PERIODIC CYCLIC HOMOLOGY AS SHEAF COHOMOLOGY

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0. Introduction. This paper continues the study of the noncommutative infinitesimal cohomology we introduced in [3]. This is the cohomology of sheaves on a noncommutative version of the commutative infinitesimal site of Grothendieck ([8]). Grothendieck showed that, for a smooth scheme X of characteristic zero, the cohomology of the structure sheaf on the infinitesimal site gives de Rham cohomology:

$$H^*_{inf}(X,O) \cong H^*_{dR}(X) \tag{1}$$

Here we prove that, for any associative, not necessarily unital algebra A over $\mathbb{Q}$, the noncommutative infinitesimal cohomology of the structure sheaf modulo commutators gives periodic cyclic homology (Theorem 3.0):

$$H^*_{inf}(A, O/[O,O]) \cong HP_*(A) \tag{2}$$

We view (2) as a noncommutative affine version of (1). Indeed, the noncommutative analogue of smoothness is quasi-freedom, and for quasi-free algebras HP_* agrees with (Karoubi's definition of) noncommutative de Rham cohomology, i.e. we have:

$$H^*_{inf}(R, O/[O,O]) \cong H^*_{dR}(R) := H^*(\Omega(R)/[\Omega(R),\Omega(R)], d) \tag{3}$$

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for R quasi-free. Here $\Omega(R)$ is the DGA of noncommutative forms. Grothendieck's theorem includes also a description of the cohomology of the de Rham complex $(E \otimes_{O_X} \Omega_X, \nabla)$ associated with a bundle E with a flat connection as the cohomology of a certain module on the infinitesimal site; in fact (1) is a obtained from this by setting $E = O_X$, $\nabla = d$. A noncommutative version of Grothendieck's theorem for the de Rham cohomology of flat bundles is proved in Theorem 2.3. It expresses the cohomology of the complex $H^*(E \otimes_{\tilde{R}} \Omega(R), \nabla)$ associated with a right module over the augmented algebra $\tilde{R}$ with a flat connection ∇ as sheaf cohomology; for $(E, \nabla) = (R, d)$ it gives:

$$H^*_{inf}(R, O) \cong H^*(\Omega(R), d) = 0$$

We also compute infinitesimal (hyper)cohomology with coefficients in algebraic K-theory. We show that, for any $\mathbb{Q}$-algebra A, we have a natural isomorphism:

$$H^n_{inf}(A, K_1) \cong HP_n(A) \qquad (n \geq 2) \tag{4}$$

In section 4, we use the case $n = 2$ of (4) to give a natural, sheaf theoretic construction for the Chern-Connes-Karoubi character:

$$ch_0 : K_0(A) \to HP_0(A) \cong H^2(A, K_1) \tag{5}$$

The cohomology groups of K_1 not listed in (4) appear in an exact sequence:
(6)

$$K_2(A) \xrightarrow{c_2} HN_2(A) \to H^0_{inf}(A, K_1) \to K_1(A) \xrightarrow{c_1} HN_1(A) \to H^1(A, K_1) \to 0$$

The map c above is the Jones-Goodwillie character to negative cyclic homology. The sequence (6) is obtained as the lower terms of the long exact sequence of homotopy groups of a homotopy fibration sequence (Th. 5.0):

$$\mathbb{H}_{inf}(A, \mathcal{K}) \to \mathcal{K}(A) \xrightarrow{c} SCN_{\geq 1}(A) \tag{7}$$

Here $\mathcal{K}$ is a (functorial, fibrant) simplicial version of the K-theory space BGl^+ and $SCN_{\geq 1}$ is the simplicial set the Dold-Kan correspondence associates with the negative cyclic complex truncated below dimension 1.

As stated above, this paper is a continuation of [3]. In op.cit. the case $* = 0$ of (2) was proved, and the case $* = 1$ was conjectured. It was further conjectured that the higher infinitesimal cohomology groups with coefficients in $O/[O, O]$ and K_1 vanished; (3) and (4) disprove both conjectures. In [3] the sequence (6) was shown to be rationally exact everywhere except at $H^1_{inf}(A, K_1)$. The fibration sequence (7) was not proved in [3] –not even rationally– and is therefore new.

The rest of this paper is organized as follows. In section 1 we give the basic definitions and notations. Section 2 concerns flat connections; no assumption on the characteristic of the ground field k is made in this section. We show that for a quasi-free algebra R, the category of pairs (E, ∇) of a right $\tilde{R}$-module and a flat connection is equivalent to a subcategory of the category of modules on the infinitesimal site (Proposition 2.1) and that the sheaf cohomology of the infinitesimal module corresponding to (E, ∇) is the cohomology of the complex $(E \otimes_{\tilde{R}} \Omega(R), \nabla)$ (Theorem 2.4). The isomorphism (2) is proved in section 3. Section 4 is devoted to the sheaf theoretic interpretation of the Chern character (5). The isomorphism (4) is proved in Proposition 4.0. In 4.1 we construct a character $ch^{inf} : K_0(A) \to H^2_{inf}(A, K_1)$ for algebras A over an arbitrary field k. In theorem 4.4. we prove that in characteristic zero, the character of 4.1 agrees with that of Connes-Karoubi up to a normalization factor of 2. The fibration sequence (7) and the exact sequence (6) are the subject of section 5 (Theorem 5.0 and Corollary 5.1.)

1. Notations and definitions.

Basic notations and conventions 1.0. We fix a field k which we call the ground field. We make no assumptions on the characteristic of k, except as stated. All tensor products are over k except as indicated. Algebras are nonunital associative k-algebras. We use the letters $A, B \dots$ for general algebras and the letters $R, S \dots$ for quasi-free algebras. We write $\tilde{A}$ for the augmented unital algebra $A \oplus k$. Ideals are always two-sided. Modules over unital algebras are right modules and are always assumed unital (i.e. $1 \cdot m = m$ for $m \in M$). Curly brackets $\{\}$ are used to denote pro-objects. Thus $\{A_n\}$ is a pro-algebra having A_n at level $n \in \mathbb{N}$ where the structure maps $A_n \to A_{n-1}$ are understood from the context.

Cosimplicial cylinder 1.1. We write Δ for the category of the finite ordered sets $[n] = \{0 < \cdots < n\}$ and nondecreasing monotone maps. Recall a cosimplicial object in a category C is a functor $\Delta \mapsto C$. Any object A in a category C with coproducts gives rise to a cosimplicial object $\Delta \to C$, by $[n] \mapsto \coprod_{i=0}^n A$; we call this the *coproduct cosimplicial object* associated to A. We write $*$ for the coproduct in the category of algebras and $Q^*A : [n] \mapsto Q^n(A) := A * \cdots * A$ $(n+1$ factors) for the coproduct cosimplicial algebra. We write ∂^i and μ^i for the cofaces and codegeneracies of this and of most cosimplicial objects appearing in this paper.

The maps $\mu^0\mu^1\dots\mu^n : Q^n(A) \to A$ $(n \geq 0)$ assemble to give a cosimplicial map μ to the constant cosimplicial algebra. We write $q^*(A)$ for its kernel and set:

$$Cyl^*(A) := Q^*(A)/q^*(A)^\infty$$

Here the exponent $^\infty$ has the same meaning as in [7]; thus $Cyl^*(A)$ is a cosimplicial pro-algebra or more precisely an inverse system of cosimplicial algebras having $Cyl^*(A)_n := Q^*A/q^*(A)^n$ at level n. We call the pro-algebra $Cyl^1(A)$ the *cylinder* of A and $Cyl^*(A)$ the *cosimplicial cylinder*. The name comes from the role these pro-algebras play in the notion of nil-homotopy developed in [2] and [3]. Roughly one thinks of $Cyl^1(A)$ as two copies of A with the 'contractible space' qA/qA^∞ stuck in between. Next we shall show that for a quasi-free algebra R, $Cyl^*(R)$ can be described in terms of –noncommutative– differential forms. By the tubular neighborhood theorem ([4, Th. 2]; see also [4, 8.7] and [2, Theorem 2.1]), we have an isomorphism of inverse systems $\alpha : Cyl^1(R)_n = Q^1(R)/q^1(R)^n \tilde{\to} \Omega(R)/\Omega^+(R)^n \cong \oplus_{i=0}^{n-1}\Omega^i(R)$, where $\Omega^+(R)$ is the ideal generated by $\Omega^1(R)$. The map α is natural on pairs (R,ϕ) of a quasi-free algebra and a 1-cochain $\phi : R \to \Omega^2(R)$ satisfying $\delta(\phi) = -d \cup d$, and is such that $\alpha\partial^1(a) = a$ and $\alpha\partial^0(a) = a + da + \phi(a) + \dots$ $(a \in R)$. This isomorphism generalizes to higher codimension as follows. Write W for the coproduct cosimplicial vectorspace on k; thus $W^m = \oplus_{i=0}^m ke_i$, and cofaces and codegeneracies are as follows: $\partial^i(e_j) = e_j$ if $j < i$ and $\partial^i(e_j) = e_{j+1}$ if $j \geq i$, $\mu^i(e_j) = e_j$ if $j \leq i$ and $\mu^i(e_j) = e_{j-1}$ if $j > i$. We write V for the kernel of the canonical map to the constant cosimplicial vectorspace k. Thus $V^m = \oplus_{i=1}^m kv_i$ where $v_i = e_i - e_{i-1}$ $(i = 1,\dots,m)$. We have an isomorphism:

$$\alpha^m : Cyl^m(R)_n \tilde{\to} \oplus_{i=0}^{n-1} \Omega^i(R) \otimes T^i(V^m) \tag{8}$$

The right hand side of (8) has the algebra structure of the quotient of the unaugmented tensor algebra $\Omega_m(R) := \ker(T_{\tilde{R}}(\oplus_{i=1}^n \Omega^1(R)) \to k)$ by the n-th power of the ideal generated by $\oplus_{i=1}^m \Omega^1(R) = \Omega^1(R) \otimes V^m$. The isomorphism α^0 is the identity map, and α^1 is the map α defined above composed with the canonical identification induced by $\Omega^i(R) \otimes T^i(V^1) = \Omega^i(R)$. To explain the isomorphism α^m for higher m, we need some more notation. Let us use the provisional notation $Cyl'^m(R)_n$ for the right hand side of (8); later on we shall identify both sides and eliminate the 's from our notation. For $m \geq 1$ and $1 \leq i \leq m+1$, let $\partial'_i : Cyl'^m(R) \to Cyl'^{m+1}(R)$, $\partial'_i(\omega \otimes x) = \omega \otimes \partial^{i^{\otimes r}}(x)$, $(\omega \otimes x \in \Omega^r(R) \otimes T^r(V^m))$. Also let $\partial'_i : R \to Cyl'^1(R)$ $\partial'_i = \alpha^1\partial^i$, $i = 0,1$. For $m \geq 2$, α^m is the map determined by $\alpha^m\partial^m \dots \partial^{i+1}\partial^{i-1} \dots \partial^0 = \partial'_m \dots \partial'_{i+1}\partial'_{i-1} \dots \partial'_0$ $(i = 0,\dots m)$. The same argument as in the proof of [2, Theorem 2.1] shows that α^m is an isomorphism for all $m \geq 0$. Thus we may –and do– identify $Cyl^m(R)$ with $Cyl'^m(R)$ and the coface and codegeneracy maps with their images through α^*. For $m \geq 1$ and $1 \leq i \leq m+1$ and for $m = 0$ and $i = 0, 1$, the i-th coface map is identified with the map ∂'_i defined above. For $m \geq 1$ the coface map $\partial^0 : Cyl^m(R) \to Cyl^{m+1}(R)$ can be written as follows:

$$\partial^0 = \sum_{n=0}^{\infty} P_n := \{\sum_{n=0}^{r} P_n : Cyl^m(R)_r \to Cyl^{m+1}(R)_r\}$$

