The product $''\delta \circ \frac{\partial}{\partial x_i}\delta''$

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Abstract

Li Banghe and Li Yaqing defined in [1] a product $S \circ T$ as a hyperdistribution for S and T in $D'(I\!R^n)$ and they calculated $\delta \circ \delta$. In this work, we shall calculate the products $\delta \circ \frac{\partial}{\partial x_i}\delta$ for $i = 1, \ldots, n$, and we obtain that for n even, this product have the Hadamard finite part nonzero.

1 Introduction

We shall define the product we shall used. Let $\mathbb{C}^*$ and $I\!R^*$ be a nonstandard models for the complex field and the real field respectively, and let ρ be a positive infinitesimal. We denote with Θ the set of all infinitesimal in $\mathbb{C}^*$ and,

$$^{\rho}C = \{x \in \mathbb{C}^* : \text{ for some finite integer } n, \mid x \mid < \rho^{-n}\}.$$

Setting

$$^{\rho}\mathbb{C}' =^{\rho} \mathbb{C}/\Theta$$

We call *hiperdistributions* to the complex linear functional of $\mathcal{D}'(I\!R^n)$ in $^{\rho}\mathbb{C}'$.

Definition 1.1 *: Suppose $T \in \mathcal{D}'(I\!R^n)$, $u(x,y)$ is a harmonic function in $I\!R^{n+1}_{+} = \{(x,y) : x \in I\!R^n,\ y > 0\}$ and its boundary values in the sense of $\mathcal{D}'(I\!R^n)$ are $T(x)$, then $u(x,y)$ is called a harmonic funtion associated with T or* harmonic representation *of T.*

Lemma 1.2 *: Let $S, T \in \mathcal{D}'(I\!R^n)$, let $\hat{S}, \hat{T}$ be harmonic representations of S and T respectively, and $^*\hat{S}$, $^*\hat{T}$ denote the nonstandard extensions of $\hat{S}$, $\hat{T}$ respectively. For any $\phi \in \mathcal{D}'(I\!R^n)$,*

$$\langle^* \hat{S}(x,\rho)^* \hat{T}(x,\rho),^* \phi(x)\rangle \in^{\rho} \mathbb{C}.$$

Lemma 1.3 *: Let $S,T \in \mathcal{D}'(I\!R^n)$, if $\hat{S}_1, \hat{S}_2$ and $\hat{T}_1, \hat{T}_2$ are two harmonic representations of S and T respectively, then for any $\phi \in \mathcal{D}'(I\!R^n)$, we have*

$$\langle {}^*\hat{S}_2(x,\rho)^*\hat{T}_2(x,\rho),^*\phi(x)\rangle - \langle {}^*\hat{S}_1(x,\rho)^*\hat{T}_1(x,\rho),^*\phi(x)\rangle \approx 0$$

Let $\Phi :^\rho \mathbb{C} \to^\rho \mathbb{C}'$ be the homomorphism modulo Θ, we define

Definition 1.4 *: For any $S,T \in \mathcal{D}'(I\!R^n)$ and any of their harmonic representations $\hat{S}, \hat{T}$, we call the complex linear functional*

$$\langle S \circ T, \phi\rangle = \Phi(\langle {}^*\hat{S}(x,\rho)^*\hat{T}(x,\rho),^*\phi(x)\rangle)$$

of $\mathcal{D}'(I\!R^n) \to^\rho \mathbb{C}'$, the product of S and T.

2 The product $\delta \circ \frac{\partial}{\partial x_i}\delta$

We consider here one of the harmonic representation of the δ-function in $\mathcal{D}'(I\!R^n)$:

$$\hat{\delta}(x,y) = \frac{y}{c_n(|x|^2+y^2)^{\frac{n+1}{2}}}, \tag{1}$$

where $c_n = \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$

The function $\hat{\delta}(x,y)$ is harmonic on $I\!R^n \times I\!R_+$ and

$$\lim_{y\to 0_+} \hat{\delta}(x,y) = \delta(x) \quad \text{in} \quad \mathcal{D}'(I\!R^n). \tag{2}$$

By differentiation to x_i, we obtain in (1):

$$\frac{\partial}{\partial x_i}\{\hat{\delta}(x,y)\} = \frac{-y(n+1)x_i}{c_n(|x|^2+y^2)^{\frac{n+3}{2}}}. \tag{3}$$

From (2) and Theorem XVIII, Chap.III, [8]; we obtain:

$$\lim_{y\to 0^+} \frac{\partial}{\partial x_i}\{\hat{\delta}(x,y)\} = \frac{\partial}{\partial x_i}\{\delta(x)\}, \quad \text{in} \quad \mathcal{D}'(I\!R^n). \tag{4}$$

Furthermore, the partial derivative of a harmonic function in a set U, is a harmonic function in the same set; therefore, the functions $\frac{\partial}{\partial x_i}\{\hat{\delta}(x,y)\},\ i = 1,2,\ldots,n$; are harmonic representations of the distributions $\frac{\partial}{\partial x_i}\{\delta(x)\}$. Our purpose is obtain the product:

$$\delta(x) \circ \frac{\partial}{\partial x_i}\{\delta(x)\}. \tag{5}$$

For $n = 1$ the product (5) has been calculated by Li Banghe and Li Yaqing in [2]. Let $\phi \in \mathcal{D}(I\!R^n)$, $supp\phi \subset \{x/ \mid x \mid < a\}$ and choose $\rho \in I\!R^*$ a positive infinitesimal, then we shall evaluate:

$$\int_{I\!R^n} \hat{\delta}(x,\rho).\frac{\partial}{\partial x_i}\{\hat{\delta}(x,\rho)\}.\phi(x)\, dx. \tag{6}$$

