HOCHSCHILD (CO)HOMOLOGY OF CROSSED PRODUCTS

Jorge A. Guccione and Juan J. Guccione

We dedicate this work to the memory of Professor Orlando Villamayor

Abstract. Let E be a crossed product of a group over an algebra and let M be an E-bimodule. We obtain a chain complex and a cochain complex, simpler than the canonical ones, given the Hochschild homology and cohomology of E with coefficients in M respectively. These complexes have natural filtrations whose spectral sequences generalize those of Hochschild-Serre.

Introduction

Let k be a commutative ring with 1, A a k-algebra, A^* the group of units of A and G a group. An associative k-algebra E, whose underlying additive group is

$$\bigoplus_{g\in G} Aw_g \simeq A\otimes_k k[G],$$

is a crossed product of G over A if it contains $A\simeq Aw_1$ as a subalgebra, and its multiplication has the following properties:

a) $w_g 1 = 1 w_g$ for all $g\in G$,

b) $w_g a \in Aw_g$ for all $a\in A$ and all $g\in G$,

c) $w_g w_{g'} \in A^* w_{gg'}$ for all $g,g'\in G$.

These algebras, which generalize the group algebras, are closely related to the study of central simple algebras, equivariant projective representations of groups, extensions of groups, graded rings, etcetera.

Certain special cases of crossed products E are important in itself and they have their own names. If the w_g's commute with the elements of A, we say that

---

1991 *Mathematics Subject Classification.* 16E40, 16S35.

Supported by UBACYT 01/X94

E is twisted group algebra and if $w_g w_{g'} = w_{gg'}$ for all $g, g' \in G$, we say that E is a skew group algebra.

The Hochschild and cyclic homology of skew group algebras have been studied in several papers, for instance in [B1], [B2], [F-T], [G-J] and [N], but we do not know a paper which describes the (co)homology of a crossed product that is not a skew group algebra, except [S], where the author study the Hochschild cohomology of these algebras under particular hypothesis.

In this paper we discuss the Hochschild (co)homology of crossed products in general. We divide the paper in three sections: In the first one, we give a method to construct (under suitable hypothesis) a projective resolution of k-algebra E as $E^e = E \otimes E^{\mathrm{Op}}$-bimodule, simpler than the canonical of Hochschild. This method can be considered as a variant of the perturbation lemma. In the second one, a resolution of a crossed product is constructed using the results of the first section. In the third one, for an E-bimodule M, we obtain a chain complex $\overline{X}_*(E, M)$ giving the Hochschild homology of E with coefficients in M, and a cochain complex $\overline{X}^*(E, M)$ giving the Hochschild cohomology. When E is a skew group algebra, $\overline{X}_*(E, E)$ is the chain complex obtained for the Hochschild homology in [G-J]. Both $\overline{X}_*(E, M)$ and $\overline{X}^*(E, M)$ have natural filtrations. The spectral sequences associated to this filtrations generalize to the Hochschild-Serre. Furthermore our construction of these spectral sequences is more straigthforward than the usual ones (c.f. [B] and [W]).

The boundary maps of the chain complex $\overline{X}^*(E, M)$ are a mixture of morphims d^0_* like the boundary maps of the normalized Hochschild complex of A with coefficients in M, morphisms d^1_* like the boundary maps of the normalized canonical complex giving the homology groups of G with coefficients in $M \otimes \overline{A}^*$ and mappings d^l_* ($l \geq 2$) depending on the 2-cocycle associated to E. Thus, it is natural to think, that the Hochschild homology of E with coefficients in M is related to the Hochschild homology of A and the homology groups of G. In the general case such relation is establish by the spectral metioned above. Under more restrictive hypothesis this relation becomes more explicit. For instance, when A is a separable k-algebra, then $\mathrm{H}_*\big(E, M\big) = \mathrm{H}_*\big(G, \frac{M}{[A,M]}\big)$ and when G is a finite group and the order of G is invertible in k, then G acts on $\mathrm{H}_*(A, M)$ and the hochschild homology of E with coefficients in M are the coinvariants of this action.

1. A method to construct resolutions

Let k be a commutative ring with unity and E a k-algebra. In this section, under suitable conditions, we construct a resolution of E as an E-bimodule. We use this result in the following section.

Let us considerer a diagram of E-bimodules and E-bimodule maps

$$
\begin{array}{ccccccccc}
\vdots & & & & & & & & \\
\downarrow{\scriptstyle d^1_{-1,4}} & & & & & & & & \\
X_{-1,3} & \xleftarrow{d^0_{03}} & X_{03} & \xleftarrow{d^0_{13}} & X_{13} & \xleftarrow{d^0_{23}} & X_{23} & \xleftarrow{d^0_{33}} & \dots \\
\downarrow{\scriptstyle d^1_{-1,3}} & & & & & & & & \\
X_{-1,2} & \xleftarrow{d^0_{02}} & X_{02} & \xleftarrow{d^0_{12}} & X_{12} & \xleftarrow{d^0_{22}} & X_{22} & \xleftarrow{d^0_{32}} & \dots \\
\downarrow{\scriptstyle d^1_{-1,2}} & & & & & & & & \\
X_{-1,1} & \xleftarrow{d^0_{01}} & X_{01} & \xleftarrow{d^0_{11}} & X_{11} & \xleftarrow{d^0_{21}} & X_{21} & \xleftarrow{d^0_{31}} & \dots \\
\downarrow{\scriptstyle d^1_{-1,1}} & & & & & & & & \\
X_{-1,0} & \xleftarrow{d^0_{00}} & X_{00} & \xleftarrow{d^0_{10}} & X_{10} & \xleftarrow{d^0_{20}} & X_{20} & \xleftarrow{d^0_{30}} & \dots,
\end{array}
$$

such that:

a) $(X_{-1,*}, d^1_{-1,*})$ is a chain complex.

b) For each $r, s \geq 0$ we have a k-module $\overline{X}_{rs}$ and E-bimodule morphisms

$$s_{rs} \colon X_{rs} \to E \otimes \overline{X}_{rs} \otimes E \qquad \text{and} \qquad \pi_{rs} \colon E \otimes \overline{X}_{rs} \otimes E \to X_{rs}$$

verifying $\pi_{rs}\, s_{rs} = id$.

c) Each row is contractible as a complex of left E-modules, with a chain contracting homotopy $\sigma^0_{rs} \colon X_{rs} \to X_{r+1,s}$ ($r \geq -1$ and $s \geq 0$).

We are going to modify this diagram adding E-bimodule morphisms

$$d^l_{rs} \colon X_{rs} \longrightarrow X_{r+l-1,s-l} \qquad (r, s \geq 0 \text{ and } 1 \leq l \leq s).$$

We will define these arrows of such form that (X_*, d_*), where

$$X_n = \bigoplus_{\substack{r+s=n \\ r \geq -1}} X_{rs} \qquad (n \geq -1),$$

$$d_n = d^1_{-1,n+1} + \sum_{\substack{r+s=n \\ r \geq 0}} \sum_{l=0}^{s} d^l_{rs} \qquad (n \geq 0),$$

is a chain complex of E-bimodules, which is contractible as a chain complex of left E-modules. In fact, we are going to build left E-module morphisms

$$\sigma^l_{rs} \colon X_{rs} \to X_{r+l+1,s-l} \qquad (r \geq -1,\ s \geq 0 \text{ and } 1 \leq l \leq s)$$

satisfying the following property:

1.1.Theorem. *The family of left E-module maps* $\sigma_n\colon X_n \to X_{n+1}$ $(n \geq -1)$, *defined by:*

$$\sigma_n = \sum_{\substack{r+s=n\\ r\geq -1}} \sum_{l=0}^{s} \sigma_{rs}^l,$$

is a chain contracting homotopy of (X_*, d_*).

1.2.Corollary. *Let* (X'_*, d'_*) *be the complex of E-bimodules defined by*

$$X'_n = \bigoplus_{\substack{r+s=n\\ r\geq 0}} X_{rs} \qquad \textit{and} \qquad d'_n = \sum_{\substack{r+s=n\\ r+l>0}} \sum_{l=0}^{s} d_{rs}^l.$$

Then $d_{0*}^0\colon (X'_*, d'_*) \to (X_{-1,*}, -d_{-1,*}^1)$ *is a morphism of complexes of E-bimodules and a chain homotopy equivalence of complexes of left E-modules.*

1.3.Corollary. *If there is an E-bimodule map* $\widetilde{\mu}\colon X_{-1,0} \to E$ *such that*

$$E \xleftarrow{\widetilde{\mu}} X_{-1,0} \xleftarrow{d_{-1,1}^1} X_{-1,1} \xleftarrow{d_{-1,2}^1} X_{-1,2} \xleftarrow{d_{-1,3}^1} X_{-1,3} \xleftarrow{d_{-1,4}^1} \dots,$$

is contractible as a complex of left E-modules, then

$$\text{(1.3.1)} \qquad E \xleftarrow{\mu} X'_0 \xleftarrow{d'_1} X'_1 \xleftarrow{d'_2} X'_2 \xleftarrow{d'_3} X'_3 \xleftarrow{d'_4} X'_4 \xleftarrow{d'_5} X'_5 \xleftarrow{d'_6} X'_6 \xleftarrow{d'_7} \dots,$$

where $\mu = \widetilde{\mu}\, d_{00}^0$, *is a relative projective resolution.*

Proof. It is immediate □

Next we define the morphisms d_{rs}^l and we prove that (X_*, d_*) is a chain complex.

