A GENERALIZATION OF A THEOREM BY CALABI TO THE PARABOLIC MONGE-AMPÈRE EQUATION

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§1. Introduction

A celebrated result of Calabi [**C**], that generalizes a two dimensional theorem by Jörgens [**J**], asserts that if u is a C^5 convex solution of the elliptic Monge-Ampère equation $\det D^2u = 1$ in $\mathbf{R}^n$ and $n \leq 5$ then u is a quadratic polynomial. This statement was extended to all dimensions by Pogorelov [**P2**] and Cheng and Yau [**Ch-Y**]. Recently, Caffarelli [**Ca3**] proved, by using the regularity theory for the Monge-Ampère equation developed in the fundamental papers [**Ca1**] and [**Ca2**], that this result holds true for viscosity solutions. The purpose of this article is to investigate the validity of results of the same nature for solutions of the parabolic Monge-Ampère equation $-u_t \det D^2u = 1$ in $\mathbf{R}^n \times (-\infty, 0]$. This type of differential operator was first considered by Krylov [**K2**]. It also appears in connection with the problem of the deformation of a surface by means of its Gauss-Kronecker curvature. Indeed, Tso [**T**] solved this problem by noting that the support function to the surface that is deforming satisfies an initial value problem involving that parabolic operator.

The function $u : \mathbf{R}^n \times (-\infty, 0] \to \mathbf{R}$, $u = u(x,t)$, is called parabolically convex if it is continuous, convex in x and nonincreasing in t. By $D^2u(x,t)$ we denote the matrix of second derivatives of u with respect to x and Du denotes the gradient of u with respect to x. We use the standard notation $C^{2k,k}(\Omega)$ to denote the class of functions u such that the derivatives $D_x^i D_t^j u$ are continuous in Ω for $i + 2j \leq 2k$. We set $\mathbf{R}_-^{n+1} = \mathbf{R}^n \times (-\infty, 0]$.

The main result of this paper is the following theorem.

Theorem 1.1. *Let $u \in C^{4,2}(\mathbf{R}_-^{n+1})$ be a parabolically convex solution to the parabolic Monge-Ampère equation*

$$-u_t \det D^2u = 1, \qquad \text{in } \mathbf{R}_-^{n+1}, \tag{1-1}$$

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such that there exist positive constants m_1 *and* m_2 *with*

$$-m_1 \leq u_t(x,t) \leq -m_2, \qquad \text{for all } (x,t) \in \mathbf{R}^{n+1}_{-}. \tag{1-2}$$

Then u *must have the form* $u(x,t) = C_1 t + p(x)$ *where* $C_1 < 0$ *is a constant and* p *is a convex quadratic polynomial.*

To prove this result we use ideas from [**Ca3**] and properties of the cross sections of solutions to the elliptic Monge-Ampère equation established in [**G-H**]. Unlike the elliptic case, in our case viscosity solutions of (1-1) may not be of the form given by Theorem 1.1.

We give a brief outline of the strategy to prove Theorem 1.1 which is based on four elements. The first one is a maximum principle for parabolic Monge-Ampère equations, Theorem 2.1, that has independent interest and generalizes an estimate first proved by Aleksandrov [**A**] which is a crucial building block in the regularity theory for solutions of the elliptic Monge-Ampère equation. The second element are the geometric properties of the cross sections to solutions of elliptic Monge-Ampère equations established in [**G-H**] which are applied to show that the level sets $Q_H = \{(x,t) : u(x,t) < H\}$ are controlled by standard cylinders. The third element in our proof is a variant of a theorem due to Pogorelov, Theorem 2.2. This result introduces very useful device to estimate second derivatives of solutions to the parabolic Monge-Ampère equation. It permits to estimate a quantity that combines u and its second derivatives in x by a quantity involving only the gradient of u (see statement of Theorem 2.2). On the other hand, if appropriate rescalings of u are sufficiently large then by Theorem 2.1 this must happen away from the boundary of Q_H. Hence from the convexity of u and properties of the cross sections we obtain an estimate of the gradient of u. Therefore a combination of these three elements yields uniformly elliptic estimates of the Hessian of u in x in the interior of Q_H. The fourth element is an Evans-Krylov type theorem that from these uniform estimates yields uniform estimates of the Hölder seminorms of the second derivatives of u in x and first derivatives in t. Finally, by using again the results in [**G-H**] we conclude that these Hölder seminorms, calculated on any cylinder of dimensions $H^{1/2}$ in x and H in t, tend to zero as $H \to \infty$ implying that the second derivatives of u in x and the first derivatives in t are constant.

The paper is organized as follows. Section 2 is divided into three subsections. Subsection 2.1 contains a maximum principle for parabolic Monge-Ampère equations. The variant of a Theorem due to Pogorelov is proved in subsection 2.2. A Theorem of Evans-Krylov type for parabolic non-linear equations is the contents of subsection 2.3. The proof of Theorem 1.1 is then carried out in §3. Finally, in §4 we show an example of a viscosity solution to (1-1) that is not of the form given in Theorem 1.1.

§2. Preliminary results

We begin introducing some notation. Given a bounded open set $D \subset \mathbf{R}^{n+1}$ and $t \in \mathbf{R}$, we denote

$$D(t) = \{x : (x,t) \in D\}.$$

Let $t_0 = \inf\{t : D(t) \neq \emptyset\}$. The parabolic boundary of the bounded domain D is defined by

$$\partial_p D = \left(\overline{D}(t_0) \times \{t_0\}\right) \cup \bigcup_{t \in \mathbf{R}} \left(\partial D(t) \times \{t\}\right),$$

where $\overline{D}$ denotes the closure of D and $\partial D(t)$ denotes the boundary of $D(t)$. We say that the set $D \subset \mathbf{R}^{n+1}$ is a bowl-shaped domain if $D(t)$ is convex for each t and $D(t_1) \subset D(t_2)$ for $t_1 \leq t_2$.

We recall the definition of cross section of a convex function, [**G-H**]. Let $u : \mathbf{R}^n \to \mathbf{R}$ be a convex function that for simplicity is assumed smooth. A cross section of u at the point $x_0 \in \mathbf{R}^n$ and with height $t > 0$ is the convex set defined by

$$S_u(x_0, t) = \{x : u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + t\}.$$

§2.1 A Maximum Principle.