Here $P_n(\Omega^r(R) \otimes T^r(V^*)) \subset \Omega^{r+n}(R) \otimes T^{r+n}(V^{*+1})$. The 0-th component is simply the map $P_0(\omega \otimes x) = \omega \otimes \partial^0(x)$; for $\omega = a_0 da_1 \dots da_p \in \Omega^p(R)$ and $x_1 \dots x_p \in T^p(V^m)$, we have:

$$(9) \quad P_1(\omega \otimes x) = d\omega \otimes v_1 \partial^1(x) + \sum_{i=1}^{p} a_0 da_1 \dots da_{i-1} \phi(a_i) da_{i+1} \dots da_p \otimes \\ \partial^0(x_1) \dots \partial^0(x_{i-1})(\partial^1(x_i) v_1 + v_1 \partial^1(x_i) - 2v_1^2) \partial^0(x_{i+1}) \dots \partial^0(x_p)$$

The formulae above are derived as follows. By definition of the canonical isomorphism α^* we have $a + da \otimes v_i + \phi(a) \otimes v_i^2 + \cdots = \partial^n \dots \partial^{i+1} \partial^{i-1} \dots \partial^0(a)$ $(i \geq 1)$; putting this together with the formula for $\alpha\partial^0$ and the cosimplicial identity $\partial^0 \partial^n \dots \partial^{i+1} \partial^{i-1} \dots \partial^0 = \partial^{n+1} \dots \partial^{i+2} \partial^i \dots \partial^0$ we obtain:

(10)

$$\partial^0(a + da \otimes v_i + \phi(a) \otimes v_i^2 + \dots) = a + da \otimes v_{i+1} + \phi(a) \otimes v_{i+1}^2 \dots \quad (i \geq 1)$$
$$\partial^0(a) = a + da \otimes v_1 + \phi(a) \otimes v_1^2 + \dots$$

The formula for P_0 is immediate from these identities; the formula for P_1 is immediate from that for P_0 and (10). It is possible to iterate this process to obtain a general formula for P_m in terms of the homogeneous parts of degrees $\leq m$ of $\partial^0 : R \to Cyl^1(R)$, but we shall have no ocassion for this. The description of $Cyl^*(R)$ is completed as one checks that the codegeneracy maps are induced by those of the cosimplicial vectorspace V; thus $\mu^i(\omega \otimes x) = \omega \otimes \mu^{i^{\otimes r}}(x)$ $(\omega \otimes x \in \Omega^r(R) \otimes T^r(V^*))$.

Adic filtration 1.2. Let R be a quasi-free algebra, $I \subset R$ an ideal, $A = R/I$, and $\pi : R \twoheadrightarrow A$ the projection map. We also write π for the map $Q^*R \to A$ to the constant cosimplicial algebra defined as either of the equal composites $Q^*R \xrightarrow{\mu} R \xrightarrow{\pi} A$, $Q^*R \xrightarrow{Q^*(\pi)} Q^*A \xrightarrow{\mu} A$. Put $q^*(R, I) := \ker \pi$, $Cyl^*(R, I) := Q^*(R)/q^*(R, I)^\infty$ and $Cyl^*(R, I)_n := Q^*(R)/q^*(R, I)^n$. We call this the *relative cosimplicial cylinder.* It is easy to see that the isomorphism α defined in 1.1 induces

$Cyl^m(R,I)_n \cong \Omega_m(R)/ < I + \Omega^+_m(R) >^n$. For $m = 1$ we get $Cyl^1(R,I)_n = \Omega(R)/\mathcal{F}^n$ where

$$\mathcal{F}^n = \mathcal{F}^n_I := < I + \Omega^+(R) >^n = \oplus_{i=0}^{\infty} F^{n-i}\Omega^i(R)$$

Here $F^p\Omega^q(R) = F^p_I\Omega^q(R) := \Omega^q(R)$ if $p \leq 0$ and is $\sum I^{i_0} dAI^{i_1} \dots I^{i_{q-1}} dAI^{i_q}$ for $p \geq 1$, with the sum taken over all multi-indices $i = (i_0, \dots, i_q)$ such that $|i| := \sum i_j \geq p$. For all $m \geq 0$, $n \geq 1$, we have:

$$\text{(11)} \qquad Cyl^m(R,I)_n = \oplus_{i=0}^{n-1} \frac{\Omega^i(R)}{F^{n-i}\Omega^i(R)} \otimes T^i(V^m)$$

For future use we also record here the behavior of the filtration $\mathcal{F}$ with respect to the Karoubi operator and the Connes, de Rham and Hochschild boundary maps:

$$\text{(12)} \qquad \kappa\mathcal{F}^n \subset \mathcal{F}^n \qquad B\mathcal{F}^n \subset \mathcal{F}^n \qquad d\mathcal{F}^n \subset \mathcal{F}^n \qquad b\mathcal{F}^n \subset \mathcal{F}^{n-1}$$

Noncommutative infinitesimal cohomology 1.3. Let $\mathcal{G}$ be a covariant functor going from the category of k-algebras to the category of abelian groups, R a quasi-free algebra, $I \subset R$ an ideal, and $A = R/I$. Consider the cosimplicial abelian group $\mathcal{G}Cyl^*(R,I)\hat{}\ : [n] \mapsto \lim_m \mathcal{G}Cyl^n(R,I)_m$. We write $H^*_{inf}(A,\mathcal{G})$ for the cohomology of $\mathcal{G}Cyl^*(R,I)\hat{}\ $; we call this the *infinitesimal cohomology* of A with values in $\mathcal{G}$. Thus $H^*_{inf}(A,\mathcal{G})$ can be computed as the cohomology of the cochain complex:

(13)

$$C(R,I,\mathcal{G}) : \lim_n \mathcal{G}(R/I^n) \xrightarrow{\partial} \lim_n \mathcal{G}(Cyl^1(R,I)_n) \xrightarrow{\partial} \lim_n \mathcal{G}(Cyl^2(R,I)_n) \xrightarrow{\partial} \dots$$

where $\partial = \sum_{i=0}^{m+1}(-1)^i\partial^i : C^m = C^m(R,I,\mathcal{G}) \to C^{m+1}$ or equivalently –by the Dold-Kan correspondence– as the cohomlogy of the normalized complex $\bar{C}^m = C^m/\sum_{i=1}^m \partial^i(C^{m-1})$ with as coboundary map the map induced –and still denoted by– ∂^0. It is implicit in our notation that the cohomology of the complex (13) depends only on A rather than on its presentation as a quotient of a quasi-free algebra. In fact the groups $H^*(A,\mathcal{G})$ have the following interpretation in terms of sheaf cohomology. Consider the category $inf(\mathrm{Alg}/A)$ having as objects the surjective algebra maps $p : B \twoheadrightarrow A$ with nilpotent kernel and as morphisms $(p : B \twoheadrightarrow A) \to (p' : B' \twoheadrightarrow A)$ the homomorphisms $B \to B'$ making the obvious diagram commute. Regard $inf(\mathrm{Alg}/A)^{op}$ as a site with the indiscrete Grothendieck topology; i.e. the coverings are the families $\{B \cong B'\}$ consisting of a single isomorphism; this site is the *noncommutative infinitesimal site* over A. Any covariant functor $inf(\mathrm{Alg}/A) \to Ab$ to abelian groups is a sheaf on $inf(\mathrm{Alg}/A)^{op}$.

It was shown in [3,§5], that the groups $H^*_{inf}(A,\mathcal{G})$ as defined above are the sheaf cohomology groups of the sheaf obtained by restriction of $\mathcal{G}$ to $inf(\mathrm{Alg}/A)$. In particular they are independent of the choice of the presentation $A = R/I$. We also consider the hypercohomology of sheaves of (cochain) complexes and of simplicial sets. If $\mathcal{G} : \dots \xrightarrow{\delta} \mathcal{G}^n \xrightarrow{\delta} \mathcal{G}^{n+1} \xrightarrow{\delta} \dots$ is a cochain complex of sheaves, we write $\mathbb{H}^*_{inf}(A,\mathcal{G})$ for the cohomology of the total complex:

$$Tot^n_\pi = \prod_{p+q=n} C^p(R,I,\mathcal{G}^q)$$

of the bicomplex $(p,q) \mapsto C^p(R,I,\mathcal{G}^q)$. For a sheaf X of fibrant simplicial sets, we put $\mathbb{H}^r_{inf}(A,X) = \pi_{-r}\,\mathrm{holim}_{\Delta\times\mathbb{N}^{op}}((m,n) \mapsto XCyl^m(R,I)_n)$; the definition can be extended to Kan spectra as in [14].

Other variants of infinitesimal cohomology 1.4. For a unital algebra A, we can consider, in addition to the infinitesimal site defined above, the indiscrete site on $inf(\mathrm{Alg}_1/A)^{op}$, where $inf(\mathrm{Alg}_1/A) \subset inf(\mathrm{Alg}/A)$ is the subcategory of unit preserving maps of unital algebras. However if $\mathcal{G} : inf(\mathrm{Alg}/A) \longrightarrow Ab$ is a functor, then the sheaf cohomology groups $H^*(inf(\mathrm{Alg}_1/A),\mathcal{G})$ agree with those defined above. Indeed, both $H^*(inf(Alg_1/A),\mathcal{G})$ and $H^*(inf(Alg/A),\mathcal{G})$ can be computed as the cohomology of $C(R,I,\mathcal{G})$ where $A = R/I$ is any presentation of A as a quotient of a unital quasi-free algebra. Similarly if A is unital and commutative, the cohomology of a functor $\mathcal{G} : inf(\mathrm{Comm}/A) \longrightarrow Ab$ is the same whether considered as a sheaf on $inf(\mathrm{Comm}/A)^{op}$ or $inf(\mathrm{Comm}_1/A)^{op}$; they are both computed as the cohomology of the cosimplicial abelian group $[n] \mapsto \lim_m \mathcal{G}(S^{\otimes n+1}/\ker(S^{\otimes n+1} \twoheadrightarrow A)^m)$ where $S \twoheadrightarrow A$ is any surjective map from a smooth unital algebra S, and $S^{\otimes n} \twoheadrightarrow A$ is the induced map. On the other hand, it follows from the discussion on page 337 of [8] that, for A commutative and unital, the cohomology of the additive group functor $inf(\mathrm{Comm}_1/A) \ni (B \twoheadrightarrow A) \mapsto B \in Ab$ agrees with the cohomology of the structure sheaf on Grothendieck's infinitesimal site $Spec(A)_{inf}$. Thus all three commutative infinitesimal cohomologies of a commutative unital algebra agree. However these do not agree with the noncommutative infinitesimal cohomology of the same algebra. In fact, we shall show further below that for any algebra A, the cohomology of the sheaf $O/[O,O] : inf(Alg/A) \longrightarrow Ab$, $(B \twoheadrightarrow A) \mapsto B/[B,B]$ is periodic cyclic homology $HP_*(A)$. Since on the other hand, for a finite type algebra over a field we have $HP_*(A) = \prod_{n=0}^\infty H^{2n+*}(SpecA_{inf},O)$ (cf. [9]), we can see that the commutative and noncommutative infinitesimal cohomologies do not agree in general. Herefrom, we shall always use H_{inf} to refer to the cohomology on the noncommutative site.