1.4.Definition. *We define the E-bimodule maps*

$$d_{rs}^l\colon X_{rs} \to X_{r+l-1,s-l} \qquad (r \geq -1 \textit{ and } 1 \leq l \leq s)$$

recursively by:

$$d_{-1,s}^l = 0 \qquad \textit{and} \qquad d_{rs}^l = \widehat{d}_{rs}^l\, s_{rs},$$

where $\widehat{d}_{rs}^l\colon E \otimes \overline{X}_{rs} \otimes E \to X_{r+l-1,s-l}$ $(r \geq 0$ *and* $1 \leq l \leq s)$ *is the E-bimodule map sending* $1 \otimes \overline{x} \otimes 1$ *to*

$$-\sum_{j=0}^{l-1} \sigma_{r+l-2,s-l}^0\, d_{r+j-1,s-j}^{l-j}\, d_{rs}^j\, \pi(1 \otimes \mathbf{x} \otimes 1),$$

for each $\overline{\mathbf{x}} \in \overline{X}_{rs}$.

1.5.Proposition. *The following equalities hold:*

$$d^0_{r+l-1,s-l}\, d^l_{rs} = -\sum_{j=0}^{l-1} d^{l-j}_{r+j-1,s-j}\, d^j_{rs} \qquad (r \geq 0 \text{ and } 1 \leq l \leq s).$$

Consequently (X_*, d_*) *is a chain complex.*

proof. We prove the proposition by induction on l and r. To abbreviate we put d^j_r instead of d^j_{rs} and σ^0_r instead of σ^0_{rs}. Let $\mathbf{x} = 1 \otimes \overline{\mathbf{x}} \otimes 1$ with $\overline{\mathbf{x}} \in \overline{X}_{0s}$. Since $\widehat{d}^1_0(\mathbf{x}) = -\sigma^0_{-1}\, d^1_{-1}\, d^0_0\, \pi(\mathbf{x})$, we have

$$d^0_0\, \widehat{d}^1_0(\mathbf{x}) = -d^0_0\, \sigma^0_{-1}\, d^1_{-1}\, d^0_0\, \pi(\mathbf{x}) = -d^1_0\, d^0_0\, \pi(\mathbf{x}),$$

which implies $d^0_0\, d^1_0 = -d^1_0\, d^0_0$. Suppose that $l \geq 0$, $l + r > 0$ and that the result is valid for d^j_p with $j \leq l$ or $j = l + 1$ and $p < r$. Let $\mathbf{x} = 1 \otimes \overline{\mathbf{x}} \otimes 1$ with $\overline{\mathbf{x}} \in \overline{X}_{rs}$. Since

$$\widehat{d}^{l+1}_r(\mathbf{x}) = -\sum_{j=0}^{l} \sigma^0_{r+l-1}\, d^{l+1-j}_{r+j-1}\, d^j_r\, \pi_{rs}(\mathbf{x}),$$

then

$$\begin{aligned} d^0_{r+l}\, \widehat{d}^{l+1}_r(\mathbf{x}) &= -\sum_{j=0}^{l} d^0_{r+l}\, \sigma^0_{r+l-1}\, d^{l+1-j}_{r+j-1}\, d^j_r\, \pi_{rs}(\mathbf{x}) \\ &= -\sum_{j=0}^{l} d^{l+1-j}_{r+j-1}\, d^j_r\, \pi_{rs}(\mathbf{x}) + \sum_{j=0}^{l} \sigma^0_{r+l-2}\, d^0_{r+l-1}\, d^{l+1-j}_{r+j-1}\, d^j_r\, \pi_{rs}(\mathbf{x}). \end{aligned}$$

Applying first the inductive hypothesis to $d^0_{r+l-1}\, d^{l+1-j}_{r+j-1}$ with $(0 \leq j \leq l)$ and then to $d^0_{r+j-1}\, d^j_r$ with $(0 < j \leq l)$ we obtain:

$$\begin{aligned} d^0_{r+l}\, \widehat{d}^{l+1}_r(\mathbf{x}) &= -\sum_{j=0}^{l} d^{l+1-j}_{r+j-1}\, d^j_r\, \pi_{rs}(\mathbf{x}) - \sum_{j=0}^{l}\sum_{i=0}^{l-j} \sigma^0_{r+l-2}\, d^{l+1-j-i}_{r+j+i-2}\, d^i_{r+j-1}\, d^j_r\, \pi_{rs}(\mathbf{x}) \\ &= -\sum_{j=0}^{l} d^{l+1-j}_{r+j-1}\, d^j_r\, \pi_{rs}(\mathbf{x}) - \sum_{j=0}^{l-1}\sum_{i=1}^{l-j} \sigma^0_{r+l-2}\, d^{l+1-j-i}_{r+j+i-2}\, d^i_{r+j-1}\, d^j_r\, \pi_{rs}(\mathbf{x}) \\ &+ \sum_{j=1}^{l}\sum_{h=0}^{j-1} \sigma^0_{r+l-2}\, d^{l+1-j}_{r+j-2}\, d^{j-h}_{r+h-1}\, d^h_r\, \pi_{rs}(\mathbf{x}) = -\sum_{j=0}^{l} d^{l+1-j}_{r+j-1}\, d^j_r\, \pi_{rs}(\mathbf{x}). \end{aligned}$$

The desired equality follows immediately from this fact □

1.6.Definition. *For $r \geq -1$, $s \geq 0$ and $0 < l \leq s$ we define $\sigma^l_{rs}\colon X_{rs} \to X_{r+l+1,s-l}$ recursively by:*

$$\sigma^l_{rs} = -\sum_{i=0}^{l-1} \sigma^0_{r+l,s-l}\, d^{l-i}_{r+i+1,s-i}\, \sigma^i_{rs}$$

Proof of Theorem 1.1. Because of the definitions of d_* and σ_* it is enough to see that

$$\sigma^0_{r-1,s}\, d^0_{rs} + d^0_{r+1,s}\, \sigma^0_{rs} = 1_{X_{rs}} \qquad \text{and}$$
$$\sum_{i=0}^{l} \sigma^{l-i}_{r+i-1,s-i}\, d^i_{rs} + \sum_{i=0}^{l} d^{l-i}_{r+i+1,s-i}\, \sigma^i_{rs} = 0 \quad \text{for } l > 0,$$

where we put $d^0_{-1,s} = 0$. The first formula simply says that σ^0_* is a chain contracting homotopy of d^0_*. Let us see the second one. To abreviate we do not write the subindexes. From the definition of σ^l we have:

$$d^0\, \sigma^l = -\sum_{i=0}^{l-1} d^0\, \sigma^0\, d^{l-i}\, \sigma^i = \sum_{i=0}^{l-1} \sigma^0\, d^0\, d^{l-i}\, \sigma^i - \sum_{i=0}^{l-1} d^{l-i}\, \sigma^i.$$

Consequently

$$\sum_{i=0}^{l} \sigma^{l-i}\, d^i + \sum_{i=0}^{l} d^{l-i}\, \sigma^i = \sum_{i=0}^{l} \sigma^{l-i}\, d^i + \sum_{i=0}^{l-1} \sigma^0\, d^0\, d^{l-i}\, \sigma^i.$$

Then, it suffices to prove that the term appearing on the right side of the equality is zero. We prove this by induction on l. For $l = 1$ we have

$$\sigma^0\, d^0\, d^1\, \sigma^0 = -\sigma^0\, d^1\, d^0\, \sigma^0 = \sigma^0\, d^1\, \sigma^0\, d^0 - \sigma^0\, d^1 = -\sigma^1\, d^0 - \sigma^0\, d^1.$$

Suppose that $l > 1$. From Proposition 1.5,

$$\sum_{i=0}^{l-1} \sigma^0\, d^0\, d^{l-i}\, \sigma^i = -\sum_{i=0}^{l-1} \sum_{j=0}^{l-i-1} \sigma^0\, d^{l-i-j}\, d^j\, \sigma^i = -\sum_{h=0}^{l-1} \sum_{i=0}^{h} \sigma^0\, d^{l-h}\, d^{h-i}\, \sigma^i.$$

So, applying the inductive hypothesis to $\sum_{i=0}^{h} d^{h-i}\, \sigma^i$ $(h \geq 0)$, we obtain:

$$\begin{aligned}\sum_{i=0}^{l-1} \sigma^0\, d^0\, d^{l-i}\, \sigma^i &= \sum_{h=0}^{l-1} \sum_{i=0}^{h} \sigma^0\, d^{l-h}\, \sigma^{h-i}\, d^i - \sigma^0\, d^l \\ &= \sum_{i=0}^{l-1} \sum_{j=0}^{l-i-1} \sigma^0\, d^{l-i-j}\, \sigma^j\, d^i - \sigma^0\, d^l = -\sum_{i=0}^{l} \sigma^{l-i}\, d^i \quad \square\end{aligned}$$

2.A resolution for a crossed product

Let A be a k-algebra, A^* the group of units of A and G a group. A weak action of G on A is a map $(g,a) \mapsto a^g$, from $G \times A$ to A, verifying:

- $a \mapsto a^g$ is a morphism of algebras for all $g \in G$,
- $a^1 = a$ for all $a \in A$.