We recall that if $u : \Omega \to \mathbf{R}$, $\Omega \subset \mathbf{R}^n$ open, $u \in C(\Omega)$, then the normal mapping of u is the set valued function $\chi_u : \Omega \to \{E : E \subset \mathbf{R}^n\}$ defined by

$$\chi_u(y) = \{p : u(x) \geq u(y) + p \cdot (x - y), \quad \forall x \in \Omega\}.$$

If $D \subset \mathbf{R}^{n+1}$ open is a bounded bowl-shaped domain and $u : D \to \mathbf{R}$ is continuous then the parabolic normal mapping of u is the set valued function $\mathcal{P}_u : D \to \{E : E \subset \mathbf{R}^{n+1}\}$ defined by

$$\mathcal{P}_u(x_0, t_0) =$$

$$\{(p,h) : u(x,t) \geq u(x_0,t_0) + p \cdot (x - x_0),\ \forall x \in D(t), \text{with } t \leq t_0,\ h = p \cdot x_0 - u(x_0,t_0)\},$$

and if $E \subset D$ then $\mathcal{P}_u(E) = \cup_{(x,t) \in E} \mathcal{P}_u(x,t)$. By $|\cdot|_k$ we denote the Lebesgue measure in $\mathbf{R}^k$. The class of sets $E \subset D$ such that $\mathcal{P}_u(E)$ is Lebesgue measurable is a Borel σ-algebra and the parabolic Monge-Ampère measure associated with u defined by $|\mathcal{P}_u(E)|_{n+1}$ is a Borel measure, see [**W-W**].

The following theorem has independent interest and it is an extension to the parabolic case of a result first proved by Aleksandrov, Theorem B, [**A**]; see also Lemma 3.5 of [**R-T**].

Theorem 2.1. *Let $D \subset \mathbf{R}^{n+1}$ be an open bounded bowl-shaped domain and $u \in C\left(\overline{D}\right)$ a parabolically convex function with $u = 0$ on $\partial_p D$. If $(x_0, t_0) \in D$ then*

$$|u(x_0, t_0)|^{n+1} \leq C_n \, dist\,(x_0, \partial D(t_0)) \;\; diam\,(D(t_0))^{n-1} \, |\mathcal{P}_u\,(D_{t_0})\,|_{n+1},$$

where $D_{t_0} = D \cap \{(x,t) : t \leq t_0\}$ and C_n is a constant depending only on the dimension n.

Proof. We may assume that $u(x_0, t_0) = -1$. Let $v(x)$ be the function whose graph is the $n+1$-dimensional inverted cone with base $\overline{D(t_0)} \times \{0\}$ and vertex at $(x_0, -1)$, i.e., $v(x_0) = -1$ and $v = 0$ on $\partial D(t_0)$. Let $p \in \chi_v(x_0)$ and the interval

$$I(p) = \left(p \cdot x_0 + \max_{x \in \overline{D(t_0)}} p \cdot (x - x_0)\,,\, p \cdot x_0 + 1 \right).$$

We claim that if $h \in I(p)$ then $(p,h) \in \mathcal{P}_u\,(D_{t_0})$. Indeed, we have $p \cdot x - h < p \cdot x - \left(p \cdot x_0 + \max_{x \in \overline{D(t_0)}} p \cdot (x - x_0)\right) \leq 0$ for $x \in \overline{D(t_0)}$ and since D is a bowl-shaped domain, the inequality holds for $x \in \overline{D(t)}$, $t \leq t_0$. Therefore $p \cdot x - h < u$ on $\partial_p D_{t_0}$. Also, $p \cdot x_0 - h \geq p \cdot x_0 - (p \cdot x_0 + 1) = -1$ implies that $p \cdot x_0 - h \geq u(x_0, t_0)$. Hence, we now slide the hyperplane $p \cdot x - h$ in the direction of $t < t_0$ until it touches the graph of u for the first time, say at a point (x_1, t_1) with $t_1 \leq t_0$. At this point we have $u(x_1, t_1) = p \cdot x_1 - h$, $p \cdot x - h \leq u(x, t_1)$ for $x \in D(t_1)$. This means that $(p,h) \in \mathcal{P}_u(x_1, t_1)$ and (x_1, t_1) is not on the parabolic boundary of D_{t_0}. Thus the claim is proved. Next we claim that

$$|\mathcal{P}_u\,(D_{t_0})\,|_{n+1} \geq \frac{C_n}{\text{dist}\,(x_0, \partial D(t_0)) \;\text{diam}\,(D(t_0))^{n-1}}.$$

By translating and rotating $D(t_0)$ we may assume that x_0 lies on the x_n-axis and it reaches its distance to $\partial D(t_0)$ at a point lying on the positive x_n-axis. We have that (see proof of Lemma 3.5 in [**R-T**])

(i) $\chi_v(x_0)$ is convex.
(ii) $p = \dfrac{e_n}{\text{dist}\,(x_0, \partial D(t_0))} \in \chi_v(x_0)$ with $e_n = (0, \cdots, 0, 1)$.
(iii) the n-dimensional ball $B_{1/\text{diam}(D(t_0))}(0) \subset \chi_v(x_0)$.

These imply that $\Gamma \subset \chi_v(x_0)$ where Γ is the n-dimensional right-cone whose base is the $(n-1)$-dimensional ball $B_{1/3\text{diam}(D(t_0))}(0)$ perpendicular to the x_n axis and whose vertex is at the point $\dfrac{e_n}{3\text{dist}\,(x_0, \partial D(t_0))}$. Let $p \in \Gamma$, $p = (p', p_n)$ and

$x_0 = (x_0', x_n^0)$. Note that $p_n > 0$. We have that

$$\max_{x\in \overline{D(t_0)}} p\cdot(x-x_0) = \max_{x\in\overline{D(t_0)}} \left(p_n(x_n - x_n^0) + p'\cdot(x'-x_0')\right)$$
$$\leq \frac{1}{3\mathrm{dist}\,(x_0,\partial D(t_0))}\mathrm{dist}\,(x_0,\partial D(t_0)) + \frac{1}{3\mathrm{diam}\,(D(t_0))}\mathrm{diam}\,(D(t_0)) = \frac{2}{3},$$

since $x_n - x_n^0 \leq \mathrm{dist}\,(x_0,\partial D(t_0))$ for all $x\in D(t_0)$ with $x_n \geq x_n^0$. Thus $|I(p)| \geq \frac{1}{3}$ for $p\in\Gamma$ and by integrating in $n+1$ dimensions we obtain

$$|\mathcal{P}_u\left(D_{t_0}\right)|_{n+1} \geq \frac{1}{3}|\Gamma|_n = C_n \frac{1}{\mathrm{dist}\,(x_0,\partial D(t_0))\ \mathrm{diam}\,(D(t_0))^{n-1}},$$

which completes the proof. $\square$

Lemma 2.1. *Let $D\subset\mathbf{R}_-^{n+1}$ be an open bounded bowl-shaped domain with $B_{\alpha_n}(0)\times[-\alpha_n,0]\subset D\subset B_1(0)\times[-1,0]$ and u a parabolically convex smooth solution of* (1-1) *in D such that $u=0$ on $\partial_p D$. Then*

$$\left|\min_D u(x,t)\right| \approx C,$$

with C a dimensional constant.

