Two basic lifting theorems in the continuous case

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Abstract

The relation between the lifting theorems due to Nagy-Foias and Cotlar-Sadosky in the continuous case is discussed.

THE NAGY-FOIAS COMMUTANT LIFTING THEOREM

The Nagy-Foias commutant lifting theorem ([N-F.1], [N-F.2]) is an abstract generalization of Sarason's generalized interpolation theorem ([S.1]) and is a basic result in Operator Theory and its applications to interpolation problems. We refer the reader to the fundamental selfcontained book [F-F] in wich is presented a unified geometric aproach, based in this theorem, to a large array of classical and modern interpolation problems arising in mathematics and engineering. To state the theorem in the continuous (monoparametric) version we need to recall a few definitions and basic results.

A strongly continuous semigroup of contractions in the Hilbert space $\mathcal{H}$ is a family $T = \{T(t)/t \geq 0\} \subset \mathcal{L}(\mathcal{H})$ of contractive operators in $\mathcal{H}$ such that $T(0) = I$ (the identity operator in $\mathcal{H}$), $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$, and $\|T(t)h - h\| \to 0$ if $t \to 0^+$, for each $h \in \mathcal{H}$. A minimal unitary dilation of such a semigroup T is a strongly continuous group $U = \{U(t)/t \in \Re\} \subset \mathcal{L}(\mathcal{F})$ of unitary operators on a Hilbert space $\mathcal{F}$ that contains $\mathcal{H}$ such that $T(t) = P_{\mathcal{H}}^{\mathcal{F}} U(t) \mid_{\mathcal{H}}$ for all $t \geq 0$, and that the minimality condition holds: $\mathcal{F} = \bigvee\{U(t)\mathcal{H}/t \in \Re\}$ (i.e., $\mathcal{F}$ is the closed linear span of $\{U(t)\mathcal{H}/t \in \Re\}$). Every strongly continuous semigroup of contractions has minimal unitary dilations and two minimal unitary dilations are always isomorphic ([N-F.2] I.8.2). If $U = \{U(t)/t \in \Re\} \subset \mathcal{L}(\mathcal{F})$ is *the* minimal unitary dilation of $T = \{T(t)/t \geq 0\} \subset \mathcal{L}(\mathcal{H})$ then a minimal isometric dilation is associated as follows: if $\mathcal{M} = \bigvee\{U(t)\mathcal{H}/t \geq 0\}$ and $W(t) = U(t) \mid_{\mathcal{M}}$ then

$W = \{W(t)/t \geq 0\} \subset \mathcal{L}(\mathcal{M})$ is a strongly continuous group of isometries such that $T(t) = P_{\mathcal{H}}^{\mathcal{M}} W(t) \mid_{\mathcal{H}}$ for all $t \geq 0$, $\mathcal{M} = \bigvee\{W(t)\mathcal{H}/t \geq 0\}$ and $T(t)P_{\mathcal{H}}^{\mathcal{M}} = P_{\mathcal{H}}^{\mathcal{M}} T(t)$, for all $t \geq 0$.

THEOREM A *For $j = 1, 2$ let $T_j = \{T_j(t)/t \geq 0\} \subset \mathcal{L}(\mathcal{H}_j)$ be a strongly continuous semigroup of contractions in the Hilbert space $\mathcal{H}_j$, $U_j = \{U_j(t)/t \in \Re\} \subset \mathcal{L}(\mathcal{F}_j)$ with $\mathcal{F}_j = \bigvee\{U_j(t)\mathcal{H}_j/t \in \Re\}$ its minimal unitary dilation and $W_j = \{W_j(t)/t \geq 0\} \subset \mathcal{L}(\mathcal{M}_j)$ with $\mathcal{M}_j = \bigvee\{U_j(t)\mathcal{H}_j/t \geq 0\}$ its minimal isometric dilation. If $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is such that $XT_1(t) = T_2(t)X$ holds for all $t \geq 0$ then:*

1. *Exists $Y \in \mathcal{L}(\mathcal{F}_1, \mathcal{F}_2)$ such that $YU_1(t) = U_2(t)Y$, $\forall t \in \Re$, $P_{\mathcal{H}_2}^{\mathcal{F}_2} Y \mid_{\mathcal{H}_1} = X$ and $\|Y\| = \|X\|$.*
2. *Exists $Z \in \mathcal{L}(\mathcal{M}_1, \mathcal{M}_2)$ such that $ZW_1(t) = W_2(t)Z$, $\forall t \geq 0$, $P_{\mathcal{H}_2}^{\mathcal{M}_2} Z = XP_{\mathcal{H}_1}^{\mathcal{M}_1}$ and $\|Z\| = \|X\|$.*

