^+ \left[M_0(f, \phi, \cdot\,)^{\frac{1}{n-r+1}} \right](x)$$

holds for $x > c$, where $\lambda = (n-r-\varepsilon)(n-r+1)/(n-r)$, $r > 0$, $\varepsilon > 0$ and $n > r + \varepsilon$.

Proof. By Theorem (3.12) we have

$$f * \psi_t(x) = \int_0^\infty \int f * \phi_s(y) \psi_t * \eta_s(x-y) dy \frac{ds}{s} .$$

The estimate of $\psi_t * \eta_s$ given in Lemma (3.18) shows that

$$|f * \psi_t(x)| \le C(n, \eta) \|\psi\|_{n,n+4} \int_0^\infty \int_x^\infty \frac{|f * \phi_s(y)|}{\left(1 + \frac{y-x}{s}\right)^{n-r+1}} \min(t/s, (s/t)^r) dy \frac{ds}{s^2} .$$

By the definitions of N^+_λ and M_0, we get

$$|f * \psi_t(x)|$$

$$\le C(n, \eta) \|\psi\|_{n,n+4} N^+_\lambda(f, \phi, x)^{\frac{n-r}{n-r+1}} \int_0^\infty \int_x^\infty \frac{M_0(f, \phi, y)^{\frac{1}{n-r+1}}}{\left(1 + \frac{y-x}{s}\right)^{1+\varepsilon}} \min(t/s, (s/t)^r) dy \frac{ds}{s^2},$$

where λ is taken as equal to $(n-r-\varepsilon)(n-r+1)/(n-r)$. The double integral above is bounded by a constant depending on ε times

$$M^+ \left[M_0(f, \phi, \cdot\,)^{\frac{1}{n-r+1}} \right](x) .$$

This ends the proof of the lemma. ∎

The proof of Theorem (3.23) requires the following vector valued extension of the Sawyer's one-sided maximal function theorem.

Lemma (3.22) *Let $1 < p < \infty$, $1 < q < \infty$ and $w \in A_q^+$. Then, if $\{f_j\}_{j=1}^{\infty}$ is any sequence of measurable functions,*

$$\int \left(\sum_{j=1}^{\infty} M^+ f_j(x)^p \right)^{q/p} w(x)dx \le C(p,q,w) \int \left(\sum_{j=1}^{\infty} |f_j(x)|^p \right)^{q/p} w(x)dx$$

holds with $C(p,q,w)$ not depending on the sequence.

Proof. This lemma is a consequence of Lemma 4 in [RS2], page 936, applied to $U_j g(x) = U g(x)$, where U is the operator $g \mapsto \left\{ \frac{1}{r} \int_x^{x+r} g(y)dy \right\}_{r>0}$, and B is the Banach space $L^\infty(0,\infty)$. ∎

Theorem (Approximation theorem) (3.23) *Let $0 < p < \infty$, $p(\gamma+1) \ge q > 1$ and $w \in A_q^+$. Given $f \in H^p_{+,\gamma}(w)$ and $\varepsilon > 0$ there exists a function $g(x)$, locally integrable on $(x_{-\infty}, +\infty)$, such that*

(i) $$\|f - g\|_{H^p_{+,\gamma}(w)} < \varepsilon \, .$$

Moreover, there exists α, $0 < \alpha < 1$, depending on w only, such that

(ii) $$\int_c^\infty \frac{|g(x)|}{(x-b)^\alpha} dx < \infty$$

holds for every $x_{-\infty} < b < c$.

Proof. We can allways assume that $q \ge p$ since otherwise $w \in A_q^+$ implies that $w \in A_p^+$. For the case $0 < p \le 1$, part (i) was proved in [RS1], Theorem (2.1), page 176. Going to the proof of the mention theorem in [RS1] with $F = f$, we see that the hypothesis $0 < p \le 1$ was used to estimate the norm $\|B\|_{H^p_{+,\gamma}(w)}$ of the distribution B defined there. In order to show that the estimate holds for any p, $0 < p < \infty$, we shall apply Lemma (3.22). In fact, taking into account part (i) of the Proposition (4.27) of [RS1] on page 175 it follows that

$$B^*_{+,\gamma}(x) \le C \left\{ f^*_{+,\gamma}(x) \, \chi_{c\Omega}(x) + \sum_{(k,i)\in\mathcal{F}} \lambda_k \sum_{j=1}^{\infty} \left[M^+ \chi_{I_{k,i,j}}(x) \right]^{\gamma+1} \right\} .$$