Proof. By Theorem 2.1, $|\min_D u(x,t)| \leq C$. To prove the reverse inequality let $w(x,t) = \epsilon\left(-t+|x|^2-\alpha\right)$. If $\epsilon = 2^{-n/(n+1)}$ then w satisfies (1-1) and if $\alpha=\alpha_n^2$ then $w\geq 0$ on $\partial_p D$. Then by the comparison principle, Proposition 2.3 [**W-W**], $u\leq w$ and we are done. $\square$

§2.2 A variant of a Theorem by Pogorelov.

We shall prove the following variant of a beautiful Theorem due to Pogorelov (see Theorem 2, [**P1**]).

Theorem 2.2. *Let $D\subset\mathbf{R}^{n+1}$ be a bounded open bowl-shaped domain and $v\in C(\overline{D})$ such that v is parabolically convex in D. Suppose that v is a smooth solution of*

$$\begin{aligned} -v_t \det D^2 v &= 1, && \text{in } \overline{D}\setminus\partial_p D\\ v(x,t) &= 0, && \text{for } (x,t)\in\partial_p D. \end{aligned}$$

Let $\alpha\in\mathbf{R}^n$, $|\alpha|=1$,

$$w(x,t) = |v(x,t)| D_{\alpha\alpha} v(x,t) e^{\frac{1}{2}(D_\alpha v(x,t))^2},$$

and $M = \max_{\bar{D}} w(x,t)$. *Then there exists* $P \in \overline{D} \setminus \partial_p D$ *where the maximum* M *is attained and the following inequality holds*

$$M \leq C_n \,(1 + |D_\alpha v(P)|)\, e^{\frac{1}{2}(D_\alpha v(P))^2},$$

with C_n *a positive constant depending only on the dimension* n.

Proof. For $M \in \mathbf{R}^{n\times n}$ positive definite, let $F(M) = \log(\det M)$. We have

$$(F_{ij}) = \frac{\partial F}{\partial M_{ij}} = M^{-1}, \quad \text{and} \quad \frac{\partial^2 F}{\partial M_{ij}\partial M_{kl}} = F_{ij,kl} = -F_{ik}F_{jl}.$$

Since $v = 0$ on $\partial_p D$ and v is strictly convex in $D \setminus \partial_p D$, if follows that the maximum M is attained at some $P \in \overline{D} \setminus \partial_p D$, $P = (p,s)$. Since $D^2 v(P) > 0$, there exists an unimodular matrix O, i.e., $\det O = 1$, such that $O^t D^2 v(P) O$ is diagonal and if $\bar{v}(x,t) = v(Ox,t)$ then $D_1 \bar{v}(x,t) = D_\alpha v(Ox,t)$ and $D_{11}\bar{v}(x,t) = D_{\alpha\alpha} v(Ox,t)$; in particular, $D^2\bar{v}(p',s)$ is diagonal where $p' = O^{-1}p$. To prove this statement, we first rotate the coordinates to have α as one of the axis (we shall omit the variable t since it is irrelevant for this purpose). That is, let Q be an orthogonal matrix such that $Qe_1 = \alpha$, and first let $u(x) = v(Qx)$. Then the first column of Q is the vector α and we have $D_1 u(x) = (D_\alpha v)(Qx)$ and $D_{11}u(x) = (D_{\alpha\alpha} v)(Qx)$. Next, given $A = (a_{ij})$, an $n \times n$ matrix positive definite and symmetric, consider

$$B = \begin{bmatrix} 1 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & -\frac{a_{14}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

We have

$$B^t A B = \begin{bmatrix} a_{11} & 0 \\ 0 & B_1 \end{bmatrix},$$

where B_1 is an $(n-1)\times(n-1)$ matrix. Since A is positive definite and symmetric, it follows that B_1 is also positive definite and symmetric. Hence there exists an orthogonal matrix O_1 such that $O_1^t B_1 O_1$ is diagonal. Let

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & O_1 \end{bmatrix}.$$

Now, we choose $A = \left(D^2 u\right)(Q^t P)$ and set $\bar{v}(x) = u(B\mathcal{O}x)$. Then $D^2\bar{v}((B\mathcal{O})^{-1} Q^t P)$ is diagonal. Combining the changes of coordinates, the matrix $O = QB\mathcal{O}$ does the job.

Therefore, we may assume that $\alpha = (1, 0, \cdots, 0)$ and so

$$w(x,t) = |v(x,t)| D_{11}v(x,t) e^{\frac{1}{2}(D_1 v(x,t))^2},$$

and the matrix $D^2 v(P)$ is diagonal.

Let L be the linearized operator at P

$$L = \frac{1}{v_t(P)} \frac{\partial}{\partial t} + F_{ij}\left(D^2 v(P)\right) D_{ij}.$$

Since w attains its maximum at P, it follows that the function

$$h = \log |v| + \log D_{11} v + \frac{1}{2}\left(D_1 v\right)^2$$

also attains its maximum at P, and consequently

$$Dh(P) = 0; \qquad h_t(P) \geq 0; \qquad \text{and} \qquad D^2 h(P) \leq 0. \tag{2-1}$$

Since $\left(F_{ij}\left(D^2 v(P)\right)\right)$ is diagonal

$$L(h)(P) = \frac{1}{v_t(P)} \frac{\partial h}{\partial t}(P) + F_{ii}\left(D^2 v(P)\right) D_{ii} h(P) \leq 0. \tag{2-2}$$

Now

$$D_i h = \frac{D_i v}{v} + \frac{D_{11i} v}{D_{11} v} + D_1 v\, D_{1i} v, \tag{2-3}$$

$$D_{ij} h = \frac{D_{ij} v}{v} - \frac{D_i v D_j v}{v^2} + \frac{D_{11ij} v}{D_{11} v} - \frac{D_{11i} v D_{11j} v}{(D_{11} v)^2} + D_{1j} v D_{1i} v + D_1 v D_{1ij} v, \tag{2-4}$$

$$h_t = \frac{v_t}{v} + \frac{D_{11t} v}{D_{11} v} + D_1 v D_{1t} v. \tag{2-5}$$

By replacing (2-4) and (2-5) into (2-2) we obtain the inequality

$$\frac{1}{v_t}\left(\frac{v_t}{v} + \frac{D_{11t} v}{D_{11} v} + D_1 v D_{1t} v\right)$$
$$+ F_{ii}\big(D^2 v(P)\big)\left(\frac{D_{ii} v}{v} - \frac{(D_i v)^2}{v^2} + \frac{D_{11ii} v}{D_{11} v} - \frac{\left(D_{11i} v\right)^2}{\left(D_{11} v\right)^2} + \left(D_{1i} v\right)^2 + D_1 v D_{1ii} v\right) \leq 0,$$