We have,

$$\hat{\delta}(x,y).\frac{\partial}{\partial x_i}\{\hat{\delta}(x,y)\} = \frac{-y^2(n+1)x_i}{c_n^2(|x|^2+y^2)^{n+2}}. \tag{7}$$

We consider the Taylor expansion of $\phi \in \mathcal{D}(I\!R^n)$ until $n+1$ order:

$$\phi(x) = \sum_{j=0}^{n+1} \sum_{(k_1,\ldots,k_n),\sum k_i=j} \frac{x_1^{k_1}\ldots x_n^{k_n}}{k_1!\ldots k_n!} \frac{\partial^j \phi}{\partial x_1^{k_1}\ldots \partial x_n^{k_n}}(0) + \mid x \mid^{n+2} \psi(x), \tag{8}$$

where ψ is continuous for $x \neq 0$ and bounded near $x = 0$. Taking into account (6),(7) and (8) we obtain:

$$\int_{I\!R^n} \hat{\delta}(x,\rho).\frac{\partial}{\partial x_i}\{\hat{\delta}(x,\rho)\}.\phi(x)\, dx =$$
$$= \int_{|x|<a} \frac{-\rho^2(n+1)x_i}{c_n^2(|x|^2+\rho^2)^{n+2}} \{\sum_{j=0}^{n+1} \sum_{(k_1,\ldots,k_n),\sum k_i=j} \frac{x_1^{k_1}\ldots x_n^{k_n}}{k_1!\ldots k_n!} \frac{\partial^j \phi}{\partial x_1^{k_1}\ldots \partial x_n^{k_n}}(0)\}\, dx +$$
$$+ \int_{|x|<a} \frac{-\rho^2(n+1)x_i}{c_n^2(|x|^2+\rho^2)^{n+2}} \mid x \mid^{n+2} \psi(x)\, dx. \tag{9}$$

We consider $n \geq 2$. Farther on we consider $n = 1$. The second term of the sum in (9) is an infinitesimal number and we can omit it, in fact:

Let $M = sup\{\psi(x) :\mid x \mid < a\}$ and we consider the polar coordinate in $I\!R^n$:

$$
\begin{array}{ccccccc}
x_1 & = & r\cos\theta_1 & & & & \\
x_2 & = & r\sin\theta_1 & \cos\theta_2 & & & \\
\vdots & & \vdots & \vdots & & & \\
x_{n-1} & = & r\sin\theta_1 & \sin\theta_2 & \cdots & \sin\theta_{n-2} & \cos\theta_{n-1} \\
x_n & = & r\sin\theta_1 & \sin\theta_2 & \cdots & \sin\theta_{n-2} & \sin\theta_{n-1}
\end{array}
$$

where

$$0 \leq \theta_1, \theta_2, \ldots, \theta_{n-2} \leq \pi, \quad 0 \leq \theta_{n-1} \leq 2\pi. \tag{10}$$

We denote $d\Omega$ the volume element of the unitary sphere S^{n-1}, then $dx = r^{n-1} dr d\Omega$ with

$$d\Omega = \sin^{n-2}\theta_1 . \sin^{n-3}\theta_2 \ldots \sin\theta_{n-2}\, d\theta_1 \ldots d\theta_{n-1}.$$

Let $B_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ the $(n-1)$-dimensional measure of the $(n-1)$-sphere unitary S^{n-1}. We obtain

$$
\begin{aligned}
&\mid \int_{|x|<a} \frac{-\rho^2(n+1)x_i}{c_n^2(|x|^2+\rho^2)^{n+2}} \mid x \mid^{n+2} \psi(x)dx \mid \leq \\
&\leq \frac{\rho^2(n+1)}{c_n^2} M \int_{|x|<a} \frac{\mid x \mid^{n+3}}{(|x|^2+\rho^2)^{n+2}}\, dx = \\
&= \frac{\rho^2(n+1)}{c_n^2} M \int_{S^{n-1}} d\Omega \int_0^a \frac{r^{n+3}}{(r^2+\rho^2)^{n+2}}\, r^{n-1}\, dr \overset{r=\rho t}{=} \\
&= \frac{\rho^2(n+1)}{c_n^2} M B_n \int_0^{\frac{a}{\rho}} \frac{(\rho t)^{2n+2}}{\{(\rho t)^2+\rho^2\}^{n+2}}\, \rho\, dt = \\
&= \rho M' \int_0^{\frac{a}{\rho}} \frac{t^{2n+2}}{(1+t^2)^{n+2}}\, dt,
\end{aligned} \tag{11}
$$

where $M' = \frac{(n+1)MB_n}{c_n^2}, \quad M' \in I\!R.$

Now, we have:

$$\int_0^{\frac{a}{\rho}} \frac{t^{2n+2}}{(1+t^2)^{n+2}} dt = \int_0^1 \frac{t^{2n+2}}{(1+t^2)^{n+2}} dt + \int_1^{\frac{a}{\rho}} \frac{t^{2n+2}}{(1+t^2)^{n+2}} dt. \tag{12}$$

Moreover:

$$| \int_1^{\frac{a}{\rho}} \frac{t^{2n+2}}{(1+t^2)^{n+2}} dt |< \int_1^{\frac{a}{\rho}} \frac{1}{t^2} dt = -((\frac{a}{\rho})^{-1} - 1) < \infty. \tag{13}$$