2. Flat connections.

Infinitesimal modules, crystals and stratifications 2.0. Let A be an algebra. By a right, *infinitesimal module* over A or $\tilde{O}_{inf(\mathrm{Alg}/A)^{op}}$-module we mean a functor $M : inf(\mathrm{Alg}/A) \to Ab$ such that $M_p \in \mathrm{Mod}-\tilde{B}$ for each $inf(\mathrm{Alg}/A) \ni p : B \twoheadrightarrow A$. We say that the infinitesimal module M is a *crystal* on the infinitesimal site if for each map $\alpha : (p : B \twoheadrightarrow A) \to (p' : B' \twoheadrightarrow A) \in inf(\mathrm{Alg}/A)$ the $\tilde{B}'$-homomorphism $\rho_\alpha : \alpha(M_p) := M_p \otimes_{\tilde{\alpha}} \tilde{B}' \to M_{p'}$ induced by $M(\alpha)$ is an isomorphism. A *stratification* on a right $\tilde{A}$-module E is a sequence of isomorphisms $\theta_n : \partial^0_n(E) \tilde{\to} \partial^1_n(E) \in \mathrm{Mod}-Cyl^1(A)_n$ $(n \geq 0)$; here ∂^i_n is the i-th coface map in the cosimplicial algebra $Cyl^*(A)_n$. The θ's are subject to the following prescriptions: we must have $\theta_1 = 1$, the identity of E; the sequence $\{\theta_n\}$ must be compatible with the projection maps $Cyl^1(A)_{n+1} \to Cyl^1(A)_n$, and finally, each θ_n must satisfy the cocycle condition:

$$\partial^2_n(\theta_n)\partial^0_n(\theta_n) = \partial^1_n(\theta_n) \tag{14}$$

For example if M is a crystal then the sequence $\theta_n(M) = \rho^{-1}_{\partial^1_n}\rho_{\partial^0_n}$ is a stratification on $E(M) = M_{A \xrightarrow{id} A}$. For a quasi-free algebra R, the assignment $M \mapsto (E(M), \theta(M))$, is an equivalence of categories between the category of crystals of $\tilde{R}$-modules and natural transformations and the category of pairs (E, θ) of an $\tilde{R}$-module together with a stratification θ, where a map $(E, \theta) \to (E', \theta')$ is a module homomorphism $f : E \to E'$ such that $\theta_n\partial^0_n(f) = \partial^1_n(f)\theta_n$ for all $n \geq 1$. An inverse for this functor was constructed in [3,7.2]; here we write E_{crys} for this inverse equivalence. If M is a crystal, then the pro-cosimplicial group $MCyl^*(R)$ of 1.3 above can be described in terms of E and θ as follows. We have $MCyl^0(R) = E$ and $MCyl^m(R) = \partial^m \dots \partial^1(E)$ for $m \geq 1$. Codegeneracies are the maps $id_E \otimes \mu^i : \partial^m \dots \partial^1(E) \to \partial^{m-1} \dots \partial^1(E)$ $(0 \leq i \leq m)$. Similarly, for $1 \leq j \leq m+1$, $id_E \otimes \partial^j$ is the j-th coface map $\partial^m \dots \partial^1(E) \to \partial^{m+1} \dots \partial^1(E)$. The 0-th coface map $E \to \partial^1(E)$ is the composite $\partial^0_\theta : E \to \partial^0(E) \xrightarrow{\theta} \partial^1(E)$. For $m \geq 1$ and $e \otimes \omega \in \partial^m_n \dots \partial^1_n(E)$ we have $\partial^0_\theta(e \otimes \omega) = \partial^0_\theta(e) \otimes \partial^0(\omega)$. Upon completion we obtain a complex $C(E, \theta)$ whose cohomology is the sheaf cohomology of the crystal M with which we started.

We have shown how one can go from crystals on a quasi-free algebra to stratifications and interpret the sheaf cohomology of the former in terms of the latter. Below we shall show how one can go from stratifications to flat connections. First we recall the definition of curvature given in [11]. Given a connection $\nabla : E \to E \otimes_{\tilde{A}} \Omega^1(A)$ on a $\tilde{A}$-module E, extend it to a map $\nabla : E \otimes_{\tilde{A}} \Omega^*(A) \to E \otimes_{\tilde{A}} \Omega^{*+1}(A)$ by the rule $\nabla(e \otimes \omega) = \nabla(e)\omega + ed\omega$. The *curvature* of ∇ is the map $\nabla^2 : E\otimes_{\tilde{A}}\Omega^*(A) \to E\otimes_{\tilde{A}}\Omega^{*+2}(A)$. We say that ∇ is *flat* if $\nabla^2 = 0$. One sees immediately that flatness is equivalent to the vanishing of the degree 2 component $\nabla^2 : E \to E\otimes_{\tilde{A}}\Omega^2(A)$. The pairs (E,∇) of a right $\tilde{A}$ module and a flat connection form a category, where a map $(E,\nabla) \to (F,D)$ is a homomorphism of modules $\alpha : E \to F$ with $D\alpha = \alpha \otimes id_{\Omega^1(A)}\nabla$.

Proposition 2.1. *Let R be a quasi-free algebra. With the definitions of 2.0 above, let (E,θ) be a stratified $\tilde{R}$-module and $\partial^0_{\theta,1} : E \to \partial^1_1(E)$ the coface map. Then the composite $\nabla : E \xrightarrow{\partial^0_{\theta,1}} \partial^1_1(E) = E \oplus E\otimes_{\tilde{A}}\Omega^1(A) \twoheadrightarrow E\otimes_{\tilde{A}}\Omega^1(A)$ is a flat connection. Moreover the assignment $(E,\theta)\mapsto(E,\nabla)$, $f\mapsto f$ is an isomorphism between the categories of stratified modules and of flat connections.*

Proof. That ∇ is a connection is immediate, as is that a homomorphism of stratifications is a morphism of modules with a connection. The proof that ∇ is flat will be easy once we have established the identity (15) below; to prove the latter, proceed as follows. Consider the map $\partial^0_\theta : E \to E\otimes_{\partial^1} Cyl^1(R) \cong E\otimes_{\tilde{R}}\Omega(R)/\Omega^+(R)^\infty$; write $\partial^0_\theta = \sum_{n=0}^\infty D_n$ where $D_n(E) \subset E\otimes_{\tilde{R}}\Omega^n(R)$. Thus $D_0 = 1$ and $D_1 = \nabla$. Similarly, the map $\partial^0_\theta : \partial^1(E) \to \partial^2\partial^1(E)$, decomposes as $\partial^0_\theta = \sum_{n=0}^\infty\sum_{i=0}^n D_i\otimes P_{n-i}$, where the P's are the components of the map $\partial^0 : Cyl^1(R) \to Cyl^2(R)$. The identity $\partial^0_\theta\partial^0_\theta = \partial^1\partial^0_\theta$ is equivalent to the sequence of identities:

$$D_n(e)\otimes v_2^n - D_n(e)\otimes v_1^n - D_n(e)\otimes(v_2-v_1)^n = \sum(D_i\otimes P_j)(D_k(e)\otimes v_1^k) \tag{15}$$

where $e\in E$, $n\geq 1$, and the sum is taken over all $i\geq 0$ and $j,k\geq 1$ such that $i+j+k=n$. The case $n=1$ of the identity (15) is tautological. That ∇ is flat follows straightforwardly from the case $n=2$ of the identity above, using the formula (9). Thus $(E,\theta)\to(E,\nabla)$ is a functor from stratified modules to modules with a flat connection. Next we construct a functor going in the opposite direction. Consider the map $t : \Omega^1(R) \to Cyl^1(R)$, $adx \mapsto a(\partial^0(x)-x)$. Define a map $\partial^0_\nabla : E \to \partial^1(E)$, $\partial^0_\nabla(e) = e + (id\otimes t)\nabla(e)$, and put $\theta : \partial^0(E) \to \partial^1(E)$, $\theta(e\otimes\omega) = \partial^0_\nabla(e)\partial^0(\omega)$. We must see now that θ is a stratification. One checks that $\partial^0_\nabla(ea) = \partial^0_\nabla(e)\partial^0(a)$ $(e\in E, a\in R)$, from which it follows that θ is a well-defined homomorphism. Of the conditions for a stratification only the cocycle condition is not immediate. To prove the latter, one does as follows. One checks that (14) is equivalent to the cosimplicial identity $\partial^0_\theta\partial^0_\theta = \partial^1\partial^0_\theta$. In turn, since $\partial^0_\theta = \partial^0_\nabla : E \to \partial^1(E)$, the latter condition is equivalent to the sequence of identities (15) with as D_n the $n-th$ component of ∂^0_∇. A straightforward induction shows that (15) holds. On the other hand it is clear from the definition of D_n that if $f : (E,\nabla) \to (E',\nabla')$ is a homomorphism of modules with a connection, and

θ and θ' are the stratifications constructed from them, then $\partial^1(f)\partial^0_\theta = \partial^0_{\theta'} f$. It follows that f is a map of stratified modules $(E, \theta) \to (E', \theta')$. Thus we have a functor from flat connections to stratifications. It is clear than if we start off with a flat connection, then take the stratification constructed from it and then go back by the functor of the proposition, we end up with the same connection we started with. That the other composition is the identity is a consequence of the recursive nature of the formula (15); indeed, the latter formula says that all of the D_n –and therefore the whole stratification– are determined by $D_1 = \nabla$. □

Corollary 2.2. *For a quasi-free algebra R, the categories of $\tilde{R}$ modules with a flat connection and of infinitesimal crystals are equivalent.*

Proof. It follows from the proposition and the equivalence between the categories of stratified modules and crystals discussed in 2.0 above. □

Theorem 2.3. *Let R be a quasi-free algebra, E a right $\tilde{R}$-module, $\nabla : E \to E \otimes_{\tilde{R}} \Omega^1(R)$ a flat connection in the sense of 2.0, and E_{crys} the infinitesimal crystal corresponding to (E, ∇) under the category equivalence of corollary 2.2. Then there is an isomorphism:*

$$H^*_{inf}(R, E_{crys}) \cong H^*(E \otimes_{\tilde{R}} \Omega(R), \nabla)$$

The proof of the theorem above uses the following variant of the perturbation lemma of [12]:

Lemma 2.4. *Let $C = (\{C^n\}_{n\geq 0}, \delta)$ be a nonnegatively graded cochain complex. Assume there is a decomposition $C^n = \prod_{p=0}^\infty C^n_p$ $(n \geq 0)$ and that δ preserves the associated filtration, i.e. $\delta(\prod_{p\geq q}^\infty (C^*_p)) \subset \prod_{p\geq q}^\infty (C^*_p)$ for all $q \geq 0$. Write $\delta = \sum_{n=0}^\infty \delta_n$ where $\delta_n(C^*_*) \subset (C^{*+1}_{*+n})$. Suppose $h_0 : C^*_* \to C^{*-1}_*$ is a contracting homotopy for δ_0 in the sense that $1 = h_0\delta_0 + \delta_0 h_0$. Define recursively:*

$$h_n = -\sum_{i=0}^{n-1} h_0 \delta_{n-i} h_i$$

Then $h = \sum_{n=0}^\infty h_n$ is a contracting homotopy for δ.