valid at the point P. By collecting terms we obtain

(2-6)
$$\frac{1}{v}+\frac{1}{D_{11}v}L\left(D_{11}v\right)+D_1vL\left(D_1v\right)+F_{ii}\left(\frac{D_{ii}v}{v}-\frac{\left(D_iv\right)^2}{v^2}-\frac{\left(D_{11i}v\right)^2}{\left(D_{11}v\right)^2}+\left(D_{1i}v\right)^2\right)\leq 0$$

valid at the point P. Since v satisfies $-v_t\det D^2v=1$ in D, by differentiating with respect to x_1 we obtain that $L\left(D_1v\right)(P)=0$. Next, let us compute $L\left(D_{11}v\right)(P)$. We have that $\log\left(-v_t\right)+\log\det D^2v=0$ in D and by differentiating this equation with respect to x_1 we obtain $\frac{1}{v_t}\left(D_1v\right)_t+F_{ij}D_{ij}\left(D_1v\right)=0$. Differentiating once more yields

$$\frac{1}{v_t}\left(D_{11}v\right)_t-\frac{\left(D_1v_t\right)^2}{v_t^2}+F_{ij}D_{ij}\left(D_{11}v\right)+F_{ij,kl}D_{kl}\left(D_1v\right)D_{ij}\left(D_1v\right)=0.$$

Therefore, at P we have

$$L\left(D_{11}v\right)=\frac{\left(D_{1t}v\right)^2}{v_t^2}+F_{ik}F_{jl}D_{kl1}vD_{ij1}v,$$

and noting again that $F_{ij}\left(D^2v(P)\right)=(D^2v)^{-1}(P)$ is diagonal, from (2-6) we obtain the inequality

$$\begin{aligned}&\frac{n+1}{v}+\frac{1}{D_{11}v}\left(\frac{\left(D_{1t}v\right)^2}{v_t^2}+\left(D_{ii}v\right)^{-1}\left(D_{jj}v\right)^{-1}\left(D_{ij1}v\right)^2\right)\\&+\frac{1}{D_{ii}v}\left(-\frac{\left(D_iv\right)^2}{v^2}-\frac{\left(D_{11i}v\right)^2}{\left(D_{11}v\right)^2}+\left(D_{1i}v\right)^2\right)\leq 0,\end{aligned}$$

that implies

$$\text{(2-7)}\quad \frac{n+1}{v}+\sum_{i=1}^n\sum_{j=2}^n\frac{\left(D_{ij1}v\right)^2}{D_{11}vD_{ii}vD_{jj}v}-\sum_{i=1}^n\frac{\left(D_iv\right)^2}{\left(D_{ii}v\right)v^2}+\sum_{i=1}^n\frac{\left(D_{1i}v\right)^2}{D_{ii}v}\leq 0,\qquad \text{at } P.$$

Since $D_ih(P)=0$ and $D^2v(P)$ is diagonal, it follows from (2-3) that

$$\frac{D_iv}{v}=-\frac{D_{11i}v}{D_{11}v},\qquad \text{at } P,\qquad i=2,\cdots,n.$$

Hence and (2-7) we get

$$\frac{n+1}{v}-\frac{\left(D_1v\right)^2}{\left(D_{11}v\right)v^2}+D_{11}v\leq 0.$$

By multiplying the last expression by $v^2 D_{11} v e^{(D_1 v)^2}$ we obtain

$$w^2 - (n+1) w e^{\frac{1}{2}(D_1 v)^2} - (D_1 v)^2 e^{(D_1 v)^2} \le 0,$$

which implies the inequality

$$w^2 \le C(n) \left(1 + (D_1 v)^2\right) e^{(D_1 v)^2}$$

valid at the point P. This completes the proof of the theorem. □

§**2.3 An Evans-Krylov type Theorem.**

Given $0 < \alpha < 1$ and a domain $D \subset \mathbf{R}^{n+1}$ we use the notation

$$[u]_{\alpha,D} = \sup_{(x,t)\neq(y,s)\in D} \frac{|u(x,t) - u(y,s)|}{\left(|x-y|^2 + |t-s|\right)^{\alpha/2}},$$

and

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(D)} = \sum_{2i+j\le 2} \|D_t^i D_x^j u\|_{C(D)} + [D^2 u]_{\alpha,D} + [u_t]_{\alpha,D},$$

with $\|u\|_{C(D)} = \sup_D |u(x,t)|$. We use the notation $\mathbf{R}_s^{n\times n}$ to denote the space of real symmetric $n \times n$ matrices.

The following theorem is stated in the form needed for the proof of Theorem 1.1.

Theorem 2.3. *Let $D = B_1(0) \times (-1, 0]$, and $u(x,t)$ is a $C^{4,2}(D)$ solution of the equation*

$$F\left(u_t, D^2 u\right) = 0 \tag{2-8}$$

in D, where $F = F(q, M)$ is defined for all $(q, M) \in \mathbf{R} \times \mathbf{R}_s^{n\times n}$ with $F(\cdot, M) \in C^1(\mathbf{R})$, for each $M \in \mathbf{R}_s^{n\times n}$, and $F \in C^2(\mathbf{R} \times \Omega)$, for some $\Omega \subset \mathbf{R}_s^{n\times n}$ neighborhood of $D^2 u(D)$. Suppose that:

1. *F is uniformly parabolic, i.e., there exist positive constants λ, Λ such that*

$$-\lambda \le F_q(q, M) \le -\Lambda,$$

$$\lambda \|N\| \le F(q, M+N) - F(q, M) \le \Lambda \|N\|,$$

   for all $q \in \mathbf{R}$ and $M, N \in \mathbf{R}_s^{n\times n}$ with $N \ge 0$.
2. *$F(q, M)$ is concave with respect to M.*

If $\|u\|_{C^{2,1}(D)} \le K$, then there exist positive constants C, depending only on λ, Λ, n, K and $F(0,0)$, and $0 < \alpha = \alpha(\lambda, \Lambda, n) < 1$ such that

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(D_{1/2})} \le C,$$

where $D_{1/2} = B_{1/2}(0) \times (-\frac{1}{2}, 0]$.