(In fact, $(\frac{a}{\rho})^{-1}$ is infinitesimal). Next, in (11) we obtain:

$$\rho M' \int_0^{\frac{a}{\rho}} \frac{t^{2n+2}}{(1+t^2)^{n+2}} dt \approx 0$$

since $\rho \approx 0$. Now, we shall calculate the first term in (9). We consider for each $j \leq n+1$, $k_1 + \ldots + k_n = j$, the following terms:

$$\begin{aligned} &\int_{|x|<a} \frac{-\rho^2(n+1)x_i}{c_n^2(|x|^2+\rho^2)^{n+2}}\, x_1^{k_1} \ldots x_n^{k_n}\, dx \;= \\ &\quad = \int_{|x|<a} \frac{-\rho^2(n+1)}{c_n^2(|x|^2+\rho^2)^{n+2}}\, x_1^{k_1} \ldots x_i^{k_i+1} \ldots x_n^{k_n}\, dx. \end{aligned} \tag{14}$$

We consider the two cases $i = n$ and $i \neq n$; in fact, we have,

1) $i = n$ (i.e the product $\delta \circ \frac{\partial}{\partial x_n}\{\delta\}$).

Taking into account (10), we obtain in (14):

$$\begin{aligned} &\int_{|x|<a} \frac{-\rho^2(n+1)}{c_n^2(|x|^2+\rho^2)^{n+2}}\, x_1^{k_1} \ldots x_{n-1}^{k_{n-1}} . x_n^{k_n+1}\, dx = \\ &\int_0^a \frac{-\rho^2(n+1)}{c_n^2(r^2+\rho^2)^{n+2}}\, r^{k_1+\ldots+k_n+1+n-1}\, dr. \int_0^\pi \cos^{k_1}\theta_1 . \sin^{j-k_1+1+n-2}\theta_1\, d\theta_1 . \\ &\int_0^\pi \cos^{k_2}\theta_2 . \sin^{j-(k_1+k_2)+1+n-3}\theta_2\, d\theta_2 \ldots \\ &\ldots \int_0^\pi \cos^{k_{n-2}}\theta_{n-2} . \sin^{j-(k_1+k_2+\ldots+k_{n-2})+1+n-(n-2+1)}\theta_{n-2}\, d\theta_{n-2} . \\ &\int_0^{2\pi} \cos^{k_{n-1}}\theta_{n-1} . \sin^{j-(k_1+k_2+\ldots+k_{n-1})+1+n-(n-1+1)}\theta_{n-1}\, d\theta_{n-1} . \end{aligned} \tag{15}$$

The integrals of type $\int_0^\pi \cos^k\theta \sin^r\theta\, d\theta$ and $\int_0^{2\pi} \cos^k\theta \sin^r\theta\, d\theta$ are zero for k odd. Next, we consider in (15) k_i even for $i = 1,\ldots,n-1$. Now, we examine the last factor in (15):

$$\int_0^{2\pi} \cos^{k_{n-1}}\theta_{n-1}.\sin^{j-(k_1+k_2+\ldots+k_{n-1})+1+n-(n-1+1)}\theta_{n-1}\, d\theta_{n-1} =$$

$$\int_0^{2\pi} \cos^{k_{n-1}}\theta_{n-1}.\sin^{k_n+1}\theta_{n-1}\, d\theta_{n-1}. \tag{16}$$

If k_{n-1} is even the function $\cos^{k_{n-1}}\theta_{n-1}$ is symmetrical with respect to $x = \pi$ and the function $\sin^{k_n+1}\theta_{n-1}$ is antisymmetrical with respect to $x = \pi$ if k_n+1 is odd. We consider in (15) $k_1,\ldots,k_{n-1}$ even and k_n odd. Let $k_i = 2s_i$, $i = 1,...,n-1$, $k_n = 2s_n + 1$, $\sum_{i=1}^n k_i = j = 2s_1 + \ldots + 2s_{n-1} + 2s_n + 1$, therefore we obtain:

$$\int_{|x|<a} \frac{-\rho^2(n+1)}{c_n^2(|x|^2+\rho^2)^{n+2}}\, x_1^{2s_1}\ldots x_{n-1}^{2s_{n-1}}.x_n^{2s_n+2}\, dx =$$

$$\int_0^a \frac{-\rho^2(n+1)}{c_n^2(r^2+\rho^2)^{n+2}}\, r^{j+n}\, dr.\frac{\Gamma(\frac{2s_1+1}{2})\Gamma(\frac{j-2s_1+1+n-2+1}{2})}{\Gamma(\frac{j+1+n-2}{2}+1)}.$$

$$\frac{\Gamma(\frac{2s_2+1}{2})\Gamma(\frac{j-(2s_1+2s_2)+1+n-3+1}{2})}{\Gamma(\frac{j-2s_1+1+n-3}{2}+1)}\ldots$$

$$\ldots\frac{\Gamma(\frac{2s_{n-2}+1}{2})\Gamma(\frac{j-(2s_1+2s_2+\ldots+2s_{n-2})+1+n-(n-2+1)+1}{2})}{\Gamma(\frac{j-(2s_1+2s_2+\ldots+2s_{n-3})+1+n-(n-2+1)}{2}+1)}.$$

$$.2\frac{\Gamma(\frac{2s_{n-1}+1}{2})\Gamma(\frac{j-(2s_1+2s_2+\ldots+2s_{n-1})+1+1}{2})}{\Gamma(\frac{j-(2s_1+2s_2+\ldots+2s_{n-2})+1}{2}+1)}, \tag{17}$$

where we have considered:

$$\int_0^\pi \cos^k\theta \sin^m\theta\, d\theta = 2\int_0^{\frac{\pi}{2}} \cos^k\theta \sin^m\theta\, d\theta = \frac{\Gamma(\frac{k+1}{2}).\Gamma(\frac{m+1}{2})}{\Gamma(\frac{k+m}{2}+1)}. \tag{18}$$