Proof. Straightforward induction. □

Proof of Theorem 2.3. Consider the complex $C = C(R, 0, E_{crys})$ of (13); this is the cochain complex associated to the completion of the pro-cosimplicial abelian group

$C(E,\theta)$ of 2.0. We shall define a map going from the normalized complex $\bar{C}$ to the complex $(E\otimes_{\tilde{R}}\Omega(R),\nabla)$ and show it is a quasi-isomorphism. First we need a better description of the complex $\bar{C}$. We have a decomposition $C^m=\prod_{n=0}^{\infty}C_n^m$, where $C_n^m:=E\otimes_{\tilde{R}}\Omega^n(R)\otimes T^n(V^m)$; cofaces and codegeneracies are as described in 2.0 above. Thus all cofaces but the 0-th coface act only on the $T(V)$ part, i.e. are of the form $id\otimes\partial^i$ where ∂^i is the coface of the completed tensor product cosimplicial vectorspace $T(V):m\mapsto\prod_{n=0}^{\infty}T^n(V^m)$. The 0-th coface map decomposes as an infinite sum of homogeneous components of nonnegative degrees of which the 0-th degree component is of the form $id\otimes\partial^0$ where ∂^0 is the coface of $T(V)$. Hence the decomposition above is preserved by normalizing and we have $\bar{C}^m=\prod_{n=0}^{\infty}\bar{C}_n^m$ where $\bar{C}_n^m=E\otimes_{\tilde{R}}\Omega^n(R)\otimes\bar{T}^n(V^m)$. Here $\bar{T}^n(V^m)$ is the normalization of the cosimplicial tensor product of the cosimplicial vectorspace of 1.1. By definition, $\bar{T}^n(V^m)$ is the quotient of $T^n(V)$ by the subspace generated by the sum of the images of all cofaces but the 0-th coface; one checks this subspace is that generated by all those words of length n on the basis elements $v_1,\dots,v_m$ of V^m in which at least one of the v_i is missing. Hence each $\bar{T}^n(V^m)$ is isomorphic to the free vectorspace on the set of all surjective maps going from a set of n elements to a set of m elements. In particular $\bar{T}^n(V^m)=0$ for $m>n$, and $\bar{T}^n(V^n)\cong k[S_n]$, the group algebra of the symmetric group. The coboundary in $\bar{C}$ is the map δ induced by ∂^0; it admits a decomposition as that of the lemma above, where δ_0 is of the form $id\otimes\partial_0'$, with ∂_0' the map induced by the 0-th coface of $T(V)$ on $\bar{T}(V)$. Consider the map $p:k[S_n]\twoheadrightarrow k$, $\sigma\mapsto sg(\sigma)$ where sg is the permutation sign. One checks that $\ker p=\partial_0'(T^n(V^{n-1}))$; thus if we regard p as a map of cochain complexes $\bar{T}^n\twoheadrightarrow k[-n]$, we see it induces an isomorphism at the H^n level. I claim it is in fact a quasi-isomorphism, i.e. we have $H^q(\bar{T}^n(V))=0$ for $q<n$. The claim follows from the fact that $\bar{V}=k[-1]$, the Künneth formula for the cohomology of cochain complexes and the fact that –as can be seen through the combined application of the Eilenberg-Zilber theorem and the Dold-Kan normalization theorem– the normalized chain complex of the tensor product $n\mapsto A^n\otimes B^n$ of two cosimplicial vectorspaces A and B is quasi-isomorphic to the tensor product of the normalized cochain complexes. Thus the claim is proved, and we may regard $p:\bar{T}^n(V)\twoheadrightarrow k$ as a free resolution of k; in particular the cochain complex $\ker p^*$ is contractible. One checks that, upon tensoring over k with $E\otimes_{\tilde{R}}\Omega(R)$ and completing, we obtain a cochain map $1\otimes p:(\bar{C},\delta)\longrightarrow(E\otimes_{\tilde{R}}\Omega(R),\nabla)$. By what we have just seen, the kernel of this map satisfies the hypothesis of the lemma above. This proves the theorem. □

3. Periodic homology v. infinitesimal cohomology. The purpose of this section is to prove the following:

Theorem 3.0. *Let k be a field of characteristic zero, A a k-algebra. Then there is an isomorphism:*

$$HP_*(A) \cong H^*_{inf}(A, O/[O,O])$$

Proof. Write $A = R/I$, with R quasi-free. The proof has two parts. First we prove that $H^*_{inf}(A, O/[O,O])$ equals the cohomology of the complex:

$$(\lim_n (\frac{\Omega^*}{\mathcal{F}^n + b\Omega^{*+1}})_\kappa, d) \tag{16}$$

where $\Omega^* = \Omega^*(R)$, $\mathcal{F}$ is the adic filtration of the ideal I, as defined in 1.2 above, and κ is the Karoubi operator. The map κ verifies $\kappa^m \equiv 1$ on $\Omega^m \mod (\mathcal{F}^n + b\Omega^{m+1})$; the subscript in (16) indicates coinvariants with respect to this action of $\mathbb{Z}/m$. The second part consists of proving that the latter complex computes $HP_*(A)$. Now to the first part. Consider the inverse systems of normalized cochain complexes $\{\bar{C}_n : n \geq 0\}$ and $\{\bar{D}_n : n \geq 0\}$ associated with the pro-cosimplicial groups

$$\{Cyl(R,I)_n/[R, Cyl(R,I)_n] : n \geq 0\}$$

and

$$\{Cyl(R,I)_n/[Cyl(R,I)_n, Cyl(R,I)_n] : n \geq 0\}.$$

Thus, by (11), $\bar{C}^m_n = \oplus_{i=m}^n \bar{C}^m_{n,i}$, where, for $\bar{T}$ as in the proof of Theorem 2.3, $\bar{C}^m_{n,i} = \frac{\Omega^i}{F^{n-i}\Omega^p + b\Omega^{i+1}} \otimes \bar{T}^i(V^m)$. We write $\delta_n = \sum_{i=0}^n \delta_{n,i}$ for the coboundary of $\bar{C}_n$. A similar argument as that in the proof of assertion c) on page 30 of [11] shows that $\bar{D}^m_n = \oplus_{i=m}^n \bar{D}^m_{n,i}$ where $\bar{D}^m_{n,i} = (\bar{C}^m_{n,i})_{\mathbb{Z}/i}$ is the coinvariants with respect to the action given on a generator ζ of $\mathbb{Z}/i$ by $\zeta(\omega \otimes x) = \kappa'(w) \otimes \zeta(x)$. Here $\kappa'(a_0 da_1 \dots da_i) = da_i a_0 da_1 \dots da_{i-1}$ is Karoubi's operator without sign, and $\zeta(x_1 \dots x_i) = x_i x_1 \dots x_{i-1}$. Proceeding as in the proof of theorem 2.3 we see that there is an inverse system of cochain maps $\pi_n : \bar{C}_n \twoheadrightarrow \Omega^*/\mathcal{F}^n$ which are 0 in codimension $> n$ and are given in codimension $\leq n$ by the composite of the projection $\bar{C}^m_n \to \bar{C}^m_{n,m} = \frac{\Omega^m}{F^{n-m}\Omega^m} \otimes k[S_m]$ and the map $(1 \otimes p)$, where p is as in the proof of theorem 2.3. Thus the restriction p_i of p to each of the complexes $\bar{T}^i(V)$ is a quasi-isomorphism of $k[\mathbb{Z}/i]$-modules. Since $char(k) = 0$, we therefore have that $\ker p_i$ is contractible as a complex of $k[\mathbb{Z}/i]$-modules, i.e. there is a contracting homotopy which commutes with the action of $\mathbb{Z}/i$. This contracting homotopy is preseved upon tensoring with $\frac{\Omega^i}{F^{n-i}\Omega^i + b\Omega^{i+1}}$ and gives a map of inverse systems $\ker(\pi_n)^* \to \ker(\pi_n)^{*-1}$ which commutes with the action of the cyclic groups and is a contracting homotopy for $\delta_{n,0}$. Now apply Lemma 2.4 to obtain an inverse system of action preserving contracting homopies for $\ker \pi_n$. Taking coinvariants and then inverse limit, we obtain a contracting homotopy for

$\ker \lim_n \pi'_n$, where π'_n is the map induced by π_n at the coinvariant level. To finish the first part of the proof it suffices to show that $\pi' = \lim_n \pi'_n$ is surjective. To see this, consider the map $s_n : \Omega^m/F^{n-m}\Omega^m \to C^m_n$, $\omega \mapsto \omega \otimes (1/m! \sum_{\sigma\in S_m} \sigma)$. We see that s_n is a right inverse for π_n, that it commutes with the action of the cyclic groups –although not with coboundaries– and that the family $\{s_n\}$ is a map of inverse systems. Hence if s'_n is the map induced by s_n at the coinvariant level and $s' = \lim_n s'_n$, then $\pi' s' = 1$ and π' is surjective.

Next we turn to the second part of the proof. First of all, note that, because the map $1-\kappa$ is invertible on the second summand of the harmonic decomposition $\Omega = P\Omega \oplus P^{\perp}\Omega$ of [6], we have an isomorphism of cochain complexes $(P\frac{\Omega}{\mathcal{F}^n}{}_\natural, d) \cong ((\frac{\Omega^m}{F_{n-m}\Omega^m + b\Omega^{m+1}})_\kappa, d)$ for each $n \geq 0$. Here $P\frac{\Omega^m}{\mathcal{F}^n}{}_\natural = \frac{P\Omega^m}{P(b\Omega^{m+1}+F_{n-m}\Omega^m)}$; we abuse notation and write d for the map it induces at the $\natural$ level. We want to compare the latter cochain complex with the supercomplex X of [5], which is known to compute the periodic cyclic groups. Since these complexes live in different categories we introduce τX, a nonnegatively graded cochain complex version of the supercomplex X. We put $\tau X^0 = \Omega^0_\natural$, and for $n \geq 1$, $\tau X^{2n} = \Omega^0 = X^{even}$, $\tau X^{2n-1} = \Omega^1_\natural = X^{odd}$. Abusing notation and writing $\mathcal{F}^\infty$ for the image of the filtration (12), we have $H^*(\tau X/\mathcal{F}^\infty) = H^*(X/\mathcal{F}^\infty)$. It follows from [2, 2.3] that the filtration $\mathcal{F}$ is equivalent to the filtration induced by $K^n = \ker(\Omega \twoheadrightarrow \Omega(R/I^n))$; in turn it was shown in [5] that for $R = TA$ and $I = JA$, the latter filtration is equivalent to the Cuntz-Quillen filtration. Thus $H^*(X/\mathcal{F}^\infty) = HP_*(A)$. We shall presently define inverse homotopy equivalences of cochain complexes $P\Omega_\natural \rightleftarrows \tau X$. First we introduce the following two operators; let $h : \Omega^* \to \Omega^{*-1}$, $h(\omega da) = (-1)^{|\omega|}\omega a$, and $\nabla : \Omega^* \to \Omega^{*+1}$, $\nabla(a_0 da_1 \dots a_n) = -a_0\phi(a_1)da_2 \dots da_n$ where ϕ is as in 1.1 above. One checks the following properties of h and ∇. We have:

$$(17) \qquad (\nabla b + b\nabla)\omega = \omega, \quad \nabla(uv) = \nabla(u)v + (-1)^{|u|}udv, \quad \nabla\mathcal{F}^n \subset \mathcal{F}^n$$

for $\omega \in \Omega^{2+*}$, $u, v \in \Omega^{1+*}$ and also:

$$(18) \qquad h^2 = 0 \quad hd + dh = 1 \quad h\mathcal{F}^n \subset \mathcal{F}^{n-1}$$