The theorem says that every operator X wich interwines the semigroups T_1 and T_2 can be *lifted* to an operator Y (with the same norm as X) wich interwines the minimal unitary dilations of these semigroups. This continuous version of the Nagy-Foias commutant lifting theorem was first proved by Arocena ([A.1]) and it was applied in [F] to the solution of an interpolation problem of Dym and Gohberg [D-G]. Combining Arocena's aproach to lifting problems in general groups developed in [A.3] (see also [A.2] and [S.2]) with the theory of extensions of local semigroups of contractions ([B]) a simpler and more conceptual proof was given in [A.3].

A COTLAR-SADOSKY LIFTING THEOREM

Let us consider two arbitrary vector spaces $\mathcal{V}_1$, $\mathcal{V}_2$ and two arbitrary subspaces $\mathcal{W}_1 \subset \mathcal{V}_1$, $\mathcal{W}_2 \subset \mathcal{V}_2$. For $j = 1, 2$ suppose that $\tau_j(t) : \mathcal{V}_j \longrightarrow \mathcal{V}_j$ ($t \in \Re$) is a group of linear isomorphisms such that:

$$\tau_1(t)\mathcal{W}_1 \subset \mathcal{W}_1 \qquad \tau_2(-t)\mathcal{W}_2 \subset \mathcal{W}_2, \qquad \forall t \geq 0.$$

$\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{W}_1, \mathcal{W}_2, \tau_1, \tau_2\}$ is called an algebraic scattering system ([C-S.2]). Let $B_j : \mathcal{V}_j \times \mathcal{V}_j \to C$ be a sesquilinear form. We recall the following definitions:

- B_j is positive $\Longleftrightarrow B_j(v,v) \geq 0, \quad \forall v \in \mathcal{V}_j$.
- B_j is $\tau_j -$ Toeplitz $\Longleftrightarrow B_j(\tau_j(t)v, \tau_j(t)w) = B_j(v,w), \quad \forall (v,w) \in \mathcal{V}_j \times \mathcal{V}_j, \quad \forall t \in \Re$.
- B_j is $\tau_j -$ continuous $\Longleftrightarrow \forall (v,w) \in \mathcal{V}_j \times \mathcal{V}_j, \quad B_j(\tau_j(.)v, w)$ is continuous.

Analogously, a form $B' : \mathcal{V}_1 \times \mathcal{V}_2 \to C$ is said (τ_1, τ_2)-Toeplitz, (τ_1, τ_2)-continuous respectively if:

- $B'(\tau_1(t)v_1, \tau_2(t)v_2) = B'(v_1, v_2), \qquad \forall (v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2, \quad \forall t \in \Re$.
- for each $(v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2$, $B'(\tau_1(.)v_1, v_2)$, $B'(v_1, \tau_2(.)v_2)$ are continuous.

We also consider sesquilinear forms $B : \mathcal{W}_1 \times \mathcal{W}_2 \to C$. Such a form is called (τ_1, τ_2)-Hankel if:

$$B(\tau_1(t)w_1, w_2) = B(w_1, \tau_2(-t)w_2), \qquad \forall (w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2, \quad \forall t \geq 0.$$

Fix an algebraic scattering system $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{W}_1, \mathcal{W}_2, \tau_1, \tau_2\}$ and two positive forms $B_1 : \mathcal{V}_1 \times \mathcal{V}_1 \to C$, $B_2 : \mathcal{V}_2 \times \mathcal{V}_2 \to C$, τ_1-Toeplitz and τ_2-Toeplitz respectively. If $B' : \mathcal{V}_1 \times \mathcal{V}_2 \longrightarrow C$ and $B : \mathcal{W}_1 \times \mathcal{W}_2 \to C$ are other two sesquilinear forms then we write:

- $B' \leq (B_1, B_2) \Longleftrightarrow \mid B'(v_1, v_2) \mid^2 \leq B_1(v_1, v_1)\ B_2(v_2, v_2), \ \forall (v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2$.
- $B \prec (B_1, B_2) \Longleftrightarrow \mid B(w_1, w_2) \mid^2 \leq B_1(w_1, w_1)\ B_2(w_2, w_2), \quad \forall (w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2$.