Proof. By the smoothness of F on the range of D^2u and differentiating (2-8) with respect to t we obtain

$$F_q(u_t, D^2u)\,(u_t)_t + F_{ij}(u_t, D^2u) D_{ij}\,(u_t) = 0,$$

where $F_{ij} = \dfrac{\partial F}{\partial M_{ij}}$. Dividing the last equation by F_q by (1) we obtain a uniformly parabolic equation and by Harnack inequality, [**K-S**], we obtain

$$[u_t]_{\gamma, D_{3/4}} \le C\|u_t\|_{L^\infty(D)},$$

where $D_{3/4} = B_{3/4}(0) \times (-3/4, 0]$ and some $0 < \gamma < 1$. For the estimation of the second x-derivatives fix t. Then $v(x) = u(x,t)$ satisfies $G(x, D^2v(x)) = F(u_t(x,t), D^2u(x,t)) = 0$. By Theorem 8.1 in [**Ca-C**], we have the estimate $\|D^2v\|_{C^\beta(B_{1/2}(0))} \le C$ uniformly in t for some $0 < \beta < 1$. To show that D^2u is Hölder continuous in t we note that by differentiating (2-8) with respect to x_k we get that D_ku satisfies

$$F_q(u_t, D^2u)\,(D_ku)_t + F_{ij}(u_t, D^2u) D_{ij}\,(D_ku) = 0,$$

and as before we get

$$[Du]_{\alpha, D_{1/2}} \le C\|Du\|_{L^\infty(D)}.$$

We then have $|Du(x,t_1) - Du(x,t_2)| \le C|t_1 - t_2|^\alpha$ and $|D^2u(x_1,t) - D^2u(x_2,t)| \le C|x_1 - x_2|^\beta$. This implies $|D^2u(x,t_1) - D^2u(x,t_2)| \le C|t_1 - t_2|^{\frac{\alpha\beta}{1+\beta}}$, [**L-S-U**] p. 78, and the desired Hölder continuity follows. □

§3. Proof of Theorem 1.1

We can assume throughout the proof that

$$u(0,0) = 0; \quad Du(0,0) = 0; \quad D^2u(0,0) = Id; \quad u_t(0,0) = -1.$$

In fact, we first show that we can assume $u_t(0,0) = -1$. Let $v(x,t) = u(\beta x, \alpha t)$ where β and α are positive numbers. Then $-v_t(x,t)\ \det D^2 v(x,t) = \alpha\,\beta^{2n}$, and we pick α and β such that $\alpha\,\beta^{2n} = 1$ and $v_t(0,0) = \alpha u_t(0,0) = -1$. Next, let $g(x,t) = v(x,t) - v(0,0) - Dv(0,0)\cdot x$. We have $g(0,0) = 0, Dg(0,0) = 0, g_t(0,0) = -1$, and $-g_t \det D^2 g(x,t) = 1$. Since $v(x,t)$ is parabolically convex, g is parabolically convex and

$$g(x,t) \geq v(x,0) - v(0,0) - Dv(0,0)\cdot x \geq 0.$$

There exists an orthogonal matrix O such that

$$O^t D^2 g(0,0) O = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

where $d_i > 0, i = 1, \cdots, n$. Let $w(x,t) = g(Ox,t)$. Then $D^2 w(x,t) = O^t D^2 g(Ox,t) O$, $w(0,0) = 0, Dw(x,t) = O^t(Dg)(Ox,t)$ and hence $Dw(0,0) = 0$. Now let $\bar{w}(x,t) = w\left(\frac{x_1}{\sqrt{d_1}}, \cdots, \frac{x_n}{\sqrt{d_n}}, t\right)$. Then $D^2\bar{w}(x,t) = \left(\frac{1}{\sqrt{d_i}\sqrt{d_j}} w_{ij}\left(\frac{x_1}{\sqrt{d_1}}, \cdots, \frac{x_n}{\sqrt{d_n}}, t\right)\right)$ and in particular $D^2\bar{w}(0,0) = Id$. Since $\det D^2 g(0,0) = \frac{1}{-g_t(0,0)} = 1$, we get that $d_1 \cdots d_n = 1$ and we then obtain

$$\begin{aligned} &-\bar{w}_t(x,t) \det D^2 \bar{w}(x,t) \\ &= -g_t\left(O\left(\frac{x_1}{\sqrt{d_1}}, \cdots, \frac{x_n}{\sqrt{d_n}}\right), t\right) \frac{1}{d_1 \cdots d_n} \left(\det D^2 g\right) \left(O\left(\frac{x_1}{\sqrt{d_1}}, \cdots, \frac{x_n}{\sqrt{d_n}}\right), t\right) = 1. \end{aligned}$$

This completes the proof of the claim.

Given $H > 0$, let

$$Q_H = \{(x,t) : u(x,t) < H\} \quad \text{and} \quad Q_H(t_0) = \{x : (x,t_0) \in Q_H\}. \tag{3-1}$$

Let x_H be the baricenter of $Q_H(0)$, E is the ellipsoid of minimum volume containing $Q_H(0)$ with center x_H and T_H is an affine transformation that normalizes $Q_H(0)$, that is $T_H(E) = B_1(0)$ and

$$B_{\alpha_n}(0) \subset T_H\left(Q_H(0)\right) \subset B_1(0),$$

with $\alpha_n = n^{-3/2}$. For the existence of T_H see [**P3**], p. 90.

The following lemma gives an estimate for the size of Q_H.

Lemma 3.1. *Let u be parabolically convex in $\mathbf{R}^{n+1}_-$, $u(0,0)=0$, $Du(0,0)=0$, and satisfying* (1-2). *Let Q_H be given be* (3-1).

Then there exist constants ϵ_0, ϵ_1 and ϵ_2 such that for all $H>0$

$$\epsilon_0\, E \times [-\epsilon_1\, H, 0] \subset Q_H \subset E \times [-\epsilon_2\, H, 0]\,. \tag{3-2}$$

Proof. By (1-2), it follows that $u(x,t) \geq u(x,0) - m_2 t$ for $t \leq 0$. Since $u(x,0) \geq 0$ for all x, we obtain $u(x,t) \geq H$ for $t < -\dfrac{H}{m_2}$ or $x \notin E$. Thus the second inclusion in (3-2) follows with $\epsilon_2 = 1/m_2$.

Due to the normalization $u(0,0)=0$ and $Du(0,0)=0$ we have that $Q_H(0)$ is a section of the convex function $u(x,0)$ at $x=0$, i.e., $Q_H(0) = S_{u(x,0)}(0,H)$. In particular, from Lemma 2.1 of [**G-H**] we have for $0<\lambda<1$ that $\lambda \alpha_n E \subset \lambda Q_H(0) \subset Q_{(1-(1-\lambda)\frac{\alpha_n}{2})H}(0)$. If $(x,t) \in \lambda \alpha_n E \times [-\epsilon_1 H, 0]$ then

$$\begin{aligned} u(x,t) = u(x,0) - \int_t^0 u_t(x,\tau)\, d\tau &\leq \left(1-(1-\lambda)\frac{\alpha_n}{2}\right) H - m_1 t \\ &\leq \left(1-(1-\lambda)\frac{\alpha_n}{2} + m_1\epsilon_1\right) H < H \end{aligned}$$

for λ and ϵ_1 sufficiently small. □

Note that $u=H$ on $\partial_p Q_H$.