Taking into account that

$$\int_0^a \frac{-\rho^2(n+1)}{c_n^2(r^2+\rho^2)^{n+2}}\, r^{j+n}\, dr \overset{r=\rho t}{=} \int_0^{\frac{a}{\rho}} \frac{-\rho^2(n+1)}{c_n^2(\rho^2t^2+\rho^2)^{n+2}}\, (\rho t)^{j+n}\rho\, dt =$$

$$= -\rho^{j-n-1}\int_0^{\frac{a}{\rho}} \frac{n+1}{c_n^2}\frac{t^{j+n}}{(1+t^2)^{n+2}}\,dt \tag{19}$$

Now, we obtain the following formulas

$$-\rho^{j-n-1}\int_0^{\frac{a}{\rho}} \frac{n+1}{c_n^2}\frac{t^{j+n}}{(1+t^2)^{n+2}}\,dt = -\rho^{j-n-1}(\int_0^{\infty} \frac{n+1}{c_n^2}\frac{t^{j+n}}{(1+t^2)^{n+2}}\,dt-$$

$$-\int_{\frac{a}{\rho}}^{\infty} \frac{n+1}{c_n^2}\frac{t^{j+n}}{(1+t^2)^{n+2}}\,dt) \tag{20}$$

and,

$$0 \le \rho^{j-n-1}\int_{\frac{a}{\rho}}^{\infty} \frac{n+1}{c_n^2}\frac{t^{j+n}}{(1+t^2)^{n+2}}\,dt \le$$

$$\rho^{j-n-1}\,\frac{n+1}{c_n^2}\int_{\frac{a}{\rho}}^{\infty} t^{j-n-4}\,dt = -\rho^2\,\frac{n+1}{c_n^2}\frac{a^{j-n-3}}{j-n-3} \approx 0, \tag{21}$$

ρ is infinitesimal.

From (20) and (21), we have

$$-\rho^{j-n-1}\int_0^{\frac{a}{\rho}} \frac{n+1}{c_n^2}\frac{t^{j+n}}{(1+t^2)^{n+2}}\,dt \approx -\rho^{j-n-1}\int_0^{\infty} \frac{n+1}{c_n^2}\frac{t^{j+n}}{(1+t^2)^{n+2}}\,dt. \tag{22}$$

Therefore, we obtain

$$\int_{|x|<a} \frac{-\rho^2(n+1)}{c_n^2(|x|^2+\rho^2)^{n+2}}\,x_1^{2s_1}\ldots x_{n-1}^{2s_{n-1}}.x_n^{2s_n+2}\,dx \approx$$

$$\approx -\rho^{j-n-1}\int_0^{\infty} \frac{n+1}{c_n^2}\frac{t^{j+n}}{(1+t^2)^{n+2}}\,dt\,.\,2\prod_{i=1}^{n-1}\frac{\Gamma(\frac{2s_i+1}{2})\Gamma(\frac{j+1-\sum_{k=1}^{i}2s_k+n-i}{2})}{\Gamma(\frac{j-\sum_{k=1}^{i-1}2s_k+n-i}{2}+1)}. \tag{23}$$

Simplifying in (23), we obtain

$$\prod_{i=1}^{n-1}\frac{\Gamma(\frac{2s_i+1}{2})\Gamma(\frac{j+1-\sum_{k=1}^{i}2s_k+n-i}{2})}{\Gamma(\frac{j-\sum_{k=1}^{i-1}2s_k+n-i}{2}+1)} =$$

$$= \frac{\prod_{i=1}^{n-1}\Gamma(\frac{2s_i+1}{2}).\prod_{i=1}^{n-2}\Gamma(\frac{j+1-\sum_{k=1}^{i}2s_k+n-i}{2})\Gamma(\frac{j+1-\sum_{k=1}^{n-1}2s_k+n-(n-1)}{2})}{\prod_{i=1}^{n-1}\Gamma(\frac{j-\sum_{k=1}^{i-1}2s_k+n-i}{2}+1)} =$$

$$= \frac{\prod_{i=1}^{n-1}\Gamma(s_i+\frac{1}{2}).\prod_{i=1}^{n-2}\Gamma(\frac{j+1-\sum_{k=1}^{i}2s_k+n-i}{2})\Gamma(\frac{2s_n+1+1+1}{2})}{\prod_{i=2}^{n-1}\Gamma(\frac{j-\sum_{k=1}^{i-1}2s_k+n-i}{2}+1)\Gamma(\frac{j+n-1}{2}+1)} =$$

$$= (s_n+\frac{1}{2})\frac{\prod_{i=1}^{n}\Gamma(s_i+\frac{1}{2})}{\Gamma(\frac{j+n+1}{2})}. \tag{24}$$