Thus,

$$\begin{aligned}
&\int_{x-\infty}^{\infty} B_{+,\gamma}^*(x)^p w(x)dx \le 2^p C^p \int_\Omega f_{+,\gamma}^*(x)^p w(x)dx \\
(3.24)\qquad &+2^p C^p \int_{x-\infty}^{\infty} \left(\sum_{(k,i)\in\mathcal{F}} \sum_{j=1}^{\infty} \left[M^+ \left(\lambda_k^{1/(\gamma+1)} \chi_{I_{k,i,j}} \right)(x) \right]^{\gamma+1} \right)^{\frac{p(\gamma+1)}{\gamma+1}} w(x)dx \ .
\end{aligned}$$

Since we assume that $p(\gamma+1) \ge q > 1$ and $w \in A_q^+$, we have $A_{p(\gamma+1)}^+$. Thus, by Lemma (3.22) we get that the second integral on the right hand side of (3.24) is bounded by a constant times

$$\int_{x-\infty}^{\infty} \left(\sum_{(k,i)\in\mathcal{F}} \sum_{j=1}^{\infty} \lambda_k \, \chi_{\, I_{k,i,j}}(x) \right)^p w(x)dx \ . \tag{3.25}$$

Since $I_{k,i,j} \subset I_{k,i} \subset \Omega_k$ we have $\lambda_k \, \chi_{\, I_{k,i,j}}(x) \le f_{+,\gamma}^*(x)$. Then, (3.25) is smaller than or equal to

$$\int_\Omega f_{+,\gamma}^*(x)^p w(x)dx \ .$$

Therefore, we have shown that

$$\int_{x-\infty}^{\infty} B_{+,\gamma}^*(x)^p w(x)dx \le C(p,q,w) \int_\Omega f_{+,\gamma}^*(x)^p w(x)dx$$

holds for $0 < p < \infty$, that is to say, we can remove the assumption $0 < p \le 1$ of Theorem (2.1) in [RS1].

As for part (ii), we observe that $(x-b)^{-1} \le C(b,c) M^- \chi_{\,(b,c)}(x)$ for $x \ge c$ and $|g(x)| \le g_{+,\gamma}^*(x)$. Then, by Hölder's inequality,

$$\begin{aligned}
\int_c^\infty \frac{|g(x)|}{(x-b)^\alpha} dx &\le C \int_c^\infty M^- \chi_{(b,c)}(x)^\alpha g_{+,\gamma}^*(x) dx \\
&\le C \left(\int M^- \chi_{(b,c)}(x)^{\alpha q'} w(x)^{-q'/q} dx \right)^{1/q'} \left(\int_c^\infty g_{+,\gamma}^*(x)^q w(x) dx \right)^{1/q}.
\end{aligned}$$

If α is smaller than one but close enough to one, the weight $w^{-q'/q}$ belongs to the class $A_{\alpha q'}^-$, thus

$$\int_c^\infty \frac{|g(x)|}{(x-b)^\alpha} dx \le C \left(\int_b^c w(x)^{-q'/q} dx \right)^{1/q'} \left(\int_c^\infty g_{+,\gamma}^*(x)^q w(x) dx \right)^{1/q} \ ,$$

where the right hand side is finite as it is shown in Theorem (2.1) of [RS1].

The following lemma is well known and shall be used in the proofs of Theorems A, B and C.

Lemma (3.26) *Let* $w \in A_q^+$, $0 < p < \infty$ *and* $f \in \left(\mathcal{S}_{x_{-\infty}}^+\right)'$. *Then, if* $\lambda > q/p$,

$$\|N_\lambda^+(f,\phi,x)\|_{L^p((x_{-\infty},\infty),w)} \leq C(p,q,w)\|M_1^+(f,\phi,x)\|_{L^p((x_{-\infty},\infty),w)}$$

holds with $C(p,q,w)$ *not depending on* f *and* ϕ.

The proof of this lemma follows the same lines of Lemma (5.6), page 243 of [GR] and shall not be given here.

