ON THE ELEMENTARY $(P \pm i0)^\lambda$ - ULTRAHYPERBOLIC SOLUTION OF THE KLEIN-GORDON OPERATOR ITERATED k-TIMES

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Abstract - Let $x=(x_1,x_2,...,x_n)$ be a point of the n-dimensional Euclidean space. Consider a non-degenerate quadratic form in n variables of the form $P=P(x)=x_1^2+...+x_p^2-x_{p+1}^2-...-x_{p+q}^2$, where $p+q=n$. The distributions $(P\pm i0)^\lambda$ are defined by $(P\pm i0)^\lambda=\lim_{\varepsilon\to 0}\{(P\pm i\varepsilon|x|^2)\}^\lambda$ where $\varepsilon>0$, $|x|^2=x_1^2+...+x_n^2$, $\lambda\in\mathbb{C}$ ([2], p. 274). We put, by definition, $W_\alpha(P\pm i0)=C(\alpha,n)[m^{-2}(P\pm i0)]^{\frac{\alpha-n}{4}}J_{\frac{\alpha-n}{2}}(m^2(P\pm i0)^{\frac{1}{2}})]$ (cf.(III,1)); where α is a complex parameter, m a real nonnegative number, n the dimension of the space and $J_\nu(z)$ the well-known Bessel function of the first kind. First, we express $W_\alpha(P\pm i0,m)$ as a infinite, linear combination of $R_\alpha(P\pm i0)$ (cf.(II,1)) of different orders; $R_\alpha(P\pm i0)$ (cf.(II,1) is the causal (anticausal) elementary solutions of the homogeneous ultrahyperbolic operator, iterated k-times (cf. formula (II,2)). From the formula (II,10) we obtain the explicit definitory formula of the kernel $W_\alpha(P\pm i0,m)$, (cf. formula (III,1)). Also we prove that $W_\alpha(P\pm i0,m=0)=R_\alpha(P\pm i0)$ (cf. formula (VII,2)) and also we give a different non-formal proof (cf. formula (VII,9)). In this Note, the following propositions have been evaluated $(\Box+m^2)^k W_{2k}(P\pm i0,m)=\delta$ (cf. formula (V,2)), where $(\Box+m^2)^k$ is the ultrahyperbolic Klein-Gordon operator iterated k-times $k=1,2,...$ (cf. formula (IV,2')); $W_{-2k}(P\pm i0,m)=(\Box+m^2)^k\delta$ (cf. formula (IV,6)); $W_0(P\pm i0,m)=\delta$ (cf. formula (VI,1)) and the composition formula $W_\alpha(P\pm i0,m)*W_\beta(P\pm i0,m)=W_{\alpha+\beta}(P\pm i0,m)$ (cf. formula (VIII,20)). Finally, by following Marcel Riesz' symbolic method (cf. paragraph (IX)), we shall justify all the steps by appealing to the theory of distributions.

I. Definitions.

Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$.

Consider a non-degenerate quadratic form in n-variables of the form

$$P = P(x) = x_1^2 + \ldots + x_p^2 - x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 \ , \qquad (\mathrm{I}, 1)$$

where $n = p + q$. The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{P \pm i\varepsilon |x|^2\}^\lambda \ , \qquad (\mathrm{I}, 2)$$

The distributions $(P \pm i0)^\lambda$ are an important contribution of Gelfand (cf. [2], p.274).

The distributions $(P \pm i0)^\lambda$ are analytic in λ everywhere except at $\lambda = -\frac{n}{2} - k$, $k = 0, 1, \ldots,$ where they have simple poles (cf. [2], p.275).

Furthermore, the Fourier transform of the distributions $(P \pm i0)^\lambda$ is (cf. [2], p.281)

$$\begin{aligned} [(P \pm i0)]^\wedge &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i<x,y>} (P \pm i0)^\lambda dx \\ &= \frac{e^{\mp i \frac{\pi}{2} q} 2^{2\lambda+n} \pi^{\frac{n}{2}} \Gamma\left(\lambda + \frac{n}{2}\right)}{(2\Pi)^{\frac{n}{2}} \Gamma(-\lambda)} (Q \mp i0)^{-\lambda - \frac{n}{2}} \ , \end{aligned} \qquad (\mathrm{I}, 3)$$

where

$$Q = Q(y) = y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_{p+q}^2 \ ,$$
$$< x, y >= \sum_{i=1}^n x_i y_i \ .$$

Furthermore, the following theorem is valid (cf. [6], (I,3;17)):

$$(P \pm i0)^\lambda \cdot (P \pm i0)^\mu = (P \pm i0)^{\lambda+\mu} \ ,$$

$\lambda, \mu \in \mathbb{C}$; λ, μ and $\lambda + \mu$ are different from $-\frac{n}{2} - k$, $k = 0, 1, \ldots.$

II. The expression of $W_\alpha(P \pm i0, m)$ as an infinite, linear combination of $R_\alpha(P \pm i0)$ of different orders.