Taking into account (23), results

$$\int_{|x|<a}\frac{-\rho^2(n+1)}{c_n^2(|x|^2+\rho^2)^{n+2}}\ x_1^{2s_1}\dots x_{n-1}^{2s_{n-1}}.x_n^{2s_n+2}\,dx \approx$$

$$-\,\rho^{j-n-1}\int_0^\infty\frac{n+1}{c_n^2}\frac{t^{j+n}}{(1+t^2)^{n+2}}\,dt\ .\ 2\ .(s_n+\frac{1}{2})\frac{\prod_{i=1}^{n}\Gamma(s_i+\frac{1}{2})}{\Gamma(\frac{j+n+1}{2})}, \tag{25}$$

with $k_i=2s_i$, $i=1,\dots,n-1$, $k_n=2s_n+1$ and $\sum_{i=1}^{n}k_i=j=(\sum_{i=1}^{n}2s_i)+1$. From (9), we have, for $i=n$

$$\int_{I\!R^n}\hat{\delta}(x,\rho).\frac{\partial}{\partial x_n}\{\hat{\delta}(x,\rho)\}.\phi(x)\,dx =$$

$$= \int_{|x|<a}\frac{-\rho^2(n+1)x_n}{c_n^2(|x|^2+\rho^2)^{n+2}}\{\sum_{j=0}^{n+1}\sum_{(k_1,\dots,k_n),\sum k_i=j}\frac{x_1^{k_1}\dots x_n^{k_n}}{k_1!\dots k_n!}\frac{\partial^j\phi}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}(0)\}dx =$$

$$= \sum_{j=0}^{n+1}\sum_{(k_1,\dots,k_n),\sum k_i=j}\frac{1}{k_1!\dots k_n!}\frac{\partial^j\phi}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}(0).$$

$$.\int_{|x|<a}\frac{-\rho^2(n+1)x_1^{k_1}\dots x_{n-1}^{k_{n-1}}.x_n^{k_n+1}}{c_n^2(|x|^2+\rho^2)^{n+2}}\,dx =$$

$$= \sum_{j=0}^{[\frac{n}{2}]}\sum_{(s_1,\dots,s_n),\sum s_i=j}\frac{1}{(2s_1)!\dots(2s_{n-1})!(2s_n+1)!}\ \frac{\partial^{2j+1}\phi}{\partial x_1^{2s_1}\dots\partial x_{n-1}^{2s_{n-1}}\partial x_n^{2s_n+1}}(0).$$

$$.-\rho^{2j+1-n-1}\int_0^\infty\frac{n+1}{c_n^2}\frac{t^{2j+1+n}}{(1+t^2)^{n+2}}\,dt\ .\ 2\ .(s_n+\frac{1}{2})\frac{\prod_{i=1}^{n}\Gamma(s_i+\frac{1}{2})}{\Gamma(\frac{2j+1+n+1}{2})} =$$

$$= \sum_{j=0}^{[\frac{n}{2}]}\sum_{(s_1,\dots,s_n),\sum s_i=j}-\rho^{2j-n}\int_0^\infty\frac{n+1}{c_n^2}\frac{t^{2j+1+n}}{(1+t^2)^{n+2}}\,dt\ \frac{\prod_{i=1}^{n}\Gamma(s_i+\frac{1}{2})}{\prod_{i=1}^{n}(2s_i)!\,\Gamma(\frac{2j+n+2}{2})}.$$

$$\cdot \frac{\partial^{2j+1}\phi}{\partial x_1^{2s_1}\dots\partial x_{n-1}^{2s_{n-1}}\partial x_n^{2s_n+1}}(0). \tag{26}$$

Finally, we arrives at the

Theorem 2.1 *:*

$$\delta\circ\frac{\partial}{\partial x_n}\{\delta\}=\sum_{j=0}^{[\frac{n}{2}]}\sum_{(s_1,\dots,s_n),\sum s_i=j}\rho^{2j-n}\, c_{s_1,\dots,s_n}\,\frac{\partial^{2j+1}\delta}{\partial x_1^{2s_1}\dots\partial x_{n-1}^{2s_{n-1}}\partial x_n^{2s_n+1}},$$

where:

$$c_{s_1,\dots,s_n}=\frac{1}{\Gamma(\frac{2j+n+2}{2})}\prod_{i=1}^{n}\frac{\Gamma(s_i+\frac{1}{2})}{(2s_i)!}\int_0^\infty\frac{n+1}{c_n^2}\frac{t^{2j+1+n}}{(1+t^2)^{n+2}}\,dt.$$

Corollary 2.2 *If n is even, the Hadamard finite part of $\delta\circ\frac{\partial}{\partial x_n}\{\delta\}$ is nonzero.*

In fact, the corresponding term for $j=[\frac{n}{2}]$ for n even is finite.

Corollary 2.3 *If $n=2$, then*

$$\delta\circ\frac{\partial}{\partial x_2}\{\delta\}=\rho^{-2}\frac{1}{16\pi}\frac{\partial}{\partial x_2}\{\delta\}+\frac{1}{64\pi}[\frac{\partial^3}{\partial x_2^3}\{\delta\}+\frac{\partial^3}{\partial x_1^2\partial x_2}\{\delta\}].$$