Given a map $f\colon G \times G \to A^*$ we consider the k-algebra (in general non associative and without 1) $A *_f G = \bigoplus_{\sigma \in G} A w_g$, whose multiplication is defined by

$$(a_1 w_{g_1})(a_2 w_{g_2}) = a_1 a_2^{g_1} f(g_1, g_2) w_{g_1 g_2}.$$

This algebra is called a crossed product if it is associative with w_1 as identity element. A well known result say that this happen if and only if

i) $(a^{g_2})^{g_1} f(g_1, g_2) = f(g_1, g_2) a^{g_1 g_2}$,

ii) $f(1, g) = f(g, 1) = 1$,

iii) $f(g_2, g_3)^{g_1} f(g_1, g_2 g_3) = f(g_1, g_2) f(g_1 g_2, g_3)$.

Condition *i*) above is called twisted module condition and conditions *ii*) and *iii*) asserts that f is a normalized 2-cocycle for the action of G on A^*.

In this section, we apply Corollary 1.3 in order to obtain a resolution of a crossed product $E = A *_f G$, considered as an E-bimódule, which is simpler than the canonical of Hochschild. Then an explicit expresion of the boundary maps of this resolution is given. To abbreviate the formulas we introduce some notations.

2.1.Notation. *Given a sequence $(g_0, \dots, g_s)$ of elements of G, $a \in A$ and indexes $-1 \le i < j \le s$, we write*

$$\mathbf{g}_s = g_0 \otimes \cdots \otimes g_s, \quad \mathfrak{g}_i^j = g_i \cdots g_j \quad \textit{and} \quad a^{\widehat{g}_j} = (\dots((a^{g_j})^{g_{j-1}})^{g_{j-2}} \dots)^{g_0}.$$

When the sequence is $(g_1, \dots, g_s)$ we put

$$\mathbf{g}_s = g_1 \otimes \cdots \otimes g_s \qquad \textit{and} \qquad a^{\widehat{g}_j} = (\dots((a^{g_j})^{g_{j-1}})^{g_{j-2}} \dots)^{g_1}.$$

2.2.Notation. *Given elements $a \in A$ and $b \in E$ invertible, we write $a^b = b^{-1} a b$ (note that $a^{w_g} = w_g^{-1} a w_g = f(g^{-1}, g)^{-1} a^{g^{-1}} f(g^{-1}, g)$ in general is different that $a^{g^{-1}}$). More generality, given $\mathbf{a} = a_1 \otimes \cdots \otimes a_r \in \overline{A}^r$ an $b \in E$ invertible, we put $\mathbf{a}^b = a_1^b \otimes \cdots \otimes a_r^b$.*

2.3.Remark. *Let*

$$X_{-1,s} = A \otimes (k[G] \otimes \overline{k[G]}^{s} \otimes k[G]) \qquad (s \geq 0),$$
$$X_{rs} = (A \otimes \overline{A}^{r} \otimes A) \otimes (k[G] \otimes \overline{k[G]}^{s} \otimes k[G]) \qquad (r, s \geq 0).$$

From now on we consider the groups $X_{-1,s}$ *($s \geq 0$) and* X_{rs} *($r, s \geq 0$) as E-bimodules via*

$$aw_g(\mathbf{a} \otimes \mathbf{g}) = \big(aa_0^g f(g, g_0) \otimes (a_1^g \otimes \cdots \otimes a_{r+1}^g)^{f(g,g_0)}\big) \otimes (gg_0 \otimes g_1 \otimes \cdots \otimes g_{s+1}),$$
$$(\mathbf{a} \otimes \mathbf{g})aw_g = \big(a_0 \otimes \cdots \otimes a_r \otimes a_{r+1} a^{\widehat{g}_{s+1}} f(g_{s+1}, g)^{\widehat{g}_s}\big) \otimes (g_0 \otimes \cdots \otimes g_s \otimes g_{s+1} g),$$

where $\mathbf{a} = a_0 \otimes \cdots \otimes a_{r+1}$ *and* $\mathbf{g} = \mathbf{g}_{s+1} = g_0 \otimes \cdots \otimes g_{s+1}$.

We define E-bimodule maps

$$\widetilde{\mu}\colon X_{-1,0} \to E, \quad d^0_{**}\colon X_{**} \to X_{*-1,*} \quad \text{and} \quad d^1_{-1,*}\colon X_{-1,*} \to X_{-1,*-1}$$

by:

$$\widetilde{\mu}(a \otimes (g_0 \otimes g_1)) = af(g_0, g_1) w_{g_0 g_1},$$
$$d^0_{rs}(\mathbf{a} \otimes \mathbf{g}) = \sum_{i=0}^{r} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{r+1}) \otimes (g_0 \otimes \cdots \otimes g_{s+1}),$$
$$d^1_{-1,s}(a \otimes \mathbf{g}) = \sum_{i=0}^{s} (-1)^{i+1} a f(g_i, g_{i+1})^{\widehat{g}_{i-1}} \otimes (g_0 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_{s+1})$$

and left E-module maps $\sigma^0_{**}\colon X_{**} \to X_{*+1,*}$, by:

$$\sigma^0_{rs}(\mathbf{a} \otimes \mathbf{g}) = (-1)^{r+1} (a_0 \otimes \cdots \otimes a_{r+1} \otimes 1) \otimes (g_0 \otimes \cdots \otimes g_{s+1}),$$

where $\mathbf{a} = a_0 \otimes \cdots \otimes a_{r+1}$ and $\mathbf{g} = \mathbf{g}_{s+1} = g_0 \otimes \cdots \otimes g_{s+1}$.

2.4.Theorem. *There is a relative projective resolution*

$$E \xleftarrow{\mu} X'_0 \xleftarrow{d'_1} X'_1 \xleftarrow{d'_2} X'_2 \xleftarrow{d'_3} X'_3 \xleftarrow{d'_4} X'_4 \xleftarrow{d'_5} X'_5 \xleftarrow{d'_6} X'_6 \xleftarrow{d'_7} \dots, \tag{2.4.1}$$

where $\mu((a_0 \otimes a_1) \otimes (g_0 \otimes g_1)) = a_0 a_1 f(g_0, g_1) w_{g_0 g_1}$,

$$X'_n = \bigoplus_{\substack{r+s=n \\ r \geq 0}} X_{rs} \qquad \text{and} \qquad d'_n = \sum_{\substack{r+s=n \\ r+l \geq 0}} \sum_{l=0}^{s} d^l_{rs},$$

withe the d^l_{rs}'s as in Definition 1.4.

Proof. It is easy to see that X_{**}, d^0_{**}, $d^1_{-1,*}$ and σ^0_{**} verify all the requested conditions at the beginning of the previous section, and that the complex

$$E \xleftarrow{\widetilde{\mu}} X_{-1,0} \xleftarrow{d^1_{-1,1}} X_{-1,1} \xleftarrow{d^1_{-1,2}} X_{-1,2} \xleftarrow{d^1_{-1,3}} X_{-1,3} \xleftarrow{d^1_{-1,4}} \dots,$$

is contractible as a complex of left E-modules, with a chain contracting homotopy $\sigma^{-1}_{-1,-1}\colon E \to X_{-1,0}$ and $\sigma^{-1}_{-1,s}\colon X_{-1,s} \to X_{-1,s+1}$ $(s \ge 0)$, given by

$$\begin{aligned} &\sigma^{-1}_{-1,-1}(aw_g) = a \otimes (g \otimes 1) \qquad \text{and} \\ &\sigma^{-1}_{-1,s}\big(a \otimes (g_0 \otimes \dots \otimes g_{s+1})\big) = (-1)^s a \otimes (g_0 \otimes \dots \otimes g_{s+1} \otimes 1). \end{aligned}$$

Hence, the desired result follows immediately from Corollario 1.3 $\square$

The boundary maps d'_*, of the relative projective resolution of E that we had just found are defined recursively. Our next objetive is to compute these morphisms. We accomplish this in Theorem 2.9.