Now let

$$\mathfrak{T}_H(x,t) = \left(T_H x, \frac{t}{H}\right), \qquad \mathfrak{T}_H(Q_H) = Q_H^*.$$

Then (3-2) implies

$$B_{\epsilon_0}(0) \times [-\epsilon_1, 0] \subset \mathfrak{T}_H\left(Q_H\right) \subset B_1(0) \times [-\epsilon_2, 0]\,. \tag{3-3}$$

Let

$$u^*(y,s) = \frac{1}{\gamma} u\left((\mathfrak{T}_H)^{-1}(y,s)\right) - \frac{H}{\gamma},$$

where $\gamma>0$ is a constant to be determined in a moment. We have $(\mathfrak{T}_H)^{-1}(y,s) = \left(T_H^{-1} y, Hs\right)$,

$$\frac{\partial u^*}{\partial s}(y,s) = \frac{1}{\gamma}\frac{\partial u}{\partial t}\left(T_H^{-1}y, Hs\right) H,$$

and

$$D^2 u^*(y,s) = \frac{1}{\gamma}\left(T_H^{-1}\right)^t \left(D^2 u\right)\left(T_H^{-1} y, Hs\right) T_H^{-1}.$$

We pick γ such that

$$-u^*_s \det D^2 u^* = 1,$$

which amounts

$$\frac{H}{\gamma^{n+1}} |\det T_H|^{-2} = 1. \tag{3-4}$$

On the other hand, the function $u^*(y,0) = \dfrac{1}{\gamma} u\left(T_H^{-1} y, 0\right) - \dfrac{H}{\gamma}$ is convex and $u^*(y,0) = 0$ for $y \in \partial\left(T_H(Q_H(0))\right)$. Since $T_H(Q_H(0))$ is normalized and

$$\det D^2 u^*(y,0) = \frac{1}{\gamma^n} |\det T_H|^{-2} \left(-\frac{1}{u_t(T_H^{-1} y, 0)} \right),$$

from (1-2) it follows that

$$\frac{1}{m_1} \frac{1}{\gamma^n} |\det T_H|^{-2} \leq \det D^2 u^*(y,0) \leq \frac{1}{m_2} \frac{1}{\gamma^n} |\det T_H|^{-2}. \tag{3-5}$$

Hence, if μ is the measure with density $\det D^2 u^*(y,0)$ then we have the doubling property

$$\mu\left(T_H(Q_H(0))\right) \leq 2^n \, \frac{m_1}{m_2} \, \mu\left(\frac{1}{2} \, T_H(Q_H(0)) \right).$$

We may then apply Proposition 1.1 of [**G-H**] to obtain

$$\mu\left(T_H(Q_H(0))\right) \approx \left| \min_{T_H(Q_H(0))} u^*(y,0) \right|^n,$$

with comparison constants depending only on the dimension n and the ratio m_1/m_2. Since $u(0,0) = 0$ and $u \geq 0$, we have that $\min_{T_H(Q_H(0))} u^*(y,0) = -\dfrac{H}{\gamma}$. On the other hand, by (3-5) and the normalization of $Q_H(0)$ we get

$$\mu\left(T_H(Q_H(0))\right) = \int_{T_H(Q_H(0))} \det D^2 u^*(y,0)\, dy \approx \frac{1}{\gamma^n} |\det T_H|^{-2}.$$

Therefore $\left(\dfrac{H}{\gamma}\right)^n \approx \dfrac{1}{\gamma^n} |\det T_H|^{-2}$, and from (3-4) we obtain $\left(\dfrac{H}{\gamma}\right)^n \approx \dfrac{\gamma}{H}$, which yields

$$\frac{H}{\gamma} \approx C = C(n, m_1, m_2). \tag{3-6}$$

Given $\epsilon > 0$, let $\Omega_\epsilon = \{(x,t) : u^*(x,t) < -\epsilon\}$. We claim that

$$|Du^*(x,t)| \leq C(\epsilon), \quad \text{for } (x,t) \in \Omega_\epsilon. \tag{3-7}$$

In fact, we apply Theorem 2.1 to the function u^* in the bowl-shaped domain Q_H^*. If $(x_0,t_0) \in \Omega_\epsilon$ then

$$\begin{aligned}
&\epsilon < |u^*(x_0,t_0)| \\
&\leq C_n \mathrm{dist}(x_0, \partial Q_H^*(t_0))^{1/n+1} \mathrm{diam}\,(Q_H^*(t_0))^{(n-1)/(n+1)} \,|\cup_{t\leq t_0} Q_H^*(t)| \\
&\leq C_n \mathrm{dist}(x_0, \partial Q_H^*(t_0))^{1/n+1},
\end{aligned}$$

by (3-3). Hence $\mathrm{dist}(x_0, \partial Q_H^*(t_0)) \geq C(\epsilon)$. The function $u^*(x,t_0)$ is convex in $Q_H^*(t_0)$ and $u^*(x,t_0) = 0$ on $\partial Q_H^*(t_0)$. Hence by Lemma 1.1 of [**G-H**] we obtain

$$|Du^*(x_0,t_0)| \leq \frac{-u^*(x_0,t_0)}{\mathrm{dist}(x_0, \partial Q_H^*(t_0))} \leq \frac{C}{C(\epsilon)},$$

where we used the fact that $-u^*(x_0,t_0) \leq C$. Thus (3-7) is proved.

Next, we prove that if $|\alpha| = 1$ then

$$D_{\alpha\alpha} u^*(x,t) \leq C(\epsilon), \qquad \forall (x,t) \in \Omega_{3\epsilon}. \tag{3-8}$$

In fact, consider the function $v(x,t) = u^*(x,t) + 2\epsilon$. We have

$$\begin{aligned}
-v_t \det D^2 v &= 1, \qquad &&\text{in } \Omega_{2\epsilon} \\
v &= 0, \qquad &&\text{on } \partial_p \Omega_{2\epsilon}.
\end{aligned}$$

We apply Theorem 2.2 to v on the set $\Omega_{2\epsilon}$ and we obtain

$$\max_{\overline{\Omega_{2\epsilon}}} h(x,t) \leq C_n \left(1 + |D_\alpha v|\right) e^{\frac{1}{2}(D_\alpha v(P))^2},$$

where $h(x,t) = |v(x,t)| D_{\alpha\alpha} v(x,t) e^{\frac{1}{2}(D_\alpha v(x,t))^2}$, and $h(P) = \max_{\overline{\Omega_{2\epsilon}}} h(x,t)$ for some $P \in \overline{\Omega_{2\epsilon}} \setminus \partial_p \Omega_{2\epsilon}$. Since $\Omega_{2\epsilon} \subset \Omega_\epsilon$, by (3-7) we get $|D_\alpha v(P)| = |D_\alpha u^*(P)| \leq C(\epsilon)$, and consequently

$$h(x,t) = |v(x,t)| D_{\alpha\alpha} u^*(x,t) e^{\frac{1}{2}(D_\alpha u^*(x,t))^2} \leq C_n C'(\epsilon), \qquad \forall (x,t) \in \Omega_{2\epsilon}. \tag{3-9}$$

If $(x,t) \in \Omega_{3\epsilon}$ then $v(x,t) = u^*(x,t) + 2\epsilon < -3\epsilon + 2\epsilon = -\epsilon$, that is $|v(x,t)| > \epsilon$ in $\Omega_{3\epsilon}$ and from (3-9) we obtain

$$\epsilon D_{\alpha\alpha} u^*(x,t) \le C_n C'(\epsilon), \qquad \forall (x,t) \in \Omega_{3\epsilon}.$$

This yields (3-8).