Thesis

$$\delta\circ\frac{\partial}{\partial x_2}\{\delta\}=\sum_{j=0}^{1}\sum_{(s_1,s_2),s_1+s_2=j}\rho^{2j-2}\, c_{s_1,s_2}\,\frac{\partial^{2j+1}\delta}{\partial x_1^{2s_1}\partial x_2^{2s_2+1}}$$

$$c_{0,0}=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{2+2}{2})}\int_0^\infty\frac{3}{c^2}\frac{t^3}{(1+t^2)^4}\,dt=\frac{1}{16\pi}.$$

$$c_{1,0}=c_{0,1}=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{2!\Gamma(\frac{2+2+2}{2})}\int_0^\infty\frac{3}{(2\pi)^2}\frac{t^5}{(1+t^2)^4}\,dt=\frac{1}{64\pi}.$$

Then,

$$\delta\circ\frac{\partial}{\partial x_2}\{\delta\}=\rho^{-2}\frac{1}{16\pi}\frac{\partial}{\partial x_2}\{\delta\}+\delta^0\frac{1}{64\pi}[\frac{\partial^3}{\partial x_2^3}\{\delta\}+\frac{\partial^3}{\partial x_1^2\partial x_2}\{\delta\}].$$

Corollary 2.4 *If $n=2$ then*

$$Hpf(\delta\circ\frac{\partial}{\partial x_2}\{\delta\})=\frac{1}{64\pi}[\frac{\partial^3}{\partial x_2^3}\{\delta\}+\frac{\partial^3}{\partial x_1^2\partial x_2}\{\delta\}].$$

2) $i \neq n$, $n \geq 2$.

Introducing the polar coordinates in (14), we obtain

$$\int_{|x|<a} \frac{-\rho^2(n+1)}{c_n^2(|x|^2+\rho^2)^{n+2}}\, x_1^{k_1}\dots x_i^{k_i+1}\dots x_n^{k_n}\, dx =$$

$$\int_0^a \frac{-\rho^2(n+1)}{c_n^2(r^2+\rho^2)^{n+2}}\, r^{k_1+\dots+k_i+1+\dots k_n+n-1}\, dr. \int_0^\pi \cos^{k_1}\theta_1. \sin^{j-k_1+1+n-2}\theta_1\, d\theta_1.$$

$$\int_0^\pi \cos^{k_2}\theta_2. \sin^{j-(k_1+k_2)+1+n-3}\theta_2\, d\theta_2\dots$$

$$\dots\int_0^\pi \cos^{k_{i-1}}\theta_{i-1}. \sin^{j-(k_1+k_2+\dots+k_{i-1})+1+n-(i)}\theta_{i-1}\, d\theta_{i-1}.$$

$$.\int_0^\pi \cos^{k_i+1}\theta_i. \sin^{j-(k_1+k_2+\dots+k_i)+n-(i+1)}\theta_i\, d\theta_i.$$

$$.\int_0^\pi \cos^{k_{i+1}}\theta_{i+1}. \sin^{j-(k_1+k_2+\dots+k_{i+1})+n-(i+2)}\theta_{i+1}\, d\theta_{i+1}\dots$$

$$\dots\int_0^{2\pi} \cos^{k_{n-1}}\theta_{n-1}. \sin^{j-(k_1+k_2+\dots+k_{n-1})+n-(n-1+1)}\theta_{n-1}\, d\theta_{n-1}. \tag{27}$$

In the same manner as (15), we consider k_r, $r \neq i$ and k_i+1 even. Taking into account (18) and $\sum k_i = j$, we obtain in (27):

$$\int_{|x|<a} \frac{-\rho^2(n+1)}{c_n^2(|x|^2+\rho^2)^{n+2}}\, x_1^{k_1}\dots x_i^{k_i+1}\dots x_n^{k_n}\, dx =$$

$$\int_0^a \frac{-\rho^2(n+1)}{c_n^2(r^2+\rho^2)^{n+2}}\, r^{j+n}\, dr. \frac{\Gamma(\frac{k_1+1}{2})\Gamma(\frac{j+1-k_1+n-2+1}{2})}{\Gamma(\frac{j+1+n-2}{2}+1)}.$$

$$\frac{\Gamma(\frac{k_2+1}{2})\Gamma(\frac{j+1-(k_1+k_2)+n-3+1}{2})}{\Gamma(\frac{j+1-k_1+n-3}{2}+1)}\dots\frac{\Gamma(\frac{k_{i-1}+1}{2})\Gamma(\frac{j+1-\sum_{r=1}^{i-1}k_r+n-i+1}{2})}{\Gamma(\frac{j+1-\sum_{r=1}^{i-2}k_r+n-i}{2}+1)}.$$

$$.\frac{\Gamma(\frac{k_i+1+1}{2})\Gamma(\frac{j-\sum_{r=1}^{i}k_r+n-(i+1)+1}{2})}{\Gamma(\frac{j+1-\sum_{r=1}^{i-1}k_r+n-(i+1)}{2}+1)}.\frac{\Gamma(\frac{k_{i+1}+1}{2})\Gamma(\frac{j-\sum_{r=1}^{i+1}k_r+n-(i+2)+1}{2})}{\Gamma(\frac{j-\sum_{r=1}^{i}k_r+n-(i+2)}{2}+1)}\dots$$

$$\dots 2\frac{\Gamma(\frac{k_{n-1}+1}{2})\Gamma(\frac{k_n+1}{2})}{\Gamma(\frac{k_{n-1}+k_n}{2}+1)}. \tag{28}$$