If B_j is τ_j-Toeplitz and τ_j-continuous for $j = 1, 2$ and $B' \leq (B_1, B_2)$ then B' is (τ_1, τ_2)-continuous. Indeed, if $(v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ then we have:

$$\mid B'(\tau_1(t)v_1, v_2) - B'(\tau_1(t_0)v_1, v_2) \mid^2 = \mid B'((\tau_1(t) - \tau_1(t_0))\, v_1, v_2) \mid^2 \leq$$

$$B_1((\tau_1(t) - \tau_1(t_0))\, v_1, ((\tau_1(t) - \tau_1(t_0))\, v_1) \quad B_2(v_2, v_2) =$$

$$[2\ B_1(v_1, v_1) - 2\ Re\ B_1(\tau_1(t)v_1, \tau_1(t_0)v_1)] \quad B_2(v_2, v_2) \longrightarrow 0 \text{ if} t \to t_0$$

Similar considerations holds for (τ_1, τ_2)-Hankel forms. With this notation we can formulate the following:

THEOREM B *For $j = 1, 2$ let $\mathcal{V}_j$ be a vector space, $\mathcal{W}_j$ a subspace, $\tau_j(t) : \mathcal{V}_j \to \mathcal{V}_j$ a group of linear isomorphisms such that $\tau_1(t)\mathcal{W}_1 \subset \mathcal{W}_1$ $\forall t \geq 0$,*

$\tau_2(-t)\mathcal{W}_2 \subset \mathcal{W}_2$ $\forall t \geq 0$ and $\mathcal{V}_j = Lin\{\tau_j(t)\mathcal{W}_j / t \in \Re\}$. Let $B_j : \mathcal{V}_j \times \mathcal{V}_j \to C$ be a positive form τ_j-Toeplitz, τ_j-continuous ($j = 1, 2$) and $B : \mathcal{W}_1 \times \mathcal{W}_2 \to C$ a (τ_1, τ_2)-Hankel form. If $B \prec (B_1, B_2)$ then there exists a form $\tilde{B} : \mathcal{V}_1 \times \mathcal{V}_2 \to C$, (τ_1, τ_2)-Toeplitz such that $\tilde{B} \leq (B_1, B_2)$ and $\tilde{B}(w_1, w_2) = B(w_1, w_2)$, $\forall (w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2$.

Theorem B is a result concerning Toeplitz extensions of generalized Hankel forms in the continuous case. It was first stated and proved (as a consecuense of the discrete version) in [C-S 2] (Theorem 2 of [C-S 2]) where several applications were also given. We are going to give a direct proof independently of the discrete case and based on the theory of unitary extensions of local semigroups of isometries [B].

Proof of Theorem B: In the vector space $E = \mathcal{W}_1 \times \mathcal{W}_2$ we set:

$$\langle (w_1, w_2), (w_1', w_2') \rangle = B_1(w_1, w_1') + B_2(w_2, w_2') + B(w_1, w_2') + \overline{B(w_1', w_2)}$$

It follows that $\langle\ ,\ \rangle : E \times E \to C$ is a sesquilinear form, wich is positive since $B \prec (B_1, B_2)$. By an standard way (quotient and completion) we obtain a Hilbert space $(\mathcal{H}\ ,\ \langle\ ,\ \rangle)$ and a natural operator π from E to a dense subspace of $\mathcal{H}$. The formulaes $\lambda_1 w_1 = \pi(w_1, 0)$, $\lambda_2 w_2 = \pi(0, w_2)$, determines two isometries $\lambda_j \in \mathcal{L}(\mathcal{W}_j, \mathcal{H})$ such that $\mathcal{H} = \lambda_1 \mathcal{W}_1 \bigvee \lambda_2 \mathcal{W}_2$. For each $t \geq 0$ set:

$$\mathcal{D}_t = \lambda_1 \mathcal{W}_1 \bigvee \lambda_2 \tau_2(-t) \mathcal{W}_2$$