Below we define maps $\alpha : P\Omega_\natural \to \tau X$, $\beta : \tau X \to P\Omega_\natural$, $\gamma : \tau X \to \tau X[-1]$, and $\epsilon : P\Omega_\natural \to P\Omega_\natural[-1]$. One checks, using (17), (18),(12) and induction, that all these maps are continuous in the adic topology, that α and β are cochain maps and that γ and ϵ are homotopies $\alpha\beta \to 1$ and $\beta\alpha \to 1$. For $j = 0, 1$, $1 \leq i \leq q$, and $n = 2q + j$, we write $c_{n,i} = n(n-2)(n-4)\dots(2i+j)$, and $c_n = c_{n,1-j}$. Let α^0, α^1, β^0 and β^1 be identity maps,

$$\alpha^n = c_n(Phb)^q, \quad \beta^{2q+1} = \frac{1}{(2q+1)!}(P(\nabla B + B\nabla))^{q-1}P, \quad \beta^{2q} = (2q+1)\beta^{2q+1}\nabla B$$

Here and below the superscrypts on expressions between parenthesis denote powers; thus for instance $(Phb)^q = Phb\circ\cdots\circ Phb$ (q times). Put also $\gamma_i = 0$ for $i \leq 1$, $\epsilon_l = 0$ for $l \leq 2$, and:

$$\gamma^{2n+1} = hP(b\nabla - 1) - \sum_{i=1}^{n}\alpha^{2i}(PhP\nabla b)\beta^{2i+1}, \qquad \gamma^{2n} = -\sum_{i=1}^{n}\alpha^{2i-1}PhP\nabla b\beta^{2i}$$

$$\epsilon^{2q+1} = (2q+1)\sum_{i=1}^{q}\frac{1}{c_{2q,i}}(P(\nabla B + B\nabla))^{q-i}P\nabla(Phb)^{q-i+1}$$

$$\epsilon^{2q} = \sum_{i=2}^{q}\frac{1}{c_{2q-1,i-1}}(P(\nabla B + B\nabla))^{q-i}P\nabla(Phb)^{n-i+1}. \quad \square$$

Remark 3.1. The cohomology isomorphism $H^n(P\frac{\Omega}{\mathcal{F}^\infty}{}_\natural) \cong HP_n(A) \cong H^{n+2}(P\frac{\Omega}{\mathcal{F}^\infty}{}_\natural)$ ($n \geq 0$) is induced by the cochain map $s' := PhbN : P\Omega_\natural \to P\Omega_\natural[-2]$, where $N(\omega) = |\omega|\omega$. This follows from the fact that, for the homotopy equivalence of the proof above, we have $\alpha^n {s'}^{n+2} = \alpha^{n+2}$.

4. A new presentation of the Chern character. The title of this section refers to the Connes-Karoubi map:

$$K_0(A) \xrightarrow{ch_0} HP_0(A) \tag{19}$$

We shall demonstrate here how this character fits into the infinitesimal cohomology framework. In 4.1 below we shall construct a character $ch^{inf} : K_0(A) \to H^2_{inf}(A, K_1)$, where K_1 is Bass's. This character is defined independently of the characteristic of k; we show in 4.4 that if $char(k) = 0$, then our character is the same as (19), up to a factor of 2. The key to proving this agreement result is provided by the following:

Proposition 4.0. *Let A be an algebra over a field k of characteristic zero; then there is a natural isomorphism:*

$$\nu : H^n_{inf}(A, K_1) \cong HP_n(A) \quad (n \geq 2)$$

NOTE: *The groups $H^n_{inf}(A, K_1)$ for $n = 0, 1$ are discussed in section 5.*

Proof. Write $R/I = A$ with R quasi-free. By virtue of (11) we have, for each $n \geq 1$, an exact sequence of cosimplicial algebras:

$$0 \to \oplus_{i=1}^{n-1}\frac{\Omega^i(R)}{F^{n-i}\Omega^i(R)} \otimes T^i(V^*) \to Cyl^*(R, I)_n \xrightarrow{\pi_n} \frac{R}{I^n} \to 0$$

here $\frac{R}{I^n}$ is the constant cosimpicial algebra. We note that we also have a natural inclusion map $\frac{R}{I^n} \subset Cyl^*(R,I)_n$ which is an algebra homomorphism (although not a cosimplicial map), and also a right inverse for π_n and a map of inverse systems. Thus if $\mathcal{G}$ is any functor going from algebras to abelian groups, the sequence

$$0 \longrightarrow \lim_n \ker \mathcal{G}(\pi_n^*) \longrightarrow \lim_n \mathcal{G}(Cyl^*(R,I)_n) \longrightarrow \lim_n \mathcal{G}(\frac{R}{I^n}) \longrightarrow 0 \tag{20}$$

is exact. The proposition follows from the isomorphism $\ker K_1(\pi_n^*) \cong \ker HC_0(\pi_n^*)$ of [13], theorem 3.0 above, and the long cohomology sequences associated to (20) for $\mathcal{G} = K_1$, and for $\mathcal{G} = HC_0 = O/[O,O]$. $\square$

The infinitesimal Chern character 4.1. The construction has two parts; first we define, for any algebra A, a character $ch^{strat} : K_0(A) \longrightarrow H^2_{strat}(A, K_1)$ taking values in the cohomology of the stratifying site of [3,5.6], i.e. in the cohomology of the complex $\lim_n K_1(Cyl(A)_n)$. Note that for A quasi-free this suffices, as $H^*_{strat}(A,-) = H^*_{inf}(A,-)$ in this case. The second part consists of lifting the construction above to the infinitesimal topology of an arbitrary algebra. Now, to the first part. Suppose a finitely generated projective right $\tilde{A}$-module E is given; then I claim that there exists, for each $n \geq 1$, an isomorphism of $\widetilde{Cyl}(A)_n$-modules $\theta_n : \partial_n^0(E) \tilde{\rightarrow} \partial_n^1(E)$ with $\theta_1 = 1$ which is compatible with the projection maps $\partial^i_{n+1}(E) \longrightarrow \partial^i_n(E)$ $(i = 0,1)$. This is best seen in terms of idempotent matrices. Let $p \in M_n(\tilde{A})$ be an idempotent matrix with $p\tilde{A}^n = E$; we have to show that there exists a compatible family of invertible matrices $u_n \in Gl_n(\widetilde{Cyl}(A)_n)$ with $u_1 = 1$ and such that $u_n\partial_n^0(p) = \partial^1(p)u_n$. Since an idempotent matrix is the same thing as a homomorphism $k \longrightarrow M_n(\tilde{A})$, it suffices to show the above for $A = k$ and $n = 1$. In this particular case, a formula for u is given in Lemma 4.2 below. The claim is thus proved. Next we consider the map $\partial^n(\theta_n) := \partial_n^2(\theta_n)\partial_n^0(\theta_n)\partial_n^1(\theta_n)^{-1}$; we observe that this is an automorphism of the $\widetilde{Cyl}^2(A)_n$-module $\partial^2\partial^1(E)_n$, and therefore has a class in $K_1(\widetilde{Cyl}^2(A)_n)$. It is immediate from a well-known form of Whitehead's lemma –which we have included below as lemma 4.3– that the class we have constructed is a 2-cocycle of the complex $K_1(\widetilde{Cyl}^*(A)_n)$. Because the θ_n are compatible, so is the sequence $\partial(\theta) = \{\partial^n(\theta_n)\}$ whence it is a 2-cocycle of the completed cochain complex $\lim_n K_1(\widetilde{Cyl}^*(A)_n)$. We write $ch^{strat}(E)$ for its cohomology class in $H^2_{strat}(A, K_1 \circ \tilde{\ }) = H^2_{strat}(A, K_1(-) \oplus K_1(k)) = H^2_{strat}(A, K_1)$. It follows by lemma 4.3 that if θ' is another inverse system of isomorphisms as above, then the automorphisms $\partial^n(\theta_n)\partial^n(\theta'_n)^{-1}$ of $\partial_n^2\partial_n^1(E)$ and $\partial_n^2(\theta_n{\theta'}_n^{-1}) \oplus \partial_n^1(\theta_n\theta_n^{-1})^{-1} \oplus \partial_n^0(\theta_n{\theta'}_n^{-1})$ of $\partial_n^2\partial_n^1(E)^2 \oplus \partial_n^0(E)$ have the same class in $K_1(\widetilde{Cyl}^1(A)_n)$. Hence the definition of $ch^{strat}(E)$ is independent of the choice of θ. Next, note that if $f : E\tilde{\to}F$ and $\alpha : \partial^0(E) \to \partial^1(E)$ are isomorphisms, then so is $\beta = \partial^1(f)\alpha\partial^0(f)^{-1}$; one checks, using lemma 4.3, that $\partial^n(\beta) \equiv \partial^n(\alpha)$ in $K_1(\widetilde{Cyl}^1(A))_n$. Hence ch^{strat} is well-defined on isomorphism classes of finitely generated projective $\tilde{A}$ modules. Furthermore it is a monoid homomorphism, because if $\alpha : \partial^0(E) \to \partial^1(E)$ and $\beta : \partial^0(F) \to \partial^1(F)$ are the isomorphisms used to define $ch^{strat}(E)$ and $ch^{strat}(F)$, then we can use $\alpha \oplus \beta$ for $E \oplus F$, and $\partial(\alpha \oplus \beta) = \partial(\alpha) \oplus \partial(\beta)$. Hence we have a well-defined group homomorphism $K_0(\tilde{A}) \xrightarrow{ch^{strat}} H^2_{strat}(A, K_1)$, and by composition with $K_0(A) \to K_0(\tilde{A})$ or – since, as one sees immediately, $ch^{strat}(\tilde{A}) = 0$– by passage to the quotient $\mod K_0(k)$, also a homomorphism $K_0(A) \xrightarrow{ch^{strat}} H^2_{strat}(A, K_1)$. This homomorphism is natural, as follows from the fact that lemma 4.2 gives a formula for an isomorphism $\theta : \partial^0(E) \to \partial^1(E)$, whence also for $ch^{strat}(E)$.