Simplifying and taking into account that $k_r = 2s_r$ for $r \neq i$ and $k_i = 2s_i + 1$, $s_r \geq 0$, $r = 1, \ldots, n$, in the same manner as (23), we have, from (28),

$$\int_{|x|<a} \frac{-\rho^2(n+1)}{c_n^2(|x|^2+\rho^2)^{n+2}}\, x_1^{k_1} \ldots x_i^{k_i+1} \ldots x_n^{k_n}\, dx =$$

$$= \int_{|x|<a} \frac{-\rho^2(n+1)}{c_n^2(|x|^2+\rho^2)^{n+2}}\, x_1^{2s_1} \ldots x_i^{2s_i+2} \ldots x_n^{2s_n}\, dx \approx$$

$$\approx -\rho^{j-n-1} \int_0^\infty \frac{n+1}{c_n^2} \frac{t^{j+n}}{(1+t^2)^{n+2}}\, dt \, . \, \frac{(2s_i+1)\prod_{r=1}^n (s_r+\frac{1}{2})}{\Gamma(\frac{j+n+1}{2})}. \tag{29}$$

Therefore, from (9), we arrive at

$$\int_{\mathbb{R}^n} \hat{\delta}(x,\rho). \frac{\partial}{\partial x_i}\{\hat{\delta}(x,\rho)\}.\phi(x)\, dx =$$

$$= \int_{|x|<a} \frac{-\rho^2(n+1)x_i}{c_n^2(|x|^2+\rho^2)^{n+2}} \{\sum_{j=0}^{n+1} \sum_{(k_1,\ldots,k_n),\sum k_i=j} \frac{x_1^{k_1}\ldots x_n^{k_n}}{k_1!\ldots k_n!} \frac{\partial^j \phi}{\partial x_1^{k_1}\ldots \partial x_n^{k_n}}(0)\} dx =$$

$$= \sum_{j=0}^{n+1} \sum_{(k_1,\ldots,k_n),\sum k_i=j} \frac{1}{k_1!\ldots k_n!} \frac{\partial^j \phi}{\partial x_1^{k_1}\ldots \partial x_n^{k_n}}(0).$$

$$. \int_{|x|<a} \frac{-\rho^2(n+1)}{c_n^2(|x|^2+\rho^2)^{n+2}}\, x_1^{k_1} \ldots x_i^{k_i+1} \ldots x_n^{k_n}\, dx =$$

$$= \sum_{j=0}^{[\frac{n}{2}]} \sum_{(s_1,\ldots,s_n),\sum s_i=j} \frac{1}{(2s_1)!\ldots(2s_i+1)!\ldots(2s_n)!} \frac{\partial^{2j+1}\phi}{\partial x_1^{2s_1}\ldots \partial x_i^{2s_i+1}\ldots \partial x_n^{2s_n}}(0).$$

$$. \int_{|x|<a} \frac{-\rho^2(n+1)}{c_n^2(|x|^2+\rho^2)^{n+2}}\, x_1^{2s_1} \ldots x_i^{2s_i+2} \ldots x_n^{2s_n}\, dx =$$

$$= \sum_{j=0}^{[\frac{n}{2}]} \sum_{(s_1,\ldots,s_n),\sum s_i=j} \frac{1}{(2s_i+1)\prod_{r=1}^n (2s_r)!} \frac{\partial^{2j+1}\phi}{\partial x_1^{2s_1}\ldots \partial x_i^{2s_i+1}\ldots x_n^{2s_n}}(0).$$

$$. -\rho^{2j+1-n-1} \int_0^\infty \frac{n+1}{c_n^2} \frac{t^{2j+1+n}}{(1+t^2)^{n+2}}\, dt \, .(2s_i+1) \frac{\prod_{r=1}^n (s_r+\frac{1}{2})}{\Gamma(\frac{2j+1+n+1}{2})} =$$

$$= \sum_{j=0}^{[\frac{n}{2}]} \sum_{(s_1,\ldots,s_n),\sum s_i=j} -\rho^{2j-n} \int_0^\infty \frac{n+1}{c_n^2} \frac{t^{2j+1+n}}{(1+t^2)^{n+2}}\, dt \cdot \frac{\prod_{r=1}^n (s_r+\frac{1}{2})}{\prod_{r=1}^n (2s_r)!\, \Gamma(\frac{2j+n+2}{2})}.$$

$$\cdot \frac{\partial^{2j+1}\phi}{\partial x_1^{2s_1}\ldots \partial x_i^{2s_i+1}\ldots \partial x_n^{2s_n}}(0). \tag{30}$$

Finally, we obtain the

Theorem 2.5

$$\delta \circ \frac{\partial}{\partial x_i}\{\delta\} = \sum_{j=0}^{[\frac{n}{2}]} \sum_{(s_1,\ldots,s_n),\sum s_i=j} \rho^{2j-n}\, c_{s_1,\ldots,s_n}\, \frac{\partial^{2j+1}\delta}{\partial x_1^{2s_1}\ldots \partial x_i^{2s_i+1}\ldots \partial x_n^{2s_n}},$$

where

$$c_{s_1,\ldots,s_n} = \frac{1}{\Gamma(\frac{2j+n+2}{2})} \prod_{r=1}^n \frac{(s_r+\frac{1}{2})}{(2s_r)!} \int_0^\infty \frac{n+1}{c_n^2} \frac{t^{2j+1+n}}{(1+t^2)^{n+2}}\, dt.$$

Corollary 2.6 *If $n=2$, then*

$$\delta \circ \frac{\partial}{\partial x_1}\{\delta\} = \rho^{-2} \frac{1}{2^6\pi^2} \frac{\partial}{\partial x_1}\{\delta\} + \frac{3}{2^7\pi^2}[\frac{\partial^3}{\partial x_1^3}\{\delta\} + \frac{\partial^3}{\partial x_1 \partial x_2^2}\{\delta\}].$$