If $w_1 \in \mathcal{W}_1$, $w_2 \in \mathcal{W}_2$ and $t \geq 0$ then we have:
$\|\lambda_1 w_1 + \lambda_2 \tau_2(-t) w_2\|^2_{\mathcal{H}}$
$= B_1(w_1, w_1) + B_2(\tau_2(-t)w_2, \tau_2(-t)w_2) + 2ReB(w_1, \tau_2(-t)w_2)$
$= B_1(\tau_1(t)w_1, \tau_1(t)w_1) + B_2(w_2, w_2) + 2ReB(\tau_1(t)w_1, w_2)$
$= \|\lambda_1 \tau_1(t) w_1 + \lambda_2 w_2\|^2_{\mathcal{H}}$.
This allows us to define an isometric operator $V(t)$ with domain $\mathcal{D}_t$ by:

$$V(t)\ (\lambda_1 w_1 + \lambda_2 \tau_2(-t) w_2) = \lambda_1 \tau_1(t) w_1 + \lambda_2 w_2$$

Remark that, $\mathcal{D}_0 = \mathcal{H}$, $V(0) = I$ (the identity operator), if $t, s \geq 0$, then $\mathcal{D}_{t+s} \subset \mathcal{D}_s$, $V_s \mathcal{D}_{t+s} \subset \mathcal{D}_t$ and $V_{t+s} = V_t\ V_s|_{\mathcal{D}t+s}$. Fix $t_0 > 0$ and put $h = \lambda_1 w_1 + \lambda_2 \tau_2(-t_0) w_2$. If $t < t_0$ we have:

$$V(t)\ (\lambda_1 w_1 + \lambda_2 \tau_2(-t_0) w_2) = \lambda_1 \tau_1(t) w_1 + \lambda_2 \tau_2(t - t_0) w_2$$

and then $\|V(t)h - h\|^2 = B_1[\tau_1(t)w_1 - w_1, \tau_1(t)w_1 - w_1] +$
$+ B_2[\tau_2(t - t_0)w_2 - \tau_2(-t_0)w_2, \tau_2(t - t_0)w_2 - \tau_2(-t_0)w_2] +$

$+2\ Re\ B[\tau_1(t)w_1 - w_1, \tau_2(t)\tau_2(-t_0)w_2 - \tau_2(-t_0)w_2]$. By the considerations we have done about continuity it follows that $\|V(t)h - h\| \to 0$ if $t \to 0^+$. Thus, we can ensure that the family $V = \{(V(t), \mathcal{D}_t)/t \geq 0\} \subset \mathcal{L}(\mathcal{H})$ is a local semigroup of isometries in the sense of [B]. Then (see theorem 1 of [B]), V can be extended to a strongly continuous group of unitary operators in a larger Hilbert space $\mathcal{F}$. There exist a Hilbert space $\mathcal{F}$ that contains $\mathcal{H}$ as a closed subspace and a strongly continuous group of unitary operators $\{U(t)/t \in \Re\} \subset \mathcal{L}(\mathcal{F})$ such that $V(t) = U(t)|_{\mathcal{D}_t}$, $\forall t \geq 0$. Define a sesquilinear form $\tilde{B} : \mathcal{V}_1 \times \mathcal{V}_2 \longrightarrow C$ by:

$$\tilde{B}\left(\tau_1(-t)w_1, \tau_2(s)w_2\right) = \langle U(-t)\lambda_1 w_1, U(s)\lambda_2 w_2\rangle$$

It is obvious that $\tilde{B}$ extends B, that $\tilde{B}$ is (τ_1, τ_2)-Toeplitz and for all $w_1 \in \mathcal{W}_1$, $w_2 \in \mathcal{W}_2$, $t, s \in \Re$ we have: $\| \tilde{B}(\tau_1(-t)w_1, \tau_2(s)w_2) \|^2 \leq \| U(-t)\lambda_1 w_1 \|^2 \| U(s)\lambda_2 w_2 \|^2 =$

$\| \lambda_1 w_1 \|^2 \| \lambda_2 w_2 \|^2 = B_1(w_1, w_1)B_2(w_2, w_2)$ and then $\tilde{B} \leq (B_1, B_2)$.

EQUIVALENCE BETWEEN BOTH THEOREMS

We shall now show that theorems A and B are equivalents.