2.5.Notation. *Given $a_0,\dots,a_{r+1} \in A$ and $b_0,\dots,b_{s+1} \in E$ invertibles, we write*

$$\begin{aligned} &(a_0 \otimes \dots \otimes a_{r+1}) * (b_0 \otimes \dots \otimes b_{s+1}) = \sum_{0 \le i_1 \le \dots \le i_r \le s} (-1)^{i_1+\dots+i_r} \times \\ &\times a_0 b_0 \otimes b_1 \otimes \dots \otimes b_{i_1} \otimes a_1^{\mathfrak{b}_0^{i_1}} \otimes b_{i_1+1} \otimes \dots \otimes b_{i_r} \otimes a_r^{\mathfrak{b}_0^{i_r}} \otimes b_{i_r+1} \otimes \dots \otimes a_{r+1}^{\mathfrak{b}_0^r} b_{s+1}, \end{aligned}$$

where $\mathfrak{b}_0^{i_j} = b_0 \cdots b_{i_j}$ and $a_j^{\mathfrak{b}_0^{i_j}} = (\mathfrak{b}_0^{i_j})^{-1} a_j \mathfrak{b}_0^{i_j}$ for $1 \le j \le r$. Moreover, for each $\mathbf{a} = a_0 \otimes \dots \otimes a_{r+1} \in A^{r+2}$, $\mathbf{b} = b_0 \otimes \dots \otimes b_{s+1} \in A^{s+2}$ with the $b_i{}'s$ invertibles and $\mathbf{g} = \mathbf{g}_{s+1} = g_0 \otimes \dots \otimes g_{s+1}$, we write

$$\mathbf{a} * (\mathbf{b} \otimes \mathbf{g}) = (\mathbf{a} * \mathbf{b}) \otimes \mathbf{g}.$$

It is easy to see that this "product" is associative and verifies

$$\begin{aligned} &(a_0 \otimes \dots \otimes a_{r+1}) * (b_0 \otimes \dots \otimes b_{s+1}) * (c_0 \otimes \dots \otimes c_r \otimes 1) \\ &\qquad = (a_0 \otimes \dots \otimes a_r \otimes 1) * (b_0 \otimes \dots \otimes b_s \otimes 1) * \left(c_0 \otimes \dots \otimes a_r \otimes a_{r+1}^{(\mathfrak{b}_0^r \mathfrak{c}_0^s)} b_{s+1}^{\mathfrak{c}_0^r}\right). \end{aligned}$$

Note that if the $b_i{}'s$ are central, then $*$ is the standard shuffle product.

2.6.Definition. *Given* $\mathbf{g} = \mathbf{g}_s = g_0 \otimes \cdots \otimes g_s$ *we define* $f^{(l)}(\mathbf{g})$ *(*$2 \le l \le s$*) recursively by:*

$$f^{(2)}(\mathbf{g}) = -f(g_{s-1}, g_s)^{\widehat{g}_{s-2}},$$
$$f^{(l+1)}(\mathbf{g}) = -\sum_{i=s-l}^{s-1} (-1)^i f(g_i, g_{i+1})^{\widehat{g}_{i-1}} \otimes f^{(l)}(g_0 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_s).$$

2.7.Lemma. *For each* $j < s - i$ *it holds that*

$$(-1)^i f^{(i)}(g_0 \otimes \cdots \otimes g_s)^{f(g_j, g_{j+1})^{\widehat{g}_{j-1}}} = f^{(i)}(g_0 \otimes \cdots \otimes g_j g_{j+1} \otimes \cdots \otimes g_s)$$

Proof. For $i = 2$ it is immediate. Suppose that $i > 2$ and that the result is valid for $i - 1$. Let

$$\mathbf{g}_s^j = g_0 \otimes \cdots \otimes g_j g_{j+1} \otimes \cdots \otimes g_s \qquad \text{for } 0 \le j < s,$$
$$\mathbf{g}_s^{j,h} = g_0 \otimes \cdots \otimes g_j g_{j+1} \otimes \cdots \otimes g_h g_{h+1} \otimes \cdots \otimes g_s \qquad \text{for } 0 \le j < h-1 < s-1.$$

Since $j < s - i$, the twisted module condition implies that

$$\begin{aligned} f^{(i)}(\mathbf{g}_s^j) &= \sum_{h=s-i}^{s-2} (-1)^{h+1} \big(f(g_{h+1}, g_{h+2})^{\widehat{g}_h}\big)^{f(g_j, g_{j+1})^{\widehat{g}_{j-1}}} \otimes f^{(i-1)}(\mathbf{g}_s^{j,h+1}) \\ &= \sum_{h=s-i+1}^{s-1} (-1)^h \big(f(g_h, g_{h+1})^{\widehat{g}_{h-1}}\big)^{f(g_j, g_{j+1})^{\widehat{g}_{j-1}}} \otimes f^{(i-1)}(\mathbf{g}_s^{j,h}). \end{aligned}$$

Consequently, because of the inductive hypothesis

$$\begin{aligned} f^{(i)}(\mathbf{g}_s^j) &= (-1)^i \sum_{h=s-i+1}^{s-1} (-1)^{h+1} \big(f(g_h, g_{h+1})^{\widehat{g}_{h-1}} \otimes f^{(i-1)}(\mathbf{g}_s^h)\big)^{f(g_j, g_{j+1})^{\widehat{g}_{j-1}}} \\ &= (-1)^i f^{(i)}(g_0 \otimes \cdots \otimes g_s)^{f(g_j, g_{j+1})^{\widehat{g}_{j-1}}} \qquad \square \end{aligned}$$

2.8.Lemma. *For each elementary tensor* $\mathbf{g} = \mathbf{g}_s = g_0 \otimes \cdots \otimes g_s$ *and each* $1 \le l \le s$*, we have*

$$\begin{aligned} &(-1)^l \otimes f^{(l+1)}(\mathbf{g}) \\ &\quad = \sum_{i=1}^{l} (-1)^i \big(1 \otimes f^{(i)}(\mathbf{g}) \otimes 1\big) * \big(1 \otimes f^{(l-i+1)}(\mathbf{g}_{s-i}) \otimes f(\mathfrak{g}_{s-l}^{s-i}, \mathfrak{g}_{s-i+1}^{s})^{\widehat{g}_{s-l-1}}\big), \end{aligned}$$

where $1\otimes f^{(1)}(\mathbf{g}_h)\otimes 1$ *denotes to* $(-1)^h\otimes 1$.

Proof. To abbreviate we write

$$\mathfrak{f}_j = f(g_j, g_{j+1})^{\widehat{g}_{j-1}} \qquad \text{and} \qquad \mathfrak{f}(h) = f(\mathfrak{g}_{s-l}^{s-h}, \mathfrak{g}_{s-h+1}^{s})^{\widehat{g}_{s-l-1}}.$$

It is clear that the lemma is valid for $l = 1$. Let $l > 1$ and suppose that the result is valid for $l-1$. With the notations introduced in the proof of Lemma 2.7 we have

$$\begin{aligned}
(-1)^l f^{(l+1)}(\mathbf{g}_s) &= \sum_{j=s-l}^{s-1} (-1)^{l+j+1}\mathfrak{f}_j \otimes f^{(l)}(\mathbf{g}_s^j)\\
&= \sum_{j=s-l}^{s-2}\sum_{i=1}^{s-j-1} (-1)^{j+i}\Big(\mathfrak{f}_j\otimes f^{(i)}(\mathbf{g}_s^j)\otimes 1\Big) * \Big(1\otimes f^{(l-i)}(\mathbf{g}_{s-i}^j)\otimes \mathfrak{f}(i)\Big)\\
&+ \sum_{j=s-l+1}^{s-1}\sum_{i=s-j}^{l-1} (-1)^{j+i}\Big(\mathfrak{f}_j\otimes f^{(i)}(\mathbf{g}_s^j)\otimes 1\Big) * \Big(1\otimes f^{(l-i)}(\mathbf{g}_{s-i-1})\otimes \mathfrak{f}(i+1)\Big)\\
&= \sum_{i=1}^{l-1}\sum_{j=s-l}^{s-i-1} (-1)^{j}\Big(1\otimes f^{(i)}(\mathbf{g}_s)\otimes 1\Big) * \Big(\mathfrak{f}_j\otimes f^{(l-i)}(\mathbf{g}_{s-i}^j)\otimes \mathfrak{f}(i)\Big)\\
&+ \sum_{i=1}^{l-1}\sum_{j=s-i}^{s-1} (-1)^{i+j}\Big(\mathfrak{f}_j\otimes f^{(i)}(\mathbf{g}_s^j)\otimes 1\Big) * \Big(1\otimes f^{(l-i)}(\mathbf{g}_{s-i-1})\otimes \mathfrak{f}(i+1)\Big)\\
&= -\sum_{i=1}^{l-1}\Big(1\otimes f^{(i)}(\mathbf{g}_s)\otimes 1\Big) * \Big(f^{(l-i+1)}(\mathbf{g}_{s-i})\otimes \mathfrak{f}(i)\Big)\\
&+ \sum_{i=2}^{l}\Big((-1)^i f^{(i)}(\mathbf{g}_s)\otimes 1\Big) * \Big(1\otimes f^{(l-i+1)}(\mathbf{g}_{s-i})\otimes \mathfrak{f}(i)\Big),
\end{aligned}$$

where the third equality follows from Lemma 2.7. It is easy to see that this fact implies the lemma for $f^{(l+1)}$ $\square$