Proof of Theorem A. Without loss of generality we can assume that $q > 1$ and $\beta < \gamma$. Since $\Phi_\beta^+ \supset \Phi_\gamma^+$ it follows that $f_{+,\gamma}^*(x) \leq f_{+,\beta}^*(x)$ for every $x > x_{-\infty}$. Let $g(x)$ be a function belonging to $H_{+,\gamma}^p(w)$ satisfying that for a given α, $0 < \alpha < 1$, and every pair (b,c), $x_{-\infty} < b < c < \infty$

$$\int_c^\infty \frac{|g(x)|}{(x-b)^\alpha}dx = C(b,c) < \infty \ . \tag{3.27}$$

By Theorem (3.23) these functions g are dense in $H_{+,\gamma}^p(w)$. Let σ belong to Φ_β^+ with support contained in $I = [-t,0]$, $t > 0$, and let $\rho(y) = t\sigma(ty)$. Then $\operatorname{supp}(\rho) \subseteq [-1,0]$ and $\|D^\beta\rho\|_\infty \leq 1$.

Let $\phi \in \mathcal{D}$, $\operatorname{supp}(\phi) \subseteq [-1,0]$ and $\int\phi = 1$. Then

$$\begin{aligned}\rho(y) &= \left[\rho(y) - \int\rho(z)dz\cdot\phi(y)\right] + \int\rho(z)dz\cdot\phi(y)\\ &= \psi(y) + \left(\int\rho\right)\phi(y) \ .\end{aligned}$$

Then, if $x_{-\infty} < x$, we have

$$< g, \sigma(x - \cdot\,) > = g * \rho_t(x) = g * \psi_t(x) + \left(\int\rho\right) g * \phi_t(x) \ .$$

By construction, the integral of ψ is equal to zero, that is to say $\hat\psi(0) = 0$. We can choose b and c satisfying $x_{-\infty} < b < c < x$. By (3.27), the distribution

$g \in \mathcal{S}'_c$ defined as $\int g\varphi dx$ satisfies the condition $| < g, \varphi > | \leq C(b,c)\|\varphi\|_{0,\alpha}$. Then applying Lemma (3.21) with $m = 0$ and $n = \beta$ to the distribution $f = g$ we get

$$|g * \psi_t(x)| \leq C(\varepsilon, \beta, \eta)\|\psi\|_{\beta,\beta+4} N^+_\lambda(g, \phi, x)^{\frac{\beta-r}{\beta-r+1}} \cdot M^+\left[M_0(g, \phi, \cdot)^{\frac{1}{\beta-r+1}}\right](x) ,$$

where $\lambda = (\beta - r - \varepsilon)(\beta - r + 1)/(\beta - r)$, $\varepsilon > 0$, $r > 0$ and $\beta > r + \varepsilon$ to be suitably chosen.

By definition of ψ it follows that $\|\psi\|_{\beta,\beta+4}$ is bounded by $2^{\beta+4}(\beta + 1) + \|\phi\|_{\beta,\beta+4} = C(\beta, \phi)$. Thus, $|g * \psi_t(x)|$ is bounded by

$$C(\varepsilon, \beta, \eta, \phi) N^+_\lambda(g, \phi, x)^{\frac{\beta-r}{\beta-r+1}} M^+\left[M_0(g, \phi, \cdot)^{1/(\beta-r+1)}\right](x) .$$

On the other hand, since $|\int \rho| \leq \|\rho\|_\infty \leq 1$ we get

$$\left|\left(\int \rho\right) g * \phi_t(x)\right| \leq M_0(g, \phi, x) .$$

Therefore,

$$g^*_{+,\beta}(x) \leq C N^+_\lambda(g, \phi, x)^{\frac{\beta-r}{\beta-r+1}} M^+\left[M_0(g, \phi, \cdot)^{1/(\beta-r+1)}\right](x) ,$$

where the constant C is independent of g, b and c. Applying Holder's inequality we get

$$\begin{aligned} \int g^*_{+,\beta}(x)^p w(x)dx &\leq C \left(\int N^+_\lambda(g, \phi, x)^p w(x)dx\right)^{\frac{\beta-r}{\beta-r+1}} \\ &\cdot \left[\int \left(M^+\left[M_0(g, \phi, \cdot)^{1/(\beta-r+1)}\right](x)\right)^{p(\beta-r+1)} w(x)dx\right]^{1/(\beta-r+1)} . \end{aligned} \tag{3.28}$$

Choosing r and ε small enough so that $p\lambda > q$ and $p(\beta - r + 1) > q$, by Lemma (3.26) and Sawyer's theorem on weighted norm estimates for the one sided maximal function, we obtain the inequality

$$\int g^*_{+,\beta}(x)^p w(x)dx \leq C \int M^+_1(g, \phi, x)^p w(x)dx . \tag{3.29}$$