Now we lift this construction to the infinitesimal site as follows. Write $A = R/I$ where R is quasi-free. Given a projective right $\tilde{A}$-module lift it succesively to an inverse system $\pi_n : E_n \to E_{n-1}$ where $E_1 = E$, and each π_n is an $\tilde{R}/I^n$ homomorphism. That this can be done is a consequence of the fact that k is quasi-free; explcit formulas for the succesive idempotents e_n for each of the E_n in terms of that of E are given in [5] for $char(k) = 0$. Using as $\theta_n : \partial^0_n(E_n) \to \partial^1_n(E_n)$ the image through the homomorphism induced by e_n of the truncation of the element u of Lemma 4.2, we obtain an element $\{\partial_n(e_n(u_n))\} \in \lim_n Gl(\partial^2_n\partial^1_n E_n)$. Taking the class of this element at each level, we obtain a 2-cocycle $\partial(e(u))$ of the complex $\lim_n K_1(\widetilde{Cyl}^*(R/I^n)_n) \cong \lim_n K_1(\widetilde{Cyl}^*(R, I)_n)$. The last isomorphism comes from the fact that, by [2, 2.3], the filtrations $(q^*(R)+ < I >)^\infty$ and $\{\ker(Q^n(R) \twoheadrightarrow Q^n(R/I^n)_n\}$ are equivalent. We write $ch^{inf}(E)$ for the cohomology class of $\partial(e(u))$. After what we have already seen, to show that $E \mapsto ch^{inf}(E)$ is well-defined on isomorphism classes of projective modules and gives a group homomorphism $ch^{inf} : K_0(A) \to H^2_{inf}(A, K_1)$, it suffices to show that any isomorphism of projective modules $\alpha_1 : E \to F$ can be lifted to an inverse system of isomorphisms $\alpha_n : E_n \to F_n$ of the chosen liftings of E and F. But this is immediate from Lemma 4.2 and the fact that the algebra $k[t, t^{-1}]$ is quasi-free.

Lemma 4.2. *Let k be the ground field; write e for its unit element. Then:*

1) *There is an isomorphism of inverse systems $Cyl^1(k)\tilde{\to}\frac{\Omega(k)}{\Omega^+(k)^\infty}$ such that the canonical coface maps $\partial^i : k \to Cyl^1(k)$ $i = 0, 1$ are $\partial^1(e) = e$ and*

$$\partial^0(e) = e + de + \sum_{n=1}^{\infty} C_n(1 - 2e)de^{2n} \tag{21}$$

Here $C_1 = 1$ *and* $C_n = \sum_{i=1}^{n-1} C_i C_{n-i}$ *is the* $n+1$*-th Catalan number (cf. [10]).*

2) The following element $u \in \frac{\widetilde{\Omega}(k)}{\Omega^+(k)^\infty}$ *verifies the identity* $u\partial^0(e) = \partial^1(e)u$*.*

$$u = 1 + \sum_{n=1}^{\infty} C_n (2e-1) de^{2n-1} \tag{22}$$

Proof. One checks that the 1-cocycle $\phi : k \longrightarrow \Omega^2(k)$, $\phi(e) = (1-2e)de^2$ satisfies $-\delta(f) = d \cup d$. Part 1) of the lemma follows from this and the recursive formula (5') of [2]. Write $D_n(e)$ for the term of degree n of (21); for $u = 1 + \sum_{n=1}^{\infty} \omega_n$, the condition $u\partial^0(e) = \partial^1(e)u$ is equivalent to the sequence of identities:

$$[e, \omega_n] = D_n(e) + \sum_{i=1}^{n-1} \omega_i D_{n-i}(e) \tag{23}$$

One checks that for $\omega_{2n-1} = C_n(2e-1)de^{2n-1}$, $\omega_{2n} = 0$, the identity above is satisfied. $\square$

The following well-known version of Whitehead's Lemma was used in 4.1 above.

Whitehead's Lemma 4.3. *Let* E *and* F *be right* $\tilde{A}$*-modules, and let* $\phi : E \tilde{\to} F$ *and* $\psi : F \tilde{\to} E$ *be module isomorphisms. Then the following automorphisms:*

$$\begin{pmatrix} \psi\phi & 0 \\ 0 & 1_{F\oplus E\oplus F} \end{pmatrix} \sim \begin{pmatrix} 1_{E\oplus F\oplus E} & 0 \\ 0 & \phi\psi \end{pmatrix}$$

are in the same commutator class of the group $Aut(E \oplus F \oplus E \oplus F)$*.*

Theorem 4.4. *Let* k *be a field of characteristic zero,* A *an algebra over* k *and* ν *and* ch^{inf} *be the homomorphisms of 4.0 and 4.1 above. Then:*

$$\nu \circ ch^{inf} = 2ch_0$$

Proof. We keep the notations of 4.1. We shall show that, for $E = e_1\tilde{A}^s$, the sequence of isomorphisms:

$$\begin{aligned} H^2_{inf}(A, K_1) &\cong H^2_{inf}(A, HC_0) \qquad \text{by 4.0} \\ &\cong H^2(\lim_n \frac{P\Omega^*}{P(\mathcal{F}^n + b\Omega^{*+1})}) \qquad \text{by the proof of 3.0} \\ &\cong H^0(\lim_n \frac{P\Omega^*}{P(\mathcal{F}^n + b\Omega^{*+1})}) \qquad \text{via 3.1} \\ &\cong H^0(\hat{X}(R, I)) \qquad \text{by the proof of 3.0} \\ &\cong HP_0(A) \qquad \text{by [5]} \end{aligned}$$

maps $ch^{inf}(E)$ to twice the class of $\{e_n\}$ in $H^0(\hat{X}(R,I))$; by [5, 12.4], the latter class coincides with $ch_0(E)$. First we need a more explicit description of ch^{inf} in terms of $Cyl^*(R,I)$. We have a natural map $Cyl^*(R/I^n) \twoheadrightarrow Cyl^*(R,I)_n$ and, by [3, 3.7] also a natural map $Cyl^*(R,I)_{2(n^2+n-1)} \twoheadrightarrow Cyl^*(R/I^n)$. Thus, if $\{u_n\}$ is a sequence of elements $u_n \in G_n^1 = Gl_s(\widetilde{Cyl}^1(R,I)_n)$ satisfying $u_1 = 1$, $u_n\partial_n^0(e_n) = \partial_n^1(e_n)u_n$, and compatible with the projections $G_n^1 \to G_{n-1}^1$, then the sequence $\{[\delta_n]\}$ of the K_1-classes of the compatible elements:

$$\delta_n = \partial_n^2(u_n)\partial_n^0(u_n)\partial_n^1(u_n)^{-1}e_n + 1 - e_n \tag{24}$$

represents $ch^{inf}(E)$. Write $\Omega = \Omega(R)$, $\partial^0 = 1 + \sum_{i=1}^{\infty} D_i : R \to Cyl(R) \cong \Omega/\Omega^{+\infty}$ with $D_1 = d$, $D_2 = \phi$, $D_i(R) \subset \Omega^i$. Thus $\partial_n^0 = 1 + \sum_{i=1}^{n-1} D_{i,n} : R/I^n \to Cyl^1(R,I)_n$ where the $D_{i,n}$ are the induced maps. As in the proof of 4.2 above, one checks that, for $u_n = 1 + \sum_{i=1}^{n-1} \omega_{i,n}$, ($\omega_{i,n} \in \frac{\Omega^i}{\mathcal{F}^{n-i}\Omega^i}$), the condition that $u_n\partial_n^0(e_n) = \partial_n^1(e_n)u_n$ is equivalent to the sequence of identities:

$$[e_n, \omega_{p,n}] = D_{p,n}(e_n) + \sum_{i=1}^{p-1} \omega_{i,n} D_{p-i,n}(e) \qquad (1 \le p \le n-1) \tag{25}$$

Next one verifies that $\omega_{1,n} = (2e_n - 1)de_n$ satisfies the identity for $p = 1$. Then one uses induction to show that the map $\psi_{p,n} : k \to \Omega^p/F_{n-p}\Omega^p$, $e \mapsto D_{p,n}(e_n) + \sum_{i=1}^{p-1} \omega_{i,n} D_{p-i,n}$ is a derivation. It follows that if $\bar{\psi}_{p,n} : \Omega^1 k \to \Omega^p/F_{n-p}\Omega^p$ is the induced homomorphism, then $\omega_{p,n} = \bar{\psi}_{p,n}(\omega_{1,n})$ satisfies (25). Hence $\{u_n = 1 + \sum_{i=1}^{n-1} \omega_{i,n}\}$ is a compatible sequence satisfying the requirements above. A formula for $\omega_{1,n}$ has been given already; we have $\omega_{2,n} = \nabla(\omega_{1,n})$, where $\nabla(xdy) = x\phi(y)$. We shall have no use for the terms of higher degree. From now on we fix n and omit it as a subscript. One uses (9) to calculate that the element (24) is:

$$\delta = 1 + e(de)^2 \otimes (v_1 v_2 + v_1^2) + \dots \tag{26}$$

Here and below, the $\dots$ stand for higher degree terms. The isomorphism:

$$H^2_{inf}(A, K_1) \cong H^2_{inf}(A, HC_0)$$

of 4.0 is induced by the logarithm map $log : \ker(Gl(Cyl^2(R,I)_n) \to Gl(R/I^n)) \to \ker(HC_0(Cyl^2(R,I)_n) \to HC_0(R/I^n))$. Applying this map to (26) we still get $log(\delta) = e(de)^2 \otimes (v_1 v_2 + v_1^2) + \dots$ where again the $\dots$ are higher degree terms, but not necessarily the same as in (26). The composite of the projection to the normalized complex and the map $1 \otimes p$ of the proof of 3.0 above applied to $log(\delta)$

yields the element $e(de)^2$. Mapping the latter through the periodicity map of 4.1 we obtain the element $2e$. □

Remark 4.5. Here is an explanation for the normalization factor appearing in theorem 4.4. For simplicity we shall restrict our attention to the quasi-free case. Let R be quasi-free, E a finitely generated right projective $\tilde{R}$-module and ∇ a connection on E. Write $\nu = s'\nu'$ where $s' : H^2_{dR}(R) \cong H^0_{dR}(R) \cong HP_0(R)$ is the periodicity isomorphism of 3.1. The proof above shows that:

$$\nu'(ch^{inf}(E)) = [tr(\nabla^2)] \in H^2_{dR}(R)$$

Here tr is the trace and [ ] denotes cohomology class. Thus the factor of 1/2 needed to normalize $ch^{inf}(E)$ is the same as that appearing in Karoubi's definition of the Chern characters to de Rham cohomology ([11, 1.17]):

$$ch_{2q}(E) = (1/2q!)[tr(\nabla^{2q})] \in H^{2q}_{dR}(R)$$

5. The Jones-Goodwillie character.

The title of this section refers to the homomorphism:

$$c_n : K_n(A) \to HN_n(A) \tag{27}$$

going from K-theory to negative cyclic homology ($n \geq 1$). The map (27) was defined for arbitrary unital rings in [13], where it was proved it induces an isomorphism at the level of the rational relative groups associated to a nilpotent ideal. Here we shall restrict our attention to the particular case when A is a $\mathbb{Q}$-algebra. Because for algebras over $\mathbb{Q}$, relative K and cyclic homology groups are $\mathbb{Q}$-spaces, (cf. [15]), it follows from [13] that if A is a unital $\mathbb{Q}$-algebra and $I \subset A$ is nilpotent then:

$$c_n : K_n(A, I) \tilde{\to} HN_n(A, I) \tag{28}$$

is an isomorphism. It is easy to extend the map (27) to nonunital algebras and to show that (28) holds nonunitally also; see [3, 4.2] for details. Also in [3, 4.2] it is shown that (27) is the homomorphism induced at the level of homotopy groups by a map of fibrant simplicial sets:

$$c : \mathbb{Z}_\infty \mathcal{N}Gl(A) \to SCN_{\geq 1}(A) \tag{29}$$

Here $\mathbb{Z}_\infty$ is the Bousfield-Kan completion [1], $\mathcal{N}Gl$ is the nerve of the general linear group, and $SCN_{\geq 1}$ is the simplicial abelian group the Dold-Kan correspondence associates with the truncation of the negative cyclic chain complex. In general, if $(C_n, \delta)_{n \in \mathbb{Z}}$ is any chain complex, and $m \in \mathbb{Z}$, we write $C_{\geq m}$ for the complex which is C_n in each degree $n \geq m$, δC_m in degree $m-1$ and 0 in degrees $< m-1$. The purpose of this section is to prove the following:

Theorem 5.0. *Let A be a $\mathbb{Q}$-algebra. Then the Jones-Goodwillie map (29) fits into a homotopy fibration sequence of fibrant simplicial sets:*

$$\mathbb{H}_{inf}(A, \mathbb{Z}_\infty \mathcal{N}Gl) \to \mathbb{Z}_\infty \mathcal{N}Gl(A) \xrightarrow{c} SCN_{\geq 1}(A)$$

Here $\mathbb{H}_{inf}(A, \mathbb{Z}_\infty \mathcal{N}Gl)$ is the infinitesimal hypercohomology simplicial set, as defined in 1.3 above.

Corollary 5.1. *Let $A = R/I$ be a presentation of A as a quotient of a quasi-free algebra R. For $n \geq 2$, write $LK_n(A) = \pi_n(\underset{m}{\operatorname{holim}} \mathbb{Z}_\infty \mathcal{N}Gl(R/I^m))$. Then there is an exact sequence:*

$$\dots \to HN_{n+1}(A) \to LK_n(A) \to K_n(A) \to HN_n(A) \to \dots \to LK_2(A) \to$$
$$K_2(A) \to HN_2(A) \to H^0_{inf}(A, K_1) \to K_1(A) \to HN_1(A) \to H^1_{inf}(A, K_1) \to 0$$

Proof of Corollary 5.1. The sequence of the corollary is the long exact sequence of homotopy groups of the fibration of the theorem. Thus it suffices to compute the hypercohomology groups $\mathbb{H}^n_{inf}(A, \mathbb{Z}_\infty \mathcal{N}Gl)$. It follows from [3, 5.5] and from [1, IX.3.1] that $LK_n(A) = H^n_{inf}(A, \mathbb{Q}_\infty \mathcal{N}E)$ where E is the elementary group. By [1,4.4] we have a functorial homotopy fibration sequence:

$$\mathbb{Z}_\infty \mathcal{N}E \to \mathbb{Z}_\infty \mathcal{N}Gl \to \mathcal{N}(K_1)$$

By [1, 7.2], $\mathbb{H}^n_{inf}(A, \mathcal{N}(K_1)) = H^{1-n}_{inf}(A, K_1)$ for $n = 0, 1$ and is zero otherwise. On the other hand, because $\mathbb{H}_{inf}(A, -)$ is a holim, it preserves homotopy fibration sequences of fibrant simplicial sets. The groups $\mathbb{H}^n_{inf}(A, \mathbb{Z}_\infty \mathcal{N}Gl)$ for $n \geq 0$ are now easily calculated from the long exact sequence of homotopy associated to the fibration sequence obtained from applying the functor $\mathbb{H}_{inf}(A, -)$ to the sequence above. □

Proof of Theorem 5.0. I claim that:

$$\mathbb{H}^n_{inf}(A, SCN_{\geq 1}) = 0 \qquad (n \leq 0) \tag{30}$$

Assume the claim holds. Let F be the homotopy fiber of the map (29). We know from [3, 5.2] that we have a map $H_{inf}(-, X) \to X$ which is natural for functors going from rings to fibrant simplicial sets, and that the functor $X \mapsto$

$H_{inf}(-,X)$ preserves homotopy fibration sequences. Using these facts, one obtains a commutative diagram where both rows are homotopy fibrations:

$$\begin{array}{ccccc} \mathbb{H}_{inf}(A,F) & \longrightarrow & \mathbb{H}_{inf}(A,\mathbb{Z}_\infty\mathcal{N}Gl) & \longrightarrow & \mathbb{H}_{inf}(A,SCN_{\geq 1}) \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & \mathbb{Z}_\infty\mathcal{N}Gl(A) & \longrightarrow & SCN_{\geq 1}(A) \end{array} \tag{31}$$

From the claim we deduce that the first map in the top row of (31) is a homotopy equivalence. Next recall from [3] that for functors X which map surjections with nilpotent kernel into homotopy equivalences the map $H_{inf}(-,X)\to X$ is a weak equivalence. By (28), this applies to F. It follows that the first vertical map of (31) is an equivalence. We have shown that the claim implies the theorem. Next note that, by [3, 5.4.1] the claim is equivalent to the assertion that the nonpositive hypercohomology groups of the chain complex $CN_{\geq 1}$ are zero. We shall actually show that $\mathbb{H}^n_{inf}(A,CN_{\geq}1)=0$ for $n\leq 0$, and equals $HP_n(A)$ for $n\geq 1$. It is proven below (Corollary 5.3) that $CN_{\geq 1}$ has the same hypercohomology as the complex having $\Omega^1_\natural$ in degree 1, $b\Omega^1$ in degree zero, and zero elsewhere. A standard spectral sequence argument shows that the hypercohomology of this complex is $H^*_{inf}(A,HH_1)$. The latter is calculated by means of the long exact sequence of cohomology groups associated with the following exact sequences of sheaves:

$$0\to HH_1\to\Omega^1_\natural\to b\Omega^1\to 0 \quad\text{and}\quad 0\to b\Omega^1\to O\to O/[O,O]\to 0$$

By Lemma 5.5 below and the first of the sequences above, we get that

$$H^0_{inf}(A,HH_1)=0 \quad\text{and}\quad H^n_{inf}(A,HH_1)=H^{n-1}(A,b\Omega^1).$$

From Theorem 2.3 and (18), we obtain $H^*_{inf}(A,O)=0$; from Theorem 3.0 and what we have just seen, we obtain that $H^1_{inf}(A,HH_1)=0$ and $H^n_{inf}(A,HH_1)=HP_n(A)$. This completes the proof. □

Lemma 5.2. *Let R be a quasi-free $\mathbb{Q}$-algebra, $\Omega^*=\Omega^*(R)$, ∇ as in* (17)*, $X=(\Omega^0 \underset{b}{\overset{\natural d}{\rightleftarrows}} \Omega^1_\natural)$ the* 2*-periodic de Rham complex of [5], $CP^*=CP^*(R)$ the (b,B) periodic cyclic complex. Define a homogeneous map $f:X\to CP$:*

$$f(\natural\omega)=\{(-\nabla B)^n(1-b\nabla)\omega\}_{n\geq 0},\quad f(a)=\{(-\nabla B)^n(a)\}_{n\geq 0}\quad (\natural\omega\in\Omega^1_\natural, a\in\Omega^0.)$$

Then:

i) *The map f is a well-defined supercomplex homomorphism.*

ii) *For the natural projection $\pi : CP \twoheadrightarrow X$, we have $\pi f = 1$.*

iii) *Put $h_m : \prod_{r\geq 0} \Omega^r \to \prod_{r\geq 0} \Omega^{r+2m+1}$,*

$$h_m(\sum_{r=0}^{\infty} \omega_r) = \sum_{r=1}^{\infty} (-1)^m \nabla (B\nabla)^m \omega_r \qquad m \geq 0$$

Then for $h = \sum_{m=0}^{\infty} h_m$, we have $1 - f\pi = (B+b)h(1-f\pi) + h(1-f\pi)(B+b)$.

iv) *For the filtration F of 1.2 above, we have:*

$$f_1(b\Omega^2 + F_m\Omega^1) \subset \prod_{p\geq 0} F_{m-(2p+1)}\Omega^{2p+1}, \quad f_0(F_m\Omega^0) \subset \prod_{p\geq 0} F_{m-2p}\Omega^{2p}$$
$$\text{and} \quad h(1-f\pi)(\prod_{p\geq 0} F_{m-p}\Omega^p) \subset \prod_{p\geq 0} F_{m-1-p}\Omega^p.$$

Proof. Part i) is straightforward; part ii) is immediate. By 2.4, the map h of iii) is a contracting homotopy for $\ker \pi$; the assertion of iii) is immediate from this. Item iv) is immediate from (12).

Corollary 5.3. *Let A be any algebra. Then the infinitesimal hypercohomology of each of the Hochschild, cyclic, negative cyclic and periodic cyclic chain complexes associated to the functorial mixed complex $\Omega^* \overset{B}{\underset{b}{\rightleftarrows}} \Omega^{*+1}$ equals the hypercohomology of the corresponding chain complex associated to the functorial mixed complex X.*

Remark 5.4. The same argument as in the proof of Theorem 5.0 shows that, for any ring A –not necessarily a $\mathbb{Q}$-algebra– we have a fibration sequence:

$$\mathbb{H}_{inf}(A, \mathbb{Q}_\infty \mathcal{N}Gl) \to \mathbb{Q}_\infty \mathcal{N}GL \xrightarrow{c} SCN_{\geq 1}(A \otimes \mathbb{Q})$$

Moreover the analogue of the exact sequence of Corollary 5.1 holds, with the same proof. The lower terms of this rational sequence give the exact sequence of [3, 6.2]), except that now we know the map $HN_1(A \otimes \mathbb{Q}) \twoheadrightarrow H^1(A, K_1 \otimes \mathbb{Q})$ is surjective. The surjectivity of this map was conjectured in [3]; this true conjecture was derived from the wrong conjecture that $H^*_{inf}(A, HN_1) = 0$. In fact it is not hard to show, using 5.3, that $\mathbb{H}^*_{inf}(A, HN_1) = \mathbb{H}^*_{inf}(A, HH_1)$ which is as calculated in the proof of 5.0, and is nonzero in general.