Thesis Let $n=2$, then:

$$\delta \circ \frac{\partial}{\partial x_1}\{\delta\} = \sum_{j=0}^{1} \sum_{(s_1,s_2),s_1+s_2=j} \rho^{2j-2}\, c_{s_1,s_2}\, \frac{\partial^{2j+1}\delta}{\partial x_1^{2s_1+1}\partial x_2^{2s_2}}$$

$$c_{0,0} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\Gamma(\frac{2+2}{2})} \int_0^\infty \frac{3}{c_2^2} \frac{t^3}{(1+t^2)^4}\, dt = \frac{1}{2^6\pi^2}.$$

$$c_{1,0} = c_{0,1} = \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!\,\Gamma(\frac{2+2+2}{2})} \int_0^\infty \frac{3}{(2\pi)^2} \frac{t^5}{(1+t^2)^4}\, dt = \frac{3}{2^7\pi^2}.$$

Then, we have,

$$\delta \circ \frac{\partial}{\partial x_1}\{\delta\} = \rho^{-2} \frac{1}{2^6\pi^2} \frac{\partial}{\partial x_1}\{\delta\} + \delta^0 \frac{3}{2^7\pi^2}[\frac{\partial^3}{\partial x_1^3}\{\delta\} + \frac{\partial^3}{\partial x_1 \partial x_2^2}\{\delta\}].$$

Corollary 2.7

$$Hpf(\delta \circ \frac{\partial}{\partial x_1}\{\delta\}) = \frac{3}{2^7\pi^2}[\frac{\partial^3}{\partial x_1^3}\{\delta\} + \frac{\partial^3}{\partial x_1 \partial x_2^2}\{\delta\}].$$

Now, we consider the case $n = 1$. From (9) we obtain

$$\int_{\mathbb{R}} \hat{\delta}(x,\rho).\hat{\delta}'(x,\rho).\phi(x)\,dx =$$

$$= \int_{-a}^{a} \frac{-2\rho^2 x}{c_1^2(x^2+\rho^2)^3}\,\{\sum_{j=0}^{2} \frac{x^j}{j!}\phi^{(j)}(0)\}\,dx =$$

$$\sum_{j=0}^{2} \frac{-2\rho^2}{c_1^2 j!}\phi^{(j)}(0)\int_{-a}^{a} \frac{x^{j+1}}{(x^2+\rho^2)^3}\,dx. \tag{31}$$

1) For $j = 0$, the integral $\int_{-a}^{a} \frac{x}{(x^2+\rho^2)^3}\,dx$ is zero because $\frac{x}{(x^2+\rho^2)^3}\,dx$ is an anti-symmetrical function respect to $x = 0$.

2) For $j = 2$, we obtain in the same maner as 1) that $\int_{-a}^{a} \frac{x^3}{(x^2+\rho^2)^3}\,dx = 0$. We have,

$$\int_{\mathbb{R}} \hat{\delta}(x,\rho).\hat{\delta}'(x,\rho).\phi(x)\,dx =$$

$$\frac{-2\rho^2}{c_1^2}\phi'(0)\int_{-a}^{a} \frac{x^2}{(x^2+\rho^2)^3}\,dx =$$

$$\frac{-2\rho^2}{c_1^2}\phi'(0)2\int_{0}^{a} \frac{x^2}{(x^2+\rho^2)^3}\,dx =$$

$$2\phi'(0)\int_{0}^{a} \frac{-2\rho^2}{c_1^2}\frac{x^2}{(x^2+\rho^2)^3}\,dx. \tag{32}$$

Using (19) and (22) for $n = 1,\ j = 1$, we obtain:

$$2\phi'(0)\int_{0}^{a} \frac{-2\rho^2}{c_1^2}\frac{x^2}{(x^2+\rho^2)^3}\,dx \approx$$

$$-2\phi'(0)\rho^{-1}\int_{0}^{\infty} \frac{2}{c_1^2}\frac{t^2}{(1+t^2)^3}\,dt =$$

$$= -\frac{4\phi'(0)\rho^{-1}}{c_1^2}\int_{0}^{\infty} \frac{t^2}{(1+t^2)^3}\,dt. \tag{33}$$

Putting

$$c_1^2 = \pi^2$$

and taking into account the following integral, we have,

$$\int_{0}^{\infty} \frac{t^2}{(1+t^2)^3}\,dt = \int_{0}^{\infty} \frac{1}{(1+t^2)^2}\,dt - \int_{0}^{\infty} \frac{1}{(1+t^2)^3}\,dt =$$

$$= \frac{t}{2(1+t^2)} + \frac{1}{2}\arctan t\,|_0^\infty - [\frac{t}{4(1+t^2)^2} + \frac{3}{4}(\frac{t}{2(1+t^2)} + \frac{1}{2}\arctan t)]\,|_0^\infty = \frac{\pi}{16}.$$

Then, we obtain

$$\int_{I\!R} \hat{\delta}(x,\rho).\hat{\delta}'(x,\rho).\phi(x)\,dx = \frac{-4\rho^{-1}\phi'(0)}{\pi^2}.\frac{\pi}{16} = \frac{-1}{4\pi}\rho^{-1}\phi'(0)$$

and finally, we arrive at the following formula

$$\delta(x) \circ \delta'(x) = \frac{1}{4\pi}\rho^{-1}\delta'(x).$$

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