Now, given x, y and t satisfying the conditions $0 \le y - x < t$, the function $\varphi(z) = \phi\left(\frac{y-x}{t} + z\right)$ has its support contained in $[-2, 0]$ and

$$g * \phi_t(y) = g * \varphi_t(x) \ .$$

Since $\varphi(z)/2^{\gamma+1}\|D^\gamma \phi\|_\infty$ belongs to Φ_γ^+, then

$$|g * \phi_t(y)| \le 2^{\gamma+1}\|D^\gamma \phi\|_\infty g^*_{+,\gamma}(x) \ ,$$

whenever $0 \le y - x < t$. Therefore,

$$M_1^+(g, \phi, x) = \sup_{0 \le y - x < t} |g * \phi_t(y)| \le 2^{\gamma+1}\|D^\gamma \phi\|_\infty g^*_{+,\gamma}(x) \ .$$

From this inequality and (3.29) it follows that

$$\int g^*_{+,\beta}(x)^p w(x) dx \le C \int g^*_{+,\gamma}(x)^p w(x) dx \ ,$$

where the constant C does not depend on g.

We have shown that the functions g satisfying (3.27) that belong, either to $H^p_{+,\beta}(w)$ or $H^p_{+,\gamma}(w)$, are the same and their norms are equivalent. Since the spaces $H^p_{+,\beta}(w)$ and $H^p_{+,\gamma}(w)$ are complete, the density of the functions g satisfying (3.27) implies that (2.1) and (2.2) hold. ∎

The next three lemmas are used in the proof of Theorem B.

Lemma (3.30) *Let $\rho(x)$ and $\phi(x)$ belong to $\mathcal{S}$, $\int \phi(x)dx \neq 0$. Given a non-negative integer m, there exist a function $a(r) \in \mathcal{D}$ with support contained in $[0, 1]$ such that the functions $\rho(x)$ and*

$$\int a(r)\phi(x + r) dr$$

have the same moments up to the order m. More precisely,

$$a(r) = \sum_{j=0}^{m} a_j(r) \int z^j \rho(z) dz \ ,$$

where the functions $a_j(r)$ belong to $\mathcal{D}$, with their supports contained in $[0, 1]$ and not depending on the function $\rho(x)$.

Proof. Let ν be a function in $\mathcal{D}$ with support contained in $[0,1]$ and $\int \nu(x)dx \neq 0$. We define the matrix $C = \{c_{k,h}\}_{k,h=0}^m$ where

$$c_{k,h} = \int x^k \left(\frac{1}{h!} \int D^h \nu(r) \phi(x+r) dr \right) dx \ .$$

Integrating by parts we get that

$$\begin{aligned} c_{k,h} &= 0 \qquad \text{if} \quad k < h, \quad \text{and} \\ c_{k,k} &= \int \nu(r) \left(\int \phi(x+r) dx \right) dr = \int \nu(r) dr \cdot \int \phi(x) dx \neq 0 \ . \end{aligned}$$

This shows that the matrix C is inversible. Let $M = C^{-1}$ and $m_{k,h}$ its entries. We define $a_j(r)$ as

$$a_j(r) = \sum_{h=0}^m \frac{m_{h,j}}{h!} D^h \nu(r) \ , \qquad\qquad 0 \leq j \leq m \ ,$$

and

$$a(r) = \sum_{j=0}^m a_j(r) \int z^j \rho(z) dz \ .$$

Then, for $0 \leq k \leq m$, we have

$$\begin{aligned} &\int x^k \left(\int a(r) \phi(x+r) dr \right) dx \\ &\qquad = \sum_{j=0}^m \sum_{h=0}^m \frac{1}{h!} \int x^k \int D^h \nu(r) \phi(x+r) dr dx \, m_{h,j} \int z^j \rho(z) dz \\ &\qquad = \sum_{j=0}^m \sum_{h=0}^m c_{k,h} \, m_{h,j} \int z^j \rho(z) dz = \int z^k \rho(z) dz \ , \end{aligned}$$

as we wanted to prove. ∎

Lemma (3.31) *Let f belong to $\mathcal{D}'(c,+\infty)$ and w a weight satisfying $w(I) > 0$ if I is a closed interval contained in $(c,+\infty)$. If $f^*_{+,\gamma}(x)$ belongs to $L^p((c,+\infty),w)$ then the distribution f can be extended to $\mathcal{S}_c^+$ and for $b > c$, $J = [\max(c, b-1), b]$ and $\varphi \in \mathcal{S}_b$*

$$| < f, \varphi > | \leq C_\gamma \left(\frac{1}{w(J)} \int_c^\infty f^*_{+,\gamma}(x)^p w(x) dx \right)^{1/p} \|\varphi\|_{\gamma,\gamma+2}$$

holds with a constant C_γ *depending on* γ *only. In particular, the extension belongs to* $(\mathcal{S}_c^+)'$.