Theorem B implies Theorem A: Assume $\|X\| = 1$ and, for $j = 1, 2$ set $\mathcal{V}_j = \mathcal{F}_j$, $\tau_j(t) = U_j(t)$, $\mathcal{W}_1 = \mathcal{M}_1$, $\mathcal{W}_2 = \bigvee\{U_2(t)\mathcal{H}_2/t \leq 0\}$ and B_j the scalar product in $\mathcal{F}_j$. Let $B : \mathcal{W}_1 \times \mathcal{W}_2 \longrightarrow C$ be given by:

$$B(w_1, w_2) = \langle X\ P_{\mathcal{H}_1}\ w_1\ ,\ w_2\rangle, \qquad (w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2.$$

For $(w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2$ and for all $t \geq 0$ we have:
$B\left(U_1(t)w_1, w_2\right) = \langle XP_{\mathcal{H}_1}W_1(t)w_1, w_2\rangle$
$= \langle T_2(t)XP_{\mathcal{H}_1}w_1, w_2\rangle = \langle XP_{\mathcal{H}_1}w_1, U_2(-t)w_2\rangle = B\left(w_1, U_2(-t)w_2\right)$ and
$\mid B(w_1, w_2) \mid^2 = \mid \langle XP_{\mathcal{H}_1}, w_2\rangle \mid \leq \langle w_1, w_1\rangle\langle w_2, w_2\rangle = B_1(w_1, w_1)B_2(w_2, w_2)$.
Thus, B is (U_1, U_2)-Hankel and $B \prec (B_1, B_2)$. Theorem B, ensures the existence of an extension $\tilde{B} : \mathcal{F}_1 \times \mathcal{F}_2 \to C$ of B such that $\tilde{B}$ is (U_1, U_2)-toeplitz and $\tilde{B} \leq (B_1, B_2)$. In this case, $\tilde{B} \leq (B_1, B_2)$ is equivalent to say that $\tilde{B}(f_1, f_2) \leq \|f_1\|^2_{\mathcal{F}_1}\|f_2\|^2_{\mathcal{F}_2}$ holds for all $(f_1, f_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ and then $\tilde{B}$ is a bounded sesquilinear form with $\| \tilde{B}\| \leq 1$. There exists an operator $Y \in \mathcal{L}(\mathcal{F}_1, \mathcal{F}_2)$ with $\|Y\| = \|\tilde{B}\|$ such that:

$$\tilde{B}(f_1, f_2) = \langle Yf_1, f_2\rangle_{\mathcal{F}_2} \qquad \forall (f_1, f_2) \in \mathcal{F}_1 \times \mathcal{F}_2$$

Since $\tilde{B}$ is (U_1, U_2)-Toeplitz we have:
$\langle YU_1(t)f_1, f_2\rangle = \tilde{B}(U_1(t)f_1, f_2) = \tilde{B}(f_1, U_2(-t)f_2) =$
$= \langle Yf_1, U_2(-t)f_2\rangle = \langle U_2(t)Yf_1, f_2\rangle$. and then $YU_1(t) = U_2(t)Y, \ \forall t \in \Re$.

For $(h_1, h_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ we have $\langle P_{\mathcal{H}_2}^{\mathcal{F}_2} Yh_1, h_2\rangle_{\mathcal{H}_2} = \langle Yh_1, h_2\rangle_{\mathcal{F}_2} = \tilde{B}(h_1, h_2) = B(h_1, h_2) = \langle Xh_1, h_2\rangle_{\mathcal{H}_2}$ and then

$$P_{\mathcal{H}_2}^{\mathcal{F}_2} \, Y|_{\mathcal{H}_1} \; = \; X$$

Since $\|X\| = 1$ and $\|Y\| \leq 1$ it follows that $\|Y\| = 1$ and $A1$ holds.

Using again the fact that $\tilde{B}$ is an extension of B and taking $(w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2$ instead of $(h_1, h_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ we obtain:

$$P_{\mathcal{W}_2}^{\mathcal{F}_2} \, Y|_{\mathcal{W}_1} = X \; P_{\mathcal{H}_1}^{\mathcal{W}_1}$$

This equality together with the known relation $\mathcal{F}_2 \ominus \mathcal{M}_2 = \mathcal{W}_2 \ominus \mathcal{H}_2$ implies $Y\mathcal{M}_1 \subset \mathcal{M}_2$. Setting $Z = Y|_{\mathcal{M}_1}$ A2 follows.