2.9.Theorem. *We have:*

$$\begin{aligned}
d_{rs}^1(\mathbf{a}\otimes\mathbf{g}) = &\sum_{i=0}^{s-1}(-1)^{i+r}\mathbf{a} * (f(g_i,g_{i+1})^{\widehat{g}_{i-1}}\otimes 1)\otimes(g_0\otimes\cdots\otimes g_ig_{i+1}\otimes\cdots\otimes g_{s+1})\\
&+ (-1)^{r+s}\mathbf{a} * \Big(1\otimes f(g_s,g_{s+1})^{\widehat{g}_{s-1}}\Big)\otimes(g_0\otimes\cdots\otimes g_{s-1}\otimes g_sg_{s+1})
\end{aligned}$$

and

$$d^l_{rs}\big(\mathbf{a}\otimes\mathbf{g}\big) = (-1)^r\mathbf{a} * \big(1\otimes f^{(l)}(\mathbf{g}_s)\otimes f(\mathfrak{g}^s_{s-l+1}, g_{s+1})^{\widehat{g}_{s-l}}\big)\otimes(\mathbf{g}_{s-l}\otimes\mathfrak{g}^{s+1}_{s-l+1}),$$

where $2\le l\le s$, $\mathbf{a} = a_0\otimes\cdots\otimes a_{r+1}$ *and* $\mathbf{g} = \mathbf{g}_{s+1} = g_0\otimes\cdots\otimes g_{s+1}$.

Proof. Without loss of generality we can assume that $a_{r+1} = 1$ and $g_{s+1} = 1$. First we suppose that the formula is valid for d^j_{rs} with $j\le l$ and we see that it is valid for d^{l+1}_{0s}. Let

$$\mathfrak{f}(i) = f(\mathfrak{g}^{s-i}_{s-l}, \mathfrak{g}^s_{s-i+1})^{\widehat{g}_{s-l-1}} \qquad\text{and}\qquad \mathbf{g}(h) = (\mathbf{g}_{s-h-1}\otimes\mathfrak{g}^s_{s-h}).$$

From the inductive hypothesis we obtain:

$$\begin{aligned}
d^{l+1}\big((1\otimes 1)\otimes\mathbf{g}\big) &= -\sum_{i=1}^{l}\sigma^0\, d^{l+1-i}\, d^i\big((1\otimes 1)\otimes\mathbf{g}\big)\\
&= -\sum_{i=1}^{l}\sigma^0\, d^{l+1-i}\Big(\big(1\otimes f^{(i)}(\mathbf{g}_s)\otimes 1\big)\otimes(\mathbf{g}_{s-i}\otimes\mathfrak{g}^s_{s-i+1})\Big)\\
&= \sum_{i=1}^{l}(-1)^i\sigma^0\Big(\big(1\otimes f^{(i)}(\mathbf{g}_s)\otimes 1\big) * \big(1\otimes f^{(l-i+1)}(\mathbf{g}_{s-i})\otimes\mathfrak{f}(i)\big)\otimes\mathbf{g}(l)\Big),
\end{aligned}$$

where $1\otimes f^{(1)}(\mathbf{g}_h)\otimes 1$ denotes $(-1)^h\otimes 1$. Because of Lemma 2.8 the last expression is equal to

$$\big(1\otimes f^{(l+1)}(\mathbf{g}_s)\otimes 1\big)\otimes(\mathbf{g}_{s-l-1}\otimes\mathfrak{g}^{s+1}_{s-l}).$$

Now we suppose that the result is valid for $d^{l+1}_{r's}$ with $r' < r$ and we show that it is valid for d^{l+1}_{rs}. We write $\mathbf{a}' = a_0\otimes\cdots\otimes a_{r-1}\otimes 1$ and $\mathbf{a}'a_r = a_0\otimes\cdots\otimes a_r$. Since for $0\le i < r$,

$$\sigma_0\, d^{l+1}\big((a_0\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_r\otimes 1)\otimes\mathbf{g}\big) = 0,$$

we have

$$\begin{aligned}
d^{l+1}\big(\mathbf{a}\otimes\mathbf{g}\big) &= -\sum_{i=0}^{l}\sigma^0\, d^{l+1-i}\, d^i\big(\mathbf{a}\otimes\mathbf{g}\big)\\
&= (-1)^{r+1}\sigma^0\, d^{l+1}\big(\mathbf{a}'a_r\otimes\mathbf{g}\big) - (-1)^r\sum_{i=1}^{l}\sigma^0\, d^{l+1-i}\Big(\mathbf{a} * d_i\big((1\otimes 1)\otimes\mathbf{g}\big)\Big).
\end{aligned}$$

Hence, by the inductive hypothesis and Lemma 2.8, we obtain

$$
\begin{aligned}
& d^{l+1}\big(\mathbf{a}\otimes\mathbf{g}\big) = (-1)^{r+1}\sigma^0\, d^{l+1}\big(\mathbf{a}' a_r\otimes\mathbf{g}\big)\\
& - \sum_{i=1}^{l}(-1)^r\sigma^0\, d^{l+1-i}\Big(\mathbf{a}*\big(1\otimes f^{(i)}(\mathbf{g}_s)\otimes 1\big)\otimes(\mathbf{g}_{s-i}\otimes\mathfrak{g}_{s-i}^{s})\Big)\\
& = \sigma^0\Big(\mathbf{a}' a_r * d^{l+1}\big((1\otimes 1)\otimes\mathbf{g}\big)\Big)\\
& + \sum_{i=1}^{l}(-1)^i\sigma^0\Big(\mathbf{a}*\big(1\otimes f^{(i)}(\mathbf{g}_s)\otimes 1\big)*d^{l+1-i}\big((1\otimes 1)\otimes(\mathbf{g}_{s-i}\otimes\mathfrak{g}_{s-i}^{s})\big)\Big)\\
& = \sigma^0\Big(\mathbf{a}' a_r * d^{l+1}\big((1\otimes 1)\otimes\mathbf{g}\big)\Big)\\
& + \sum_{i=1}^{l}\sigma^0\Big(\mathbf{a}*\big((-1)^i\otimes f^{(i)}(\mathbf{g}_s)\otimes 1\big)*\big(1\otimes f^{(l+1-i)}(\mathbf{g}_{s-i})\otimes\mathfrak{f}(i)\big)\otimes\mathbf{g}(l)\\
& = \sigma^0\Big(\mathbf{a}' a_r * d^{l+1}\big((1\otimes 1)\otimes\mathbf{g}\big)\Big) + (-1)^l\sigma^0\Big(\mathbf{a}*\big(1\otimes f^{(l+1)}(\mathbf{g}_s)\big)\Big)\otimes\mathbf{g}(l)\\
& = (-1)^r\mathbf{a}*d_{0s}^{l}\big((1\otimes 1)\otimes\mathbf{g}\big) \quad \square
\end{aligned}
$$

3. The Hochschild (co)homology of a crossed product

3.1. Hochschild homology. Let $E = A *_f G$ and M an E-bimodule. We use Theorem 2.4 in order to construct a complex simpler than the canonical one giving the Hochschild homology of E with coefficients in M. This complex has a natural filtration. When A is a group algebra, the spectral sequence of this filtration is the Hochschild-Serre spectral sequence.

3.1.1. Notation. *Given $a_1,\dots,a_r\in A$ and $b_1,\dots,b_l\in E$ invertibles, we put*

$$
\begin{aligned}
(a_1\otimes\cdots\otimes a_r)*(b_1\otimes\cdots\otimes b_l) = & \sum_{0\le i_1\le\cdots\le i_r\le l}(-1)^{i_1+\cdots+i_l}\times\\
& \times b_1\otimes\cdots\otimes b_{i_1}\otimes a_1^{\mathfrak{b}_1^{i_1}}\otimes b_{i_1+1}\otimes\cdots\otimes b_{i_r}\otimes a_r^{\mathfrak{b}_1^{i_r}}\otimes b_{i_r+1}\otimes\cdots,
\end{aligned}
$$

where $\mathfrak{b}_1^{i_j} = b_1\cdots b_{i_j}$ and $a_j^{\mathfrak{b}_1^{i_j}} = (\mathfrak{b}_1^{i_j})^{-1}a_j\mathfrak{b}_1^{i_j}$ for $1\le j\le r$. Note that if the $b_i{}'s$ are central, then $*$ is the standard shuffle product.