The proof of this lemma can be found in [RS2] as part of the proof of Theorem 4, page 943.

Lemma (3.32) *Let* ϕ *belong to* $\mathcal{S}$, $\operatorname{supp}(\phi) \subset (-\infty, 0]$ *and* $f \in (\mathcal{S}_d^+)'$. *Then*

$$M_1^+(f, \phi, x) \le C_\gamma \|\phi\|_{\gamma,\gamma+2} f_{+,\gamma}^*(x)$$

holds for every $x > d$.

Proof. Let $\{\eta_k(x)\}_{k=0}^\infty$ be a sequence of C^∞ functions satisfying the following properties:

$$\begin{aligned}
&\eta_k(x) \ge 0 \ , \\
&\operatorname{supp}(\eta_0) \subset [-2, 0] \ , \\
&\operatorname{supp}(\eta_k) \subset \left[-2^{k+1}, -2^{k-1}\right] \quad \text{for} \quad k \ge 1 \ , \\
&\|D^\ell \eta_k\|_\infty \le C_\ell 2^{-k\ell} \quad \text{for} \quad k \ge 0 \ , \quad \text{and} \\
&\sum_{k=0}^\infty \eta_k(x) = 1 \quad \text{if} \quad x \le 0 \ .
\end{aligned}$$

Since $\operatorname{supp}(\phi)$ is contained in $(-\infty, 0]$, it follows that

$$\phi(y) = \sum_{k=0}^\infty \eta_k(y)\phi(y) \ . \tag{3.33}$$

We shall show that this series converges in $\mathcal{S}$. Let ℓ and h be non-negative integers, then

$$(1+|y|)^h D^\ell(\eta_k\phi)(y) = \sum_{\ell_1+\ell_2=\ell} C_{\ell_1,\ell_2}(1+|y|)^h D^{\ell_1}\eta_k(y) D^{\ell_2}\phi(y) \ .$$

Thus, by the properties of the sequence $\{\eta_k(x)\}_{k=0}^\infty$ stated above, we have

$$\begin{aligned}
(1+|y|)^h \left|D^{\ell_1}\eta_k(y) D^{\ell_2}\phi(y)\right| &\le C_{\ell_1} 2^{-\ell_1 k} \frac{\|\phi\|_{\ell,h+1}}{(1+|y|)} \chi_{[-2^{k+1},-2^{k-1}]}(y) \\
&\le C_\ell 2^{-(\ell_1+1)k} \|\phi\|_{\ell,h+1} \ .
\end{aligned}$$

This implies that $\|\eta_k\phi\|_{\ell,h} \leq C_\ell 2^{-k}\|\phi\|_{\ell,h+1}$. Therefore,

$$\sum_{k=0}^{\infty} \|\eta_k\phi\|_{\ell,h} \leq C_\ell\|\phi\|_{\ell,h+1} \ .$$

Thus, the series (3.33) converges in $\mathcal{S}$ as we announce. Now, given $x > d$ and z, $0 \leq z - x < t$, we have

$$f * \phi_t(z) = \sum_{k=0}^{\infty} f * (\eta_k\phi)_t(z) \ .$$

For $k \geq 0$ we define $\mu_k(y) = (\eta_k\phi)\left(\frac{z-x}{t} + y\right)$. Then,

$$f * (\eta_k\phi)_t(z) = f * (\mu_k)_t(x) \ .$$

For $k \geq 0$ the support of $(\mu_k)_t(y)$ is contained in $I_k = \left[-t2^{k+2}, 0\right]$. Thus,