Theorem A implies Theorem B: For $j = 1, 2$ let $(\mathcal{F}_j, \langle , \rangle_j)$ be the Hilbert space generated (after quotient and completion) by the vector space $\mathcal{V}_j$ and the positive form B_j. Let us identify $\mathcal{V}_j$ as a dense subspace of $\mathcal{F}_j$. For each $t \in \Re$ and for all $v \in \mathcal{H}_j$ we have:

$$\|\tau_j(t)v\|_j^2 = B_j(\tau_j(t)v, \tau_j(t)v) = B_j(v, v) = \|v\|_j^2$$

and then $\tau_j(t)$ can be extended to an unitary operator $U_j(t) \in \mathcal{L}(\mathcal{F}_j)$. It is clear that $U_j = \{U_j(t)/t \in \Re\} \subset \mathcal{L}(\mathcal{F}_j)$ is a unitary group. Let us see the continuous property, for each $v \in \mathcal{V}_j$ we have:

$$\|U_j(t)v - v\|_j^2 = 2\|v\|^2 - 2Re\langle U_j(t)v, v\rangle = 2\|v\|^2 - 2ReB_j(\tau_j(t)v, v) \longrightarrow 0 \; if \; t \to 0^+$$

It follows that U_j is strongly continous. If $\mathcal{M}_j$ is the closure of $\mathcal{W}_j$ in $\mathcal{F}_j$ then:

$$U_1(t)\mathcal{M}_1 \subset \mathcal{M}_1 \qquad U_2(-t)\mathcal{M}_2 \subset \mathcal{M}_2 \qquad \forall t \geq 0.$$

$$\mathcal{F}_1 = \bigvee\{U_1(t)\mathcal{M}_1/t \leq 0\} \qquad \mathcal{F}_2 = \bigvee\{U_2(t)\mathcal{M}_2/t \geq 0\}$$

For each $t \geq 0$ set:

$$T_1(t) \; = \; U_1(t)|_{\mathcal{M}_1}, \qquad T_2(t) \; = \; P_{\mathcal{M}_2}^{\mathcal{F}_2} \, U_2(t)|_{\mathcal{M}_2}.$$

Then $T_j = \{T_j(t)/t \geq 0\} \subset \mathcal{L}(\mathcal{M}_j)(j = 1, 2)$ is a strongly continuous semi-group of contractions. This is obvious for T_1. For T_2 we only have to show that

$P^{\mathcal{F}_2}_{\mathcal{M}_2}\, U_2(t)\, P^{\mathcal{F}_2}_{\mathcal{M}_2}\, U_2(s)|_{\mathcal{M}_2} = P^{\mathcal{F}_2}_{\mathcal{M}_2}\, U_2(t)\, U_2(s)|_{\mathcal{M}_2}$, $\forall t, s \geq 0$ but this equality holds since $U_2(t)\mathcal{M}_2^{\perp} \subset \mathcal{M}_2^{\perp}$ for $t \geq 0$.

The form B verifies $|B(w_1, w_2)|^2 \leq \|W_1\|_1^2 \|W_2\|_2^2$, for all $(w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2$ and then it can be extended to a bounded form $B : \mathcal{M}_1 \times \mathcal{M}_2 \to C$ with $\|B\| \leq 1$. There exists an operator $X \in \mathcal{L}(\mathcal{M}_1, \mathcal{M}_2)$ such that $\langle Xm_1, m_2\rangle = B(m_1, m_2)$, and $\|X\| = \|B\|$. For $(w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2$ and $t \geq 0$ we have:

$$\langle XT_1(t)w_1, w_2\rangle_{\mathcal{M}_2} = \langle XU_1(t)w_1, w_2\rangle = B(U_1(t)w_1, w_2) = B(w_1, U_2(-t)w_2)$$

$$= \langle Xw_1, U_2(-t)w_2\rangle_{\mathcal{M}_2} = \langle U_2(t)Xw_1, w_2\rangle_{\mathcal{F}_2} = \langle P^{\mathcal{F}_2}_{\mathcal{M}_2} U_2(t)Xw_1, w_2\rangle_{\mathcal{M}_2}$$

$$= \langle T_2(t)Xw_1, w_2\rangle$$

and then X interwines the semigruoups T_1 and T_2. By Theorem B, X can be lifted to an operator $Y \in \mathcal{L}(\mathcal{F}_1, \mathcal{F}_2)$ that interwines (U_1, U_2) with $\|Y\| = \|X\| = \|B\|$. Setting $\tilde{B}(f_1, f_2) = \langle Yf_1, f_2\rangle_{\mathcal{F}_2}$ we obtain a form $\tilde{B} : \mathcal{F}_1 \times \mathcal{F}_2 \longrightarrow C$ wich verifies the desire conditions.

For an enlightening discussion between generalizations of these theorems in the discrete case we refer to [A.4].

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