Lemma 5.5. *Let A be a $\mathbb{Q}$-algebra. Then:*

$$H^*_{inf}(A, \Omega^1_\natural) = 0$$

Proof. Let $A = R/I$ be a presentation of A as a quotient of a quasi-free algebra. We have to show that the cohomology of the complex $C(R, I, \Omega^1_\natural)$ is trivial. By definition the latter complex is the inverse limit of the complexes $\Omega^1_\natural Cyl(R, I)_n$ The proof has two main parts. The first part consists of showing that there is an isomorphism of pro-vectorspaces:

$$\Omega^1_\natural Cyl^m(R, I) \cong \{\oplus_{r=1}^n \frac{(\Omega^{r+1} \oplus \Omega^r) \otimes T^r(V^m)}{M_r^m + {}_r\mathcal{G}^{m,n}}\} \tag{32}$$

Here $\Omega = \Omega(R)$, $M_r^m = \{(b\omega \otimes x + u \otimes y + (-1)^r \kappa(u) \otimes \zeta(y), (-1)^{r+1} b(u) \otimes y) : \omega \in \Omega^{r+1}, u \in \Omega^r, x, y \in T^r(V^m)\}$, and:

$${}_r\mathcal{G}^{m,n} = (\sum_{i_0 + \dots i_{r+1} + j \geq n-r} I^{i_0} dAI^{i_1} \dots I^{i_r} d(I^j) I^{i_{r+1}} \oplus F^{n-r}\Omega^r) \otimes T^r(V^m)$$

Now to the first part. We consider first the case when the ideal $I = 0$. We have:

$$\begin{aligned}
\Omega^1(Cyl^m(R)) &= \{\Omega^1(Cyl^m(R)_n)\} \\
&= \{\oplus_{r=0}^{n-1} \oplus_{p+q=r} \widetilde{\Omega}^p \otimes \Omega^q \otimes T^r(V^m)\} \\
&= \{\Omega^1(\Omega_m)_{deg<n}\}
\end{aligned}$$

Here $\tilde{\Omega}^0 = \tilde{R}$, $\tilde{\Omega}^p = \Omega^p$ if $p \geq 1$, $\Omega_m = \Omega_m(R) = \oplus_{p\geq 0} \Omega^p \otimes T^p(V^m)$ is the graded algebra of 1.1 above, and ${}_{deg<n}$ indicates a truncation of the graded Ω_m-module $\Omega^1(\Omega_m)$. Because the commutator subspace is graded, we have $\{\Omega^1_\natural(Cyl^m(R)_n)\} \cong \{\Omega^1_\natural(\Omega_m)_{deg<n}\}$. We write δ for the de Rham differential of Ω_m. Fix a degree r. By repeated application of the Leibniz formula satisfied by δ, we see that every form of degree r can be written as a sum of of elements of the following types:

$$\begin{aligned}
&\text{Type}(p, 0): \quad a_0 da_1 \dots da_p \delta a_{p+1} da_{p+2} \dots da_{r+1} \otimes x \\
&\text{Type}(p, 1): \quad a_0 da_1 \dots da_{p-1} \delta(da_p) da_{p+1} \dots da_r \otimes x
\end{aligned} \tag{33}$$

Here $x \in T^r(V^m)$, $a_i \in R$, and if $\epsilon = 0, 1$, then for elements of type (p, ϵ), the index p runs between ϵ and r. One checks furthermore that the sum of the subspaces generated by the elements of each type is direct, i.e. that we have an isomorphism:

$$(\oplus_{p=0}^r \Omega^{r+1} e_p \oplus \oplus_{p=1}^r \Omega^r e_p) \otimes T^r(V^m) \cong_r \Omega^1(\Omega_m) \tag{34}$$

Here we write ωe_p for the element ω in the p-th summand; the isomorphism maps $a_0 da_1 \dots da_{r+\epsilon} e_p$ to the element of type (p, ϵ) of (33) ($\epsilon = 0, 1$), and ${}_r\Omega^1(\Omega_m)$ is the homogenous summand of degree r. To compute the degree r part of $\Omega^1(\Omega)_\natural$, we must divide ${}_r\Omega^1(\Omega_m)$ by the subspace generated by the elements of the form $[a, \omega \otimes x]$ and $[da \otimes v, u \otimes y]$ for $a \in R$, $\omega \in_r \Omega^1(\Omega_1)$, $u \in_{r-1} \Omega^1(\Omega_1)$, $x \in T^r(V^m)$, $v \in V^m$ and $y \in T^{r-1}(V^m)$. One calculates each of these commutators for each of the types of ω and u, and then pulls them back through the isomorphism (34) to obtain that ${}_r\Omega^1(\Omega_m)_\natural$ is isomorphic to the quotient of the left hand side of (34) by the subspace M'_r generated by the following elements:

$$[\omega, a]e_p \otimes x, \quad ([u, a]e_q + (-1)^{q+1}(udae_q + udae_{q-1})) \otimes x,$$
$$\eta \otimes xe_s - (-1)^{r-\epsilon}\kappa(\eta) \otimes \zeta(x)e_{s+1}$$

Here $0 \leq p \leq r$, $1 \leq q \leq r$, $\epsilon \leq s \leq r-1$, $\omega \in \Omega^{r+1}$, $u \in \Omega^r$, $\eta \in \Omega^{r+1-\epsilon}$, $\epsilon = 0, 1$, $x \in T^r(V^m)$, and ζ is as in the proof of theorem 3.0 above. Thus $\eta \otimes xe_s \equiv (-1)^{r-\epsilon}\kappa(\eta) \otimes \zeta(x)e_{s+1} \pmod{M'_r}$ whence every element in ${}_r\Omega^1(\Omega_m)_\natural$ is the class of some element in $(\Omega^r e_r \oplus \Omega^{r+1} e_r) \otimes T^r(V^m)$, or in other words there is a surjective homomorphism going from the latter to the former. One has to check that, upon identifying $\Omega^{r+1} \oplus \Omega^r$ with $\Omega^{r+1}e_{r+1} \oplus \Omega^r e_r$, we have $M^m_r = {M'}^m_r \cap ((\Omega^{r+1}e_r \oplus \Omega^r e_r) \otimes T^r(V^m))$. The inclusion $\subset$ is immediate. To prove the other inclusion, let $\omega^0 e_r + \omega^1 e_r \in ((\Omega^{r+1}e_r \oplus \Omega^r e_r) \otimes T^r(V^m)) \cap {M'}^m_r$. Then there exist elements $u^0_i \in \Omega^{r+2} \otimes T^r(V^m)$, $u^1_j \in \Omega^{r+1} \otimes T^r(V^m)$, $v^0_k \in \Omega^{r+1} \otimes T^r(V^m)$, and $v^1_l \in \Omega^r \otimes T^r(V^m)$, $0 \leq i \leq r$, $1 \leq j \leq r$, $0 \leq k \leq r-1$, $1 \leq l \leq r-1$, such that:

$$\omega^1 e_r = \sum_{i=1}^{r} (b \otimes 1)(u^1_i)e_i + \sum_{i=1}^{r-1} (v^1_i e_i + (-1)^r (\kappa \otimes \zeta)(v^1_i)e_{i+1})$$

and:

$$\omega^0 e_r = \sum_{i=0}^{r} (b \otimes 1)(u^0_i)e_i + \sum_{i=1}^{r} (-1)^{i+1}(u^1_i e_i + u^1_i e_{i-1}) + \sum_{i=0}^{r-1} (v^0_i + (-1)^{r+1}(\kappa \otimes \zeta)(v^0_i))$$

In the first identity, for each $i \neq r$ the coefficient of e_i in the right hand side must be zero; thus if we apply $((-1)^{r+1}\kappa \otimes \zeta)^{r-i}$ to it and then take the sum over all i, we obtain the identity:

$$\omega^1 = b \otimes 1 (\sum_{i=1}^{r} ((-1)^{r+1}\kappa \otimes \zeta)^{r-i}(u^1_i))$$

By a similar procedure, the second identity yields:

$$\omega^0 = (b\otimes 1)(\sum_{i=0}^{r}((-1)^r\kappa\otimes\zeta)^{r-i}(u_i^0)+$$
$$\sum_{i=1}^{r}(1+(-1)^r(\kappa\otimes\zeta))(-1)^{i+1}((-1)^r\kappa\otimes\zeta)^{r-i}(u_i^1)$$

This proves the inclusion $M'_r\cap(\Omega^{r+1}e_r\oplus\Omega^r e_r)\subset M_r$. Thus the proof of (32) for the case $I=0$ is complete. Now let $I\subset R$ be any ideal. We have:

$$\Omega^1(Cyl^m(R,I)_n) = \oplus_{0\le p,q<n}\frac{\Omega^p\delta\Omega^q}{F^{n-p}\Omega^p\delta\Omega^q+\Omega^p\delta F^{n-q}\Omega^q}T^{p+q}(V^m)$$

One checks that the filtrations

$$\Omega^1(\Omega_m)\supset\mathcal{G}''^{m,n} = \sum_{p,q\ge 0}(F^{n-p}\Omega^p\delta\Omega^q+\Omega^p\delta F^{n-q}\Omega^q)\otimes T^{p+q}(V^m)$$
$$\text{and}\qquad \mathcal{G}'^{m,n} = \sum_{n\le p+q+i+j}(F^i\Omega^p\delta F^j\Omega^q)\otimes T^{p+q}(V^m)$$

are equivalent, whence $\Omega^1(Cyl^m(R,I))\cong\frac{\Omega^1(\Omega_m)}{\mathcal{G}'^{m,\infty}}$ as pro-vectorspaces. For the degreewise identification (34), we have: $\mathcal{G}'^{m,n}\cap(\Omega^r e_s\otimes T^r(V^m)) = F^{n-r}\Omega^r e_s\otimes T^r(V^m)$ and

$$\mathcal{G}'^{m,n}\cap\Omega^{r+1}e_s\otimes T^r(V^m) =$$
$$\sum_{i_0+\dots i_{r+1}+j\ge n-r} I^{i_0}dAI^{i_1}\dots I^{i_{s-1}}d(I^j)I^{i_s}\dots dAI^{i_{r+1}}$$

We note that $\mathcal{G}'^{m,n}$ is closed under $\kappa\otimes\zeta$, and if $u\otimes x\in\mathcal{G}'^{m,n}\cap\Omega^{r+1}e_s$ $(s\ge 1)$, then $b\otimes 1(u\otimes x)\in\mathcal{G}'^{m,n}\cap(\Omega^r e_s\otimes T^r(V^m))$. From these observations, we see that the same argument as that given in the proof of (32) for the case $I=0$ can be applied here, to prove now that $(\Omega^{r+1}\oplus\Omega^r e_r)\otimes T^r(V^m)\cap(M'_r+\mathcal{G}'^{m,n}) = M_r +_r\mathcal{G}^{m,n}$. This finishes the first part of the proof. For the second part, consider the map p of the proof of theorem 3.0 above. Using the calculation of the first part of the current proof and the arguments of the proof of 3.0, one deduces that $1\otimes p$ induces a homotopy equivalence between the pro-complex $\Omega^1_\natural(Cyl(R,I))$ and the pro-complex having $\frac{\Omega^{m+1}\oplus\Omega^m}{N^m+\mathcal{G}^{1,\infty}}$ in codimension m, with $N^m=\{(b\omega+u-\kappa(u),(-1)^{m+1}b(u)) : \omega\in\Omega^{m+2}, u\in\Omega^{m+1}\}$ and with as coboundary map that induced by $\Omega^{m+1}\oplus\Omega^m\to\Omega^{m+2}\oplus\Omega^{m+1}$, $(\omega,u)\mapsto(d\omega,(-1)^m\omega+du)$. The latter complex is contractible: a contracting homotopy is given by $(\omega,u)\mapsto((-1)^{|\omega|}u,0)$. We see that this homotopy maps N^* to N^{*-1} and is continuous for the topology of the filtration $\mathcal{G}^1$. This concludes the second and last part of the proof. □

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