$$\begin{aligned} |I_k|^{\gamma+1}|D^\gamma(\mu_k)_t(y)| &\leq \frac{t^{\gamma+1}2^{(k+2)(\gamma+1)}}{t^{\gamma+1}} \sum_{\gamma_1+\gamma_2=\gamma} C_{\gamma_1,\gamma_2}\left|(D^{\gamma_1}\eta_k D^{\gamma_2}\phi)\left(\frac{z-x+y}{t}\right)\right| \\ &\leq C_\gamma 2^{k(\gamma+1)}\chi_{[-2^{k+1},-2^{k-1}]}\left(\frac{z-x+y}{t}\right)\frac{\|\phi\|_{\gamma,\gamma+2}}{\left(1+|\frac{z-x+y}{t}|\right)^{\gamma+2}} \\ &\leq C_\gamma 2^{-k}\|\phi\|_{\gamma,\gamma+2} \ . \end{aligned}$$

This estimates implies that for $0 \leq z - x < t$

$$|f * (\eta_k\phi)_t(z)| \leq C_\gamma 2^{-k}\|\phi\|_{\gamma,\gamma+2} f^*_{+,\gamma}(x) \ .$$

Suming up in k, we get that

$$|f * \phi_t(z)| \leq C_\gamma\|\phi\|_{\gamma,\gamma+2} f^*_{+,\gamma}(x)$$

holds for $0 \leq z - x < t$. ∎

Proof of Theorem B. Let $f \in H^p_{+,\phi}(w)$. Given c, $x_{-\infty} < c < \infty$ there exist $K > 0$ and, ℓ and γ non-negative integers, such that

$$|<f,\varphi>| \leq K\|\varphi\|_{\ell,\gamma} \tag{3.34}$$

holds for any $\varphi \in \mathcal{S}_c$. Enlarging γ, if necessary, we can assume that $p(\gamma+1) > q$. Take $\sigma \in \Phi_\gamma^+$ with support contained in $I = [-t, 0]$ and $t^{\gamma+1}\|D^\gamma \sigma\|_\infty \leq 1$. Let $\rho(y) = t\sigma(ty)$. The support of ρ is contained in $[-1, 0]$ and $\|D^\gamma \rho\|_\infty \leq 1$. By Lemma (3.30), the $\rho(y)$ can be written as

$$\rho(y) = \psi(y) + \int_0^1 a(r)\phi(y+r)dr$$

where ψ has all its moments equal to zero up to the order γ. Since $|\int z^j \rho(z)dz| \leq \|\rho\|_\infty \leq 1$, we have $\int_0^1 |a(r)|dr \leq C_\gamma$. Then,

$$\begin{aligned} \|\psi\|_{\gamma,\gamma+4} &\leq \|\rho\|_{\gamma,\gamma+4} + \int_0^1 |a(r)| \|\phi(\,\cdot\, + r)\|_{\gamma,\gamma+4} dr \\ &\leq 2^{\gamma+4}(\gamma+1) + 2^{\gamma+4} C_\gamma \|\phi\|_{\gamma,\gamma+4} = C(\phi,\gamma)\ . \end{aligned} \tag{3.35}$$

It is easy to see that

$$\begin{aligned} (f * \sigma)(x) &= (f * \rho_t)(x) \\ &= (f * \psi_t)(x) + \int_0^1 a(r) f * \phi_t(x + rt)dr = I + II\ . \end{aligned}$$

For II we have the estimate

$$|II| \leq \int_0^1 |a(r)|dr M_1^+(f,\phi,x) \leq C_\gamma M_1^+(f,\phi,x)\ .$$

As for I, by Lemma (3.21) with $n = \gamma$, $m = \gamma$ and $\alpha = 0$ and (3.35), we have

$$|I| = C(\varepsilon,\gamma,\eta)C(\phi,\gamma)N_\lambda^+(f,\phi,x)^{\frac{\gamma-r}{\gamma-r+1}} M^+\left[M_0(f,\phi,\,\cdot\,)^{\frac{1}{\gamma-r+1}}\right](x)\ ,$$

where $\lambda = (\gamma - r - \varepsilon)(\gamma - r + 1)/(\gamma - r)$. Therefore, the estimates just obtained for I and II, show that

$$\begin{aligned} f^*_{+,\gamma}(x) &= \sup_{\sigma \in \Phi_\gamma^+} |f * \sigma(x)| \\ &\leq C(\varepsilon,\gamma,\eta,\phi)N_\lambda^+(f,\phi,x)^{\frac{\gamma-r}{\gamma-r+1}} M^+\left[M_0(f,\phi,\,\cdot\,)^{\frac{1}{\gamma-r+1}}\right](x) + M_1^+(f,\phi,x)\ . \end{aligned}$$

for $x > c$. Then, by Hölder's inequality, Lemma (3.26) and Sawyer's theorem on weighted norm estimates for the one sided maximal function, we get

$$\int_c^\infty f^*_{+,\gamma}(x)^p w(x)dx \leq C(\gamma,\eta,\phi,p,q,w)\int_{x_{-\infty}}^\infty M_1^+(f,\phi,x)^p w(x)dx < \infty\ . \tag{3.36}$$