Recall that E is the ellipsoid of minimum volume containing $Q_H(0)$ centered at x_H the baricenter of $Q_H(0)$. Note that if O is a rotation then $O(Q_H(0)) = \{z : u(O^{-1}z, 0) < H\}$. By changing u by $u(O^{-1}\cdot, \cdot)$, we may assume that the axis of the ellipsoid E lie on the coordinate axes. If $T = T_H$ is an affine transformation that normalizes $Q_H(0)$ then $T(E) = B_1(0)$, $T(x_H) = 0$ and $Tx = A\,(x - x_H)\,, A = A_H$ with A a diagonal matrix

$$A = \begin{bmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n \end{bmatrix}.$$

We claim that

$$\mu_i \approx H^{-1/2}, \qquad i = 1, \cdots, n. \tag{3-10}$$

We note that $T^{-1}y = A^{-1}y + x_H$. By (3-8) and (1-2) we obtain

$$C_2(\epsilon) Id \le D^2 u^*(x,t) \le C_1(\epsilon) Id, \qquad \forall (x,t) \in \{(x,t) : u^*(x,t) < -\epsilon\}. \tag{3-10'}$$

Since $T = T_H$ normalizes $Q_H(0)$ and by (1-2) $\det D^2 u(x,0)$ is doubling, by Theorem 2.3 of [**G-H**] applied to the sections $Q_H(0), Q_{\tau H}(0)$ with $0 < \tau < 1$ we get that

$$B(T(0), K_2\tau) \subset T(Q_{\tau H}(0)).$$

Let $\eta > 0$ then as in the proof of Lemma 3.1, we get that $Q_{\tau H}(0) \times [-\epsilon_1\, \eta\, H, 0] \subset Q_{(\tau+\eta)H}$ for some $\epsilon_1 > 0$ depending on m_1. By applying $\mathfrak{T}_H$ we obtain

$$B(T_H(0), K_2\tau) \times [-\epsilon_1 \eta, 0] \subset \mathfrak{T}_H(Q_{(\tau+\eta)H}),$$

for $\eta > 0$. If we pick η such that $\tau + \eta < 1$ then

$$\mathfrak{T}_H(Q_{(\tau+\eta)H}) \subset \{(x,t) : u^*(x,t) < -(1 - \tau - \eta)\frac{H}{\gamma}\}.$$

Hence, from (3-6) we obtain that there exists constants $\delta_0 > 0$ and $c_0, c_1 > 0$ such that

$$B(T(0), c_0) \times [-c_1, 0] \subset \Omega^*_\epsilon = \{(x,t) : u^*(x,t) < -\epsilon\} \tag{3-11}$$

for all $\epsilon \leq \delta_0$. On the other hand,

$$D^2u^*(y,s) = \frac{1}{\gamma}(A^{-1})^t D^2u\left(A^{-1}y + x_H, Hs\right)A^{-1},$$

and

$$u_s^*(y,s) = \frac{H}{\gamma}u_t\left(A^{-1}y + x_H, Hs\right).$$

Hence letting $y = T(0) = -Ax_H$ and $s = 0$ we obtain

$$D^2u^*(T(0),0) = \frac{1}{\gamma}\left(A^{-1}\right)^t D^2u(0,0)A^{-1} = \frac{1}{\gamma}\left(A^{-1}\right)^t A^{-1}.$$

Consequently

$$C_2(\epsilon)Id \leq \frac{1}{\gamma}\left(A^{-1}\right)^t A^{-1} \leq C_1(\epsilon)Id.$$

Now

$$A^{-1} = \begin{bmatrix} \frac{1}{\mu_1} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \frac{1}{\mu_1} \end{bmatrix},$$

and therefore $C_2 \leq \frac{1}{\gamma}\frac{1}{\mu_i^2} \leq C_1$, for $i = 1, \cdots, n$. Since $\frac{H}{\gamma} \approx C_n$, (3-10) follows.

We have

$$u^*(y,s) = \frac{1}{\gamma}u\left(\left(\frac{1}{\mu_1}y_1, \cdots, \frac{1}{\mu_n}y_n\right) + x_H, Hs\right) - \frac{H}{\gamma},$$

and to estimate the second derivatives of u^* we apply Theorem 2.3 to u^* in the following way. Let $L(q,M) = \log(-q) + \log(\det M)^{1/n}$ be defined for $-\lambda < q < -\Lambda$ and $M \in \mathbf{R}_s^{n\times n}$, $\lambda Id \leq M \leq \Lambda Id$. There exists an extension F of L to $\mathbf{R}\times\mathbf{R}_s^{n\times n}$ such that F satisfies the hypotheses of Theorem 2.3. An extension can be constructed, for example, as follows. Let $G(M) = \frac{1}{n}\inf\{\text{trace}AM : \det A = 1, A = A^t, \frac{\lambda}{\Lambda}Id \leq A \leq \frac{\Lambda}{\lambda}Id\}$. The function G is concave and satisfies the second inequality in (1) Theorem 2.3. If $\lambda Id \leq M \leq \Lambda Id$ is symmetric then $(\det M)^{1/n} = G(M)$. Now let $f(t) = \log t$ for $\lambda^n \leq t \leq \Lambda^n$ and such that $f \in C^1(\mathbf{R})$, f is concave, and $\Lambda^{-n} \leq f'(t) \leq \lambda^{-n}$. Then the function $f(G(M))$ is concave, uniformly elliptic and extends $\log(\det M)^{1/n}$. Also, let $h(q)$ be an extension to $\mathbf{R}$ of the function

$\log(-q)$ restricted to $-\lambda < q < -\Lambda$, such that h is concave, $-\frac{1}{\Lambda} \le h'(q) \le -\frac{1}{\lambda}$ and $C^1(\mathbf{R})$. The desired extension is then $F(q,M) = h(q) + f(G(M))$. Then, by (1-2), (3-10') and Theorem 2.3, we obtain

$$C(n) \ge [D_{ij}u^*]_{\alpha, B(T(0),c_0)\times[-c_1,0]},$$

and

$$C(n) \ge [D_t u^*]_{\alpha, B(T(0),c_0)\times[-c_1,0]}.$$

We now observe that if A is any invertible matrix and $w(x,t) = v\left((\mathfrak{T}_H)^{-1}(x,t)\right)$ with $(\mathfrak{T}_H)^{-1}(x,t) = (A^{-1}x + x_H, Ht)$ then

$$[w]_{\alpha,D} \ge \frac{1}{\left(\|A\|^2 + \dfrac{1}{H}\right)^{\alpha/2}} [v]_{\alpha,(\mathfrak{T}_H)^{-1}(D)}.$$