Let

$$\overline{d}_{rs}^{l}\colon M\otimes\overline{A}^{r}\otimes\overline{k[G]}^{s}\to M\otimes\overline{A}^{r+l-1}\otimes\overline{k[G]}^{s-l}\quad (r,s\ge 0,\ 0\le l\le s \text{ and } r+l>0)$$

be the morphisms defined by

$$\begin{aligned}\overline{d}^0_{rs}(m\otimes \mathbf{a}\otimes \mathbf{g}) &= ma_1\otimes(a_2\otimes\cdots\otimes a_r)\otimes(g_1\otimes\cdots\otimes g_s)\\ &+\sum_{i=1}^{r-1}(-1)^i m\otimes(a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_r)\otimes(g_1\otimes\cdots\otimes g_s)\\ &+(-1)^r a_r m\otimes(a_1\otimes\cdots\otimes a_{r-1})\otimes(g_1\otimes\cdots\otimes g_s),\\ \overline{d}^1_{rs}(m\otimes \mathbf{a}\otimes \mathbf{g}) &= (-1)^r\Big(w_{g_1}^{-1} m w_{g_1}\otimes\big(a_1^{w_{g_1}}\otimes\cdots\otimes a_r^{w_{g_1}}\big)\otimes(g_2\otimes\cdots\otimes g_s)\\ &+\sum_{i=1}^{s-1}(-1)^i (m\otimes\mathbf{a})^{f(g_i,g_{i+1})^{\widehat{g}_{i-1}}}\otimes(g_1\otimes\cdots\otimes g_ig_{i+1}\otimes\cdots\otimes g_s)\\ &+(-1)^s m\otimes(a_1\otimes\cdots\otimes a_r)\otimes(g_1\otimes\cdots\otimes g_{s-1})\Big),\\ \overline{d}^l_{rs}(m\otimes \mathbf{a}\otimes \mathbf{g}) &= (-1)^r f(g_{s-1},g_s)^{\widehat{g}_{s-2}}\cdots f(g_{s-l+1},\mathfrak{g}^s_{s-l+2})^{\widehat{g}_{s-l}} m\\ &\otimes\big(a_1\otimes\cdots\otimes a_r\big)*f^{(l)}(\mathbf{g}_s)\otimes(g_1\otimes\cdots\otimes g_{s-l}),\end{aligned}$$

where

$$\mathbf{a}=a_1\otimes\cdots\otimes a_r,\qquad \mathbf{g}=g_1\otimes\cdots\otimes g_s\qquad\text{and}$$

$$(m\otimes\mathbf{a})^{f(g_i,g_{i+1})^{\widehat{g}_{i-1}}}=\big(f(g_i,g_{i+1})^{\widehat{g}_{i-1}}\big)^{-1} m f(g_i,g_{i+1})^{\widehat{g}_{i-1}}\otimes \mathbf{a}^{f(g_i,g_{i+1})^{\widehat{g}_{i-1}}}.$$

By tensoring M on the left with the complex (X'_*, d'_*) of Theorem 2.4 over E^e, and using the identifications

$$\theta_{rs}\colon M\otimes\overline{A}^r\otimes\overline{k[G]}^s\longrightarrow M\otimes_{E^e}(A\otimes\overline{A}^r\otimes A)\otimes(k[G]\otimes\overline{k[G]}^s\otimes k[G]),$$

given by

$$\begin{aligned}&\theta_{rs}\big(m\otimes(a_1\otimes\cdots\otimes a_r)\otimes(g_1\otimes\cdots\otimes g_s)\big)\\ &\qquad = w_{g_s}^{-1}\cdots w_{g_1}^{-1} m\otimes(1\otimes a_1\otimes\cdots\otimes a_r\otimes 1)\otimes(1\otimes g_1\otimes\cdots\otimes g_s\otimes 1),\end{aligned}$$

we obtain the complex

$$\overline{X}_*(E,M)=\quad \overline{X}_0\xleftarrow{\overline{d}_1}\overline{X}_1\xleftarrow{\overline{d}_2}\overline{X}_2\xleftarrow{\overline{d}_3}\overline{X}_3\xleftarrow{\overline{d}_4}\overline{X}_4\xleftarrow{\overline{d}_5}\overline{X}_5\xleftarrow{\overline{d}_6}\overline{X}_6\xleftarrow{\overline{d}_7}\dots,$$

where

$$\overline{X}_n=\bigoplus_{\substack{r+s=n\\ r\ge 0}} M\otimes\overline{A}^r\otimes\overline{k[G]}^s\qquad\text{and}\qquad \overline{d}_n=\sum_{\substack{r+s=n\\ r+l\ge 0}}\sum_{l=0}^{s}\overline{d}^l_{rs}.$$

Note that when f take its values in k, then $\overline{X}_*(E,M)$ is the total complex of the double complex $\big(\overline{X}_{**},\overline{d}^0_{**},\overline{d}^1_{**}\big)$, where $\overline{X}_{rs}=M\otimes\overline{A}^r\otimes\overline{k[G]}^s$.

3.1.2.Theorem. *The Hochschild homology of E with coefficients in M is the homology of* $\overline{X}_*(E,M)$.

Proof. It is an immediate consequence of the above discussion $\square$

3.1.3.Lemma. *Let $c \in A$ invertible. The morphism of chain complexes*

$$\vartheta_*^c \colon (M \otimes \overline{A}^*, b_*) \longrightarrow (M \otimes \overline{A}^*, b_*),$$

defined by:

$$\vartheta_r^c(m \otimes a_1 \otimes \cdots \otimes a_r) = c^{-1}mc \otimes c^{-1}a_1c \otimes \cdots \otimes c^{-1}a_rc.$$

induce the identity map on $\mathrm{H}_*(A,M)$.

Proof. By a standard argument it is sufficient to prove it for $\mathrm{H}_0(A,M)$, and in this case the result it is immediate $\square$

For each $g \in G$ we have the morphism

$$\theta_*^g \colon (M \otimes \overline{A}^*, b_*) \longrightarrow (M \otimes \overline{A}^*, b_*),$$

defined by:

$$\theta_r^g(m \otimes a_1 \otimes \cdots \otimes a_r) = w_g^{-1}mw_g \otimes a_1^{w_g} \otimes \cdots \otimes a_r^{w_g}.$$

3.1.4.Proposition. *Let $g, g' \in G$. The endomorphism of $\mathrm{H}_*(A,M)$ induced by $\theta_*^{g'}\,\theta_*^g$ and by $\theta_*^{gg'}$ coincide. Consequently $\mathrm{H}_*(A,M)$ is a right G-module.*

Proof. It is sufficient to see that

$$(\theta_*^{gg'})^{-1}\,\theta_*^{g'}\,\theta_*^g$$

induce the identity map on $\mathrm{H}_*(A,M)$, which is an immediate consequence of Lemma 3.1.3 $\square$

The chain complex $\overline{X}_*(E,M)$ has a filtration $F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$, where

$$F_i(\overline{X}_n) = \bigoplus_{\substack{r+s=n \\ r\ge 0,\, 0\le s\le i}} M \otimes \overline{A}^r \otimes \overline{k[G]}^s$$

Using this fact, Lemma 3.1.3 and Proposition 3.1.4 we obtain the following

3.1.5.Corollary. *There is a converging spectral sequence*

$$E^2_{rs} = \mathrm{H}_s(G, \mathrm{H}_r(A, M)) \Rightarrow \mathrm{H}_{r+s}(E, M)$$

Given an A-bimodule M we write $[A, M]$ to denote the k-submodule of M generated by the commutators $am - ma$ where $a \in A$ and $m \in M$, and given a G-module N we write N_G to denote the coinvariants.