This inequality shows that $f \in H^p_{+,\gamma}(w_c)$, where $w_c(x) = w(x)\chi_{(c,\infty)}(x)$ belongs to A^+_q. By Theorem A, we have that $f \in H^p_{+,\gamma_0}(w_c)$ where γ_0 is the least of the integers $\gamma > 0$ satisfying $p(\gamma+1) > q$. If $b > x_{-\infty}$ we choose c, $x_{-\infty} < c < b$. Then, by Lemma (3.31), we have that

$$|<f,\varphi>| \le C_{\gamma_0} \frac{1}{w(J)} \left(\int_c^\infty f^*_{+,\gamma_0}(x)^p w(x)dx \right)^{1/p} \|\varphi\|_{\gamma_0,\gamma_0+2} \; .$$

Since γ_0 does not depend on b and b can be taken as any number greater than $x_{-\infty}$, we can take $\gamma = \gamma_0 + 2$ in (3.34) obtaining (3.36) for any $c > x_{-\infty}$. Thus, taking the limit of (3.36) for $c \to x_{-\infty}$ we obtain that

$$\int_{x_{-\infty}}^\infty f^*_{+,\gamma_0+2}(x)^p w(x)dx \le C(\gamma_0+2,\eta,\phi,p,q,w) \int_{x_{-\infty}}^\infty M_1^+(f,\phi,x)^p w(x)dx < \infty \; ,$$

which shows that f belongs to $H^p_{+,\gamma_0+2}(w)$. Applying Theorem A again we obtain that f belongs to $H^p_{+,\gamma}(w)$ for every γ, $p(\gamma+1) > q$.

Since, as we have already show in Lemma (3.32),

$$M_1^+(f,\phi,x) \le C_\gamma \|\phi\|_{\gamma,\gamma+2} f^*_{+,\gamma}(x)$$

holds for $x > x_{-\infty}$, we get that also

$$\int_{x_{-\infty}}^\infty M_1^+(f,\phi,x)^p w(x)dx \le C(\gamma,\phi) \int_{x_{-\infty}}^\infty f^*_{+,\gamma}(x)^p w(x)dx$$

holds. This ends the proof of Theorem B. ■

Proof of Theorem C. Without loss of generality we can assume that $\int \phi = 1$. In order to prove the theorem it is enough to consider f in the dense class of $H^p_{+,\gamma}(w)$ described in Theorem (3.23). Procceding exactly as in the proof of Theorem A we get (3.28) with $\beta = \gamma$. That is to say

$$\int g^*_{+,\gamma}(x)^p w(x)dx \le C \left(\int N^+_\lambda(g,\phi,x)^p w(x)dx \right)^{\frac{\gamma-r}{\gamma-r+1}}$$
$$\cdot \left(\int M^+ \left[M_0(g,\phi,\cdot\,)^{\frac{1}{\gamma-r+1}} \right](x)^{p(\gamma-r+1)} w(x)dx \right)^{\frac{1}{\gamma-r+1}} \; .$$

By Lemma (3.26) and Sawyer's theorem on weighted norm estimates for the one sided maximal function, we obtain

$$\int g^*_{+,\gamma}(x)^p w(x)dx \le C\left(\int M_1^+(g,\phi,x)^p w(x)dx\right)^{\frac{\gamma-r}{\gamma-r+1}} \cdot \left(\int M_0(g,\phi,x)^p w(x)dx\right)^{\frac{1}{\gamma-r+1}} .$$

Since by Theorem B

$$\int M_1^+(g,\phi,x)^p w(x)dx \le C\int g^*_{+,\gamma}(x)^p w(x)dx \ ,$$

we get

$$\int g^*_{+,\gamma}(x)^p w(x)dx \le C\int M_0(g,\phi,x)^p w(x)dx \ ,$$

which is the first inequality in (2.3). The second inequality follows from Lemma (3.32). ∎

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