We have

$$D_{ij}u^*(y,s) = \frac{1}{\gamma}\frac{1}{\mu_i\mu_j} D_{ij}u\left(\left(\frac{1}{\mu_1}y_1, \cdots, \frac{1}{\mu_n}y_n\right) + x_H, Hs\right),$$

and so by (3-11)

$$\begin{aligned} C(n) &\ge [D_{ij}u^*]_{\alpha, B(T(0),c_0)\times[-c_1,0]} \\ &\ge \frac{1}{\gamma\mu_i\mu_j}\frac{1}{(\max_i \mu_i)^\alpha}[D_{ij}u]_{\alpha,(\mathfrak{T}_H)^{-1}(B(T(0),c_0)\times[-c_1,0])}, \end{aligned}$$

and analogously

$$C(n) \ge [D_t u^*]_{\alpha, B(T(0),c_0)\times[-c_1,0]} \ge CH^{\alpha/2}[D_t u]_{\alpha,(\mathfrak{T}_H)^{-1}(B(T(0),c_0)\times[-c_1,0])}.$$

Since $T(0) = -Ax_H$, it follows that $(\mathfrak{T}_H)^{-1}(B(T(0),c_0)\times[-c_1,0]) \approx B(0, H^{1/2}c_0)\times[-c_1H, 0]$ and consequently,

$$C(n) \ge H^{\alpha/2}[D_{ij}u]_{\alpha, B(0,H^{1/2}c_0)\times[-c_1H,0]},$$

and

$$C(n) \ge H^{\alpha/2}[D_t u]_{\alpha, B(0,H^{1/2}c_0)\times[-c_1H,0]},$$

By letting $H \to \infty$, we obtain that $D_{ij}u$ and $D_t u$ are constant on each bounded set and the proof is complete. $\square$

We finish this section by comparing the condition on u_t given by the second inequality in (1-2) with the size estimate for Q_H given by (3-2). By Lemma 3.1, (1-2) implies (3-2). Conversely, and as a byproduct of the proof of Theorem 1.1 the following proposition holds.

Proposition 3.1. *Let $u(x,t) \in C^{4,2}$ be parabolically convex in $\mathbf{R}^{n+1}_-$, $u(0,0)=0$, and $Du(0,0)=0$. Suppose that* (3-2) *and the first inequality in* (1-2) *hold, and u solves* (1-1)*. Then the second inequality in* (1-2) *also holds.*

Proof. With the notation used in the proof of Theorem 1.1, we have that $u^*(y,s) = 0$ on $\partial_p Q^*_H$ and by (3-4) $-u^*_s \det D^2 u^* = 1$. Since $\min_{Q^*_H} u = -\dfrac{H}{\gamma}$, by application of Lemma 2.1 to u^* it follows that $\left|\min_{Q^*_H} u\right| \approx C_1$ and therefore we get $\dfrac{H}{\gamma} \approx C_2$ like in (3-6). Hence from (3-7), (3-8) and the first inequality in (1-2), we get that $|u^*_s(y,s)| \leq C$ for all $(y,s) \in \mathbf{R}^{n+1}_-$. Therefore $-u^*_s = \dfrac{1}{\det D^2 u^*} \geq \dfrac{1}{C_\epsilon}$ for all $(y,s) \in \Omega_\epsilon$. Now, given $0 < \epsilon < 1$ there exists $0 < \delta = \delta(\epsilon) < 1$ such that $\mathfrak{T}_H\left(Q_{\delta H}\right) \subset \Omega_\epsilon$. Hence, $u_t(x,t) \leq -C'_\epsilon$ with $C'_\epsilon > 0$ for all $(x,t) \in Q_{\delta H}$. By letting $H \to \infty$ we obtain the proposition. □

§4. A Counterexample

Let $g(r)$ be a C^2 function on $(0,+\infty)$ and $v(x_1,\cdots,x_n) = g(r)$ with $r = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, $n \geq 1$. An easy but tedious computation yields

$$\det D^2 v(x) = g''(r)\left(\frac{g'(r)}{r}\right)^{k-1}, \tag{4-1}$$

for $r > 0$. Let $\epsilon > 0$ and $h_\beta(x) = \left(|x|^2 + \epsilon\right)^\beta$, $x \in \mathbf{R}^n$, $n \geq 2$. If $\beta \geq 1/2$ then the function h_β is convex in $\mathbf{R}^n$ because from (4-1), all the principal minors of the matrix $D^2 h_\beta$ are positive. Let $u_\epsilon(x,t) = \delta\,(-t+\epsilon)^\alpha h_\beta(x)$ be defined for $t \leq 0$ and $x \in \mathbf{R}^n$ with $\delta > 0$. If $\alpha > 0$ then u_ϵ is parabolically convex in $\mathbf{R}^{n+1}_-$. By using (4-1) we get

$$\begin{aligned} &-(u_\epsilon)_t \det D^2 u_\epsilon \\ &= \delta^{n+1}\alpha(2\beta)^n(-t+\epsilon)^{\alpha(n+1)-1}(r^2+\epsilon)^{(\beta-1)(n+1)+1}\left(\frac{r^2+\epsilon+2(\beta-1)r^2}{r^2+\epsilon}\right). \end{aligned}$$

If we pick $\alpha = \dfrac{1}{n+1}$ and $\beta = \dfrac{n}{n+1}$ then

$$-(u_\epsilon)_t \det D^2 u_\epsilon = \delta^{n+1}\alpha(2\beta)^n\left(1 - \frac{2}{n+1}\left(\frac{r^2}{r^2+\epsilon}\right)\right) = f_\epsilon(x).$$

If we choose $\delta > 0$ such that $\delta^{n+1}\alpha(2\beta)^n \left(1 - \dfrac{2}{n+1}\right) = 1$ then the function $u(x,t) = \delta\,(-t)^{1/(n+1)}\,|x|^{2n/(n+1)}$ is parabolically convex in $\mathbf{R}^{n+1}_-$ and is a viscosity solution of $-u_t \det D^2u = 1$. Indeed, we have that $u_\epsilon \to u$ in $C_{loc}(\mathbf{R}^{n+1}_-)$ as $\epsilon \to 0$, and since u_ϵ is a classical solution it is also a viscosity solution, see [**W-W**] for the definition. Also $f_\epsilon \to 1$ and since the class of viscosity solutions is closed under locally uniform limits the claim is proved. □

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