3.1.6.Corollary. *If A is separable, then $\mathrm{H}_*\big(E, M\big)$ equals to $\mathrm{H}_*\big(G, \frac{M}{[A,M]}\big)$.*

3.1.7.Corollary. *If A is quasi-free, then there is a long exact sequence*

$$\begin{aligned} \dots \longrightarrow \mathrm{H}_{n+1}(G, \mathrm{H}_0(A, M)) \longrightarrow \mathrm{H}_{n-1}(G, \mathrm{H}_1(A, M)) \longrightarrow \mathrm{H}_n(E, M) \longrightarrow \\ \mathrm{H}_n(G, \mathrm{H}_0(A, M)) \longrightarrow \mathrm{H}_{n-2}(G, \mathrm{H}_1(A, M)) \longrightarrow \mathrm{H}_{n-1}(E, M) \longrightarrow \dots. \end{aligned}$$

3.1.8.Corollary. *If G is a finite group and the order of G is invertible in k, then $\mathrm{H}_*\big(E, M\big)$ equals to $\mathrm{H}_*(A, M)_G$.*

3.1.9.Corollary. *If G is a finite group, the order of G is invertible in k, A is a smooth commutative k-algebra and M is a E-bimodule verifying $ma = am$ for all $a \in A$, then $\mathrm{H}_*\big(E, M\big)$ equals to $\big(M \otimes_A \Omega^*_{A/k}\big)_G$, where the action of G on $M \otimes_A \Omega^*_{A/k}$, is given by*

$$(m da_1 \wedge \dots \wedge da_n) \cdot g = w_g^{-1} m w_g d(a_1^{w_g}) \wedge \dots \wedge d(a_n^{w_g}).$$

Let S be a separable subalgebra of A. Next we prove that if the 2-cocycle f take its values in S, then the Hochschild homology of E with coefficients in M is the homology of G with coefficients in a chain complex. When S equals A we recover Corollary 3.1.6.

Assume that $f(g, g') \in S$ for all $g, g' \in G$. Let

$$\widetilde{A} = A/S, \qquad \widetilde{A}^0 = S, \qquad \widetilde{A}^r = \widetilde{A} \otimes_S \dots \otimes_S \widetilde{A} \quad (r\text{-times})$$

for $r > 0$ and let $M \otimes_S \widetilde{A}^r \otimes_S = M \otimes_S \widetilde{A}^r \otimes_{S^e} S$ be the cyclic tensor product over S of M and $\widetilde{A}^r$ (see [C] or [Q]). Using that f take its values on S it is easy to see that G acts on $(M \otimes_S \widetilde{A}^r \otimes S, b_*)$ via

$$\big(m \otimes_S a_1 \otimes_S \dots \otimes_S a_r \otimes_S\big) \cdot g = w_g^{-1} m w_g \otimes_S a_1^{w_g} \otimes_S \dots \otimes_S a_r^{w_g} \otimes_S .$$

3.1.10.Theorem. *The Hochschild homology* $\mathrm{H}_*(E,M)$ *of* E *with coefficients in* M *is the homology of* G *with coefficients in* $(M\otimes_S \widetilde{A}^r\otimes S, b_*)$.

Proof. It is sufficient to prove that $\mathrm{H}_*(E,M)$ is the homology of the total complex $\overline{X}^S_*(E,M)$ of the double complex $(\widetilde{X}_{**},\widetilde{d}^0_{**},\widetilde{d}^1_{**})$, with objects

$$\widetilde{X}_{**}=(M\otimes_S \widetilde{A}^r\otimes_S)\otimes\overline{k[G]}^s,$$

horizontal differentials

$$\begin{aligned}\widetilde{d}^0_{rs}(m\otimes_S \mathbf{a}\otimes\mathbf{g}) &= (ma_1\otimes_S a_2\otimes_S\cdots\otimes_S a_r\otimes_S)\otimes(g_1\otimes\cdots\otimes g_s)\\ &+\sum_{i=1}^{r-1}(-1)^i(m\otimes_S a_1\otimes_S\cdots\otimes_S a_ia_{i+1}\otimes_S\cdots\otimes_S a_r\otimes_S)\otimes(g_1\otimes\cdots\otimes g_s)\\ &+(-1)^r(a_rm\otimes_S a_1\otimes_S\cdots\otimes_S a_{r-1}\otimes_S)\otimes(g_1\otimes\cdots\otimes g_s),\end{aligned}$$

and vertical differentials

$$\begin{aligned}\widetilde{d}^1_{rs}(m\otimes_S \mathbf{a}\otimes\mathbf{g}) &= (-1)^r\Big(w_{g_1}^{-1}mw_{g_1}\otimes_S a_1^{w_{g_1}}\otimes_S\cdots\otimes a_r^{w_{g_1}}\otimes_S\Big)\otimes(g_2\otimes\cdots\otimes g_s)\\ &+\sum_{i=1}^{s-1}(-1)^i(m\otimes_S \mathbf{a})\otimes(g_1\otimes\cdots\otimes g_ig_{i+1}\otimes\cdots\otimes g_s)\\ &+(-1)^s(m\otimes_S a_1\otimes_S\cdots\otimes_S a_r\otimes_S)\otimes(g_1\otimes\cdots\otimes g_{s-1})\Big),\end{aligned}$$

where $\mathbf{a}=a_1\otimes_S\cdots\otimes_S a_r\otimes_S$ and $\mathbf{g}=g_1\otimes\cdots\otimes g_s$.

Let $\pi_*\colon \overline{X}_*(E,M)\to\overline{X}^S_*(E,M)$ be the map

$$m\otimes(a_1\otimes\cdots\otimes a_r)\otimes(g_1\otimes\cdots\otimes g_s)\mapsto(m\otimes_S a_1\otimes_S\cdots\otimes_S a_r\otimes_S)\otimes(g_1\otimes\cdots\otimes g_s).$$

Consider the filtration $F^S_0\subseteq F^S_1\subseteq F^S_2\subseteq\dots$ of $\overline{X}^S_*(E,M)$, given by:

$$F^S_i\left(\bigoplus_{\substack{r+s=n\\ r,s\ge 0}}\widetilde{X}_{rs}\right)=\bigoplus_{\substack{r+s=n\\ r\ge 0,\,0\le s\le i}}(M\otimes_S \widetilde{A}^r\otimes_S)\otimes\overline{k[G]}^s.$$

From Lemma 2.3 of [C] follows that π_* is a morphism of filtrated complexes that induces an isomorphism between the graded complexes associated to the filtrations of $\overline{X}_*(E,M)$ and $\overline{X}^S_*(E,M)$. Consequently, π_* is a quasi-isomorphism. The proof can be finish by applying Theorem 3.1.2 □

3.2.Hochschild cohomology. Let $E = A *_f G$ and M an E-bimodule. Using again Theorem 2.4 we construct a complex simpler than the canonical giving the Hochschild cohomology of E with coefficients in M. This complex is provided with a natural filtration. When A is a group algebra the spectral sequence of this filtration is the Hochschild-Serre spectral sequence.

Let

$$\overline{d}_l^{rs}\colon \operatorname{Hom}_k\bigl(\overline{A}^{r+l-1}\otimes\overline{k[G]}^{s-l}, M\bigr) \to \operatorname{Hom}_k\bigl(\overline{A}^{r}\otimes\overline{k[G]}^{s}, M\bigr) \quad (0\le l\le s,\ r+l>0)$$

be the morphisms defined by

$$\begin{aligned}\overline{d}_0^{rs}(\varphi)(\mathbf{a}\otimes\mathbf{g}) &= a_1\varphi\bigl((a_2\otimes\cdots\otimes a_r)\otimes(g_1\otimes\cdots\otimes g_s)\bigr)\\ &\quad + \sum_{i=1}^{r-1}(-1)^i\varphi\bigl((a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_r)\otimes(g_1\otimes\cdots\otimes g_s)\bigr)\\ &\quad + (-1)^r\varphi\bigl((a_1\otimes\cdots\otimes a_{r-1})\otimes(g_1\otimes\cdots\otimes g_s)\bigr)a_r,\end{aligned}$$

$$\begin{aligned}\overline{d}_1^{rs}(\varphi)(\mathbf{a}\otimes\mathbf{g}) &= (-1)^r\Bigl(w_{g_1}\varphi\bigl((a_1^{w_{g_1}}\otimes\cdots\otimes a_r^{w_{g_1}})\otimes(g_2\otimes\cdots\otimes g_s)\bigr)w_{g_1}^{-1}\\ &\quad + \sum_{i=1}^{s-1}(-1)^i\varphi\bigl(\mathbf{a}^{f(g_i,g_{i+1})^{\widehat{g}_{i-1}}}\otimes(g_1\otimes\cdots\otimes g_ig_{i+1}\otimes\cdots\otimes g_s)\bigr)^{f(g_i,g_{i+1})^{\widehat{g}_{i-1}}}\\ &\quad + (-1)^s\varphi\bigl((a_1\otimes\cdots\otimes a_r)\otimes(g_1\otimes\cdots\otimes g_{s-1})\bigr)\Bigr),\end{aligned}$$

$$\begin{aligned}\overline{d}_l^{rs}(\varphi)(\mathbf{a}\otimes\mathbf{g}) &= (-1)^r\varphi\Bigl(\mathbf{a}*f^{(l)}(\mathbf{g}_s)\otimes(g_1\otimes\cdots\otimes g_{s-l})\Bigr)\times\\ &\quad\times f(g_{s-l+1},\mathbf{g}_{s-l+2}^s)^{\widehat{g}_{s-l}}f(g_{s-l+2},\mathbf{g}_{s-l+2}^s)^{\widehat{g}_{s-l+1}}\cdots f(g_{s-1},g_s)^{\widehat{g}_{s-1}},\end{aligned}$$

where $\mathbf{a} = a_1\otimes\cdots\otimes a_r$ and $\mathbf{g} = g_1\otimes\cdots\otimes g_s$. Applying the functor $\operatorname{Hom}_{E^e}(-,M)$ to the complex (X'_*, d'_*) of Theorem 2.4 and using the identifications

$$\theta^{rs}\colon \operatorname{Hom}_k\bigl(\overline{A}^{r}\otimes\overline{k[G]}^{s}, M\bigr) \longrightarrow \operatorname{Hom}_{E^e}\bigl((A\otimes\overline{A}^{r}\otimes A)\otimes(k[G]\otimes\overline{k[G]}^{s}\otimes k[G]), M\bigr),$$

given by

$$\begin{aligned}&\theta^{rs}(\varphi)\bigl((1\otimes a_1\otimes\cdots\otimes a_r\otimes 1)\otimes(1\otimes g_1\otimes\cdots\otimes g_s\otimes 1)\bigr)\\ &\qquad = \varphi\bigl((a_1\otimes\cdots\otimes a_r)\otimes(g_1\otimes\cdots\otimes g_s)\bigr)w_{g_1}\cdots w_{g_s},\end{aligned}$$

we obtain the complex

$$\overline{X}^*(E,M) = \quad \overline{X}^0 \xrightarrow{\overline{d}^1} \overline{X}^1 \xrightarrow{\overline{d}^2} \overline{X}^2 \xrightarrow{\overline{d}^3} \overline{X}^3 \xrightarrow{\overline{d}^4} \overline{X}^4 \xrightarrow{\overline{d}_5} \overline{X}^5 \xrightarrow{\overline{d}^6} \overline{X}^6 \xrightarrow{\overline{d}^7} \dots,$$

where

$$\overline{X}^n = \bigoplus_{\substack{r+s=n\\ r\geq 0}} \operatorname{Hom}_k\big(\overline{A}^r \otimes \overline{k[G]}^s, M\big) \qquad \text{and} \qquad \overline{d}^n = \sum_{\substack{r+s=n\\ r+l\geq 0}} \sum_{l=0}^{s} \overline{d}_l^{rs}.$$

Note that when f take its values in k, then $\overline{X}^*(E,M)$ is the total complex of the double complex $\big(\overline{X}^{**}, \overline{d}_0^{**}, \overline{d}_1^{**}\big)$, where $\overline{X}^{rs} = \operatorname{Hom}_k\big(\overline{A}^r \otimes \overline{k[G]}^s, M\big)$.

3.2.1.Theorem. *The Hochschild cohomology of E with coefficients in M is the homology of $\overline{X}^*(E,M)$.*

Proof. It is an immediate consequence of the above discussion □

3.2.2.Lemma. *Let $c \in A$ invertible. The morphism of cochain complexes*

$$\vartheta_c^*\colon (\operatorname{Hom}_k(\overline{A}^*, M), b^*) \to (\operatorname{Hom}_k(\overline{A}^*, M), b^*),$$

defined by:

$$\vartheta_c^r(\varphi)(a_1 \otimes \cdots \otimes a_r) = \varphi\big(a_1^c \otimes \cdots \otimes a_r^c\big)^c,$$

induce the identity map on $\mathrm{H}^*(A,M)$.

Proof. By a standard argument it is sufficient to prove it for $\mathrm{H}^0(A,M)$, and in this case the result it is immediate □

For each $g \in G$ we have the morphism

$$\theta_g^*\colon (\operatorname{Hom}_k(\overline{A}^*, M), b^*) \to (\operatorname{Hom}_k(\overline{A}^*, M), b^*),$$

defined by:

$$\theta_g^r(\varphi)(a_1 \otimes \cdots \otimes a_r) = w_g \varphi\big(a_1^{w_g} \otimes \cdots \otimes a_r^{w_g}\big) w_g^{-1}.$$

3.2.3.Proposition. *Let $g, g' \in G$. The endomorphisms of $\mathrm{H}^*(A,M)$ induced by $\theta_g^*\,\theta_{g'}^*$ and by $\theta_{gg'}^*$ coincide. Consequently $\mathrm{H}^*(A,M)$ is a left G-module.*

Proof. It is sufficient to see that

$$(\theta_{gg'}^*)^{-1}\,\theta_g^*\,\theta_{g'}^*$$

induce the identity map on $\mathrm{H}^*(A,M)$, which is an immediate consequence of Lemma 3.2.2 □

The cochain complex $\overline{X}^*(E,M)$ has a filtration $F^0 \subseteq F^1 \subseteq F^2 \subseteq \dots$, where

$$F^i(\overline{X}^n) = \bigoplus_{\substack{r+s=n\\ r\geq 0,\, s\geq i}} \operatorname{Hom}_k\big(\overline{A}^r \otimes \overline{k[G]}^s, M\big).$$

From this fact we obtain the following:

3.2.4.Corollary. *There is a converging spectral sequence*

$$E_2^{rs} = \mathrm{H}^s(G, \mathrm{H}^r(A,M)) \Rightarrow \mathrm{H}^{r+s}(E,M)$$

Given an A-bimodule M we write M^A to denote the k-submodule of M consisting of the elements m verifying $am = ma$ for all $a \in A$, and given a G-module N we write N^G to denote the invariants.

3.2.5.Corollary. *If A is separable, then* $\mathrm{H}^*\big(E,M\big) = \mathrm{H}^*\big(G, M^A\big)$.

3.2.6.Corollary. *If A is quasi-free, then there is a long exact sequence*

$$\begin{aligned} \dots \to \mathrm{H}^{n-2}(G,\mathrm{H}^1(A,M)) \to \mathrm{H}^n(G,\mathrm{H}^0(A,M)) \to \mathrm{H}^n(E,M) \to \\ \mathrm{H}^{n-1}(G,\mathrm{H}^1(A,M)) \to \mathrm{H}^{n+1}(G,\mathrm{H}^0(A,M)) \to \mathrm{H}^{n+1}(E,M) \to \dots . \end{aligned}$$

3.2.7.Corollary. *If G is a finite group and the order of G is invertible in k, then* $\mathrm{H}^*\big(E,M\big)$ *equals to* $\mathrm{H}^*(A,M)_G$.

Let S be a separable subalgebra of A and let $\widetilde{A}^r$ $(r \geq 0)$ be as in Theorem 3.1.10. Assume that $f(g,g') \in S$ for all $g, g' \in G$. Using that f take its values on S it is easy to see that G acts on $\big(\mathrm{Hom}_{S^e}(\widetilde{A}^r, M), b^*\big)$ via

$$\big(g\cdot\varphi\big)(a_1 \otimes_S \cdots \otimes_S a_r \otimes_S) = w_g \varphi\big(a_1^{w_g} \otimes_S \cdots \otimes_S a_r^{w_g} \otimes_S\big) w_g^{-1}.$$

3.2.8.Theorem. *The Hochschild cohomology* $\mathrm{H}^*(E,M)$ *of E with coefficients in M is the cohomology of G with coefficients in* $\big(\mathrm{Hom}_{S^e}(\widetilde{A}^r, M), b^*\big)$

Proof. It is similar to the proof of Theorem 3.1.10 □

References

[B] Benson, *Representations and Cohomology*, Cambridge University Press, 1991.

[B1] J. L. Brylinski, *Algebras associated with group actions and their homology*, Brown preprint (1987).

[B2] J. L. Brylinski, *Cyclic homology and equivariant theories*, Ann. Inst. Fourier **37** (1987), 15–28.

[C] C. Cibils, *Cyclic and Hochschild homology of 2-nilpotent algebras*, K-theory **4** (1990), 131–141.

[F-T] B. L. Feigin and B. L. Tsygan, *Additive K-theory*, Lect. Notes Math. **1289** (1986), 67–209.

[G-J] E. Getzler and D. S. Jones, *The cyclic homology of crossed products algebras*, J. Reine Angew. Math. **445** (1993), 161–174.

[N] V. Nistor, *Group homology and the cyclic homology of crossed products*, Invent. Math. **99** (1989), 411–424.

[Q] D. Quillen, *Cyclic cohomology and algebra extensions*, K-theory **3** (1989), 205–246.

[S] K. Sanada, *On the Hochschild homology of crossed products*, Comm. in Algebra **21** (1993), 2727–2748.

[W] C. Weibel, *An Introduction to Homological Algebra*, vol. 38, Cambridge Studies in Advanced Mathematics, 1994.

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Pabellón 1 - Ciudad Universitaria, (1428) Buenos Aires, Argentina. E-mail: vander @ mate.dm.uba.ar

Received in July 1998.

Instituto Argentino de Matemática
Saavedra 15 - 3er. Piso
1083 - Buenos Aires
Argentina.