We define the causal (anticausal) distributions $R_\alpha(P \pm i0)$ as follows:

$$R_\alpha(P \pm i0) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} (P \pm i0)^{\frac{\alpha - n}{2}} \ , \qquad (\mathrm{II}, 1)$$

where $\alpha \in \mathbb{C}$ and P is defined by (I,1). The distributinal functions $R_\alpha(P \pm i0)$ are causal (anticausal) analogues of the elliptic kernel of Marcel Riesz (cf. [5], pp.1-223) and have analogue properties, that we use to obtain causal (anticausal) solutions of the n-dimensional ultrahyperbolic operator, iterated k-times (k integer ≥ 1), $p + q = n$,

$$\square^{\,k} = \left\{ \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \ldots - \frac{\partial^2}{\partial x_{p+q}^2} \right\}^k \ . \qquad (\mathrm{II}, 2)$$

We know (cf. [6], p.39), formula (II,3;3)) that

$$\{R_\alpha(P \pm i0)\}^\wedge = \frac{(Q \mp i0)^{-\frac{\alpha}{2}}}{(2\pi)^{\frac{n}{2}}} \ , \qquad (\mathrm{II}, 3)$$

where, as always, we write

$$Q = y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_{p+q}^2 \ .$$

The following theorems are valid when α, β and $\alpha + \beta$ are different from $n + 2r$, $r = 0, 1, \ldots$,

1. $R_\alpha * R_\beta = R_{\alpha+\beta}$; (II,4)
2. $[R_\alpha * R_\beta]^\wedge = (2\pi)^{\frac{n}{2}} \{R_\alpha\}^\wedge \cdot \{R_\beta\}^\wedge$; (II,5)
3. $\square^{\,k} [R_{2k}(P \pm i0)] = \delta$; (II,6)
4. $R_{-2k}(P \pm i0) = \square^{\,k} \delta$; (II,7)
5. $R_0(P \pm i0) = \delta$; (II,8)
6. $\square^{\,k} \{Pf R_{2k}(P \pm i0)\} = \delta$, (II,9)

for all values of $k = 1, 2, \ldots$; here Pf is the finite part.

We begin by writing the following formal expression:

$$W_\alpha(P \pm i0, m) = \sum_{\nu=0}^{\infty} \binom{-\frac{\alpha}{2}}{\nu} m^{2\nu} R_{\alpha+2\nu}(P \pm i0) \ . \qquad (\text{II}, 10)$$

Taking into account the definitory formula of $R_\alpha(P \pm i0)$ (cf. formula (II,1)), we have

$$W_\alpha(P \pm i0, m) = \sum_{\nu=0}^{\infty} \binom{-\frac{\alpha}{2}}{\nu} m^{2\nu} \frac{(P \pm i0)^{\frac{\alpha-n+2\nu}{2}}}{H_n(\alpha+2\nu)} \ , \qquad (\text{II}, 11)$$

here

$$H_n(\alpha+2\nu) = \frac{2^{\alpha+2\nu} \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha+2\nu}{2}\right)}{\Gamma\left(\frac{n-\alpha-2\nu}{2}\right)} \ . \qquad (\text{II}, 12)$$

From [3], p.3, formula (1), we know that

$$\Gamma\left(\frac{\alpha-n}{2} + \nu + 1\right) = \left(\frac{\alpha-n}{2} + \nu\right) \Gamma\left(\frac{\alpha-n}{2} + \nu\right) \ , \qquad (\text{II}, 13)$$

Also, taking into account the formula (5), page 3 of [3], we have

$$\Gamma\left(\frac{\alpha-n}{2} + \nu\right) \Gamma\left(\frac{n-\alpha}{2} - \nu\right) = \frac{-\pi}{\left(\frac{\alpha-n+2\nu}{2}\right)} \frac{1}{\sin\left[\pi\left(\frac{\alpha-n+2\nu}{2}\right)\right]} \ . \qquad (\text{II}, 14)$$

By remembering the classical expression of $J_\nu(z)$, we have

$$J_{\frac{\alpha-n}{2}}\left[m(P \pm i0)^{\frac{1}{2}}\right] = \frac{m^{\frac{\alpha-n}{2}} (P \pm i0)^{-\frac{\alpha-n}{4}}}{2^{\frac{\alpha-n}{2}}} \cdot \sum_{\nu=0}^{\infty} (-1)^\nu \frac{m^{2\nu} (P \pm i0)^{\frac{\alpha-n+2\nu}{2}}}{2^{2\nu} \nu! \Gamma\left(\frac{\alpha-n}{2} + \nu + 1\right)} \ . \qquad (\text{II}, 15)$$

From (II,14), we have

$$\Gamma\left(\frac{\alpha-n}{2} + \nu\right) = \frac{-\pi}{\left(\frac{\alpha-n+2\nu}{2}\right) \sin\left[\pi\left(\frac{\alpha-n+2\nu}{2}\right)\right] \Gamma\left(\frac{n-\alpha}{2} - \nu\right)} \ . \qquad (\text{II}, 16)$$

From (II,12), we have,

$$\Gamma\left(\frac{n-\alpha-2\nu}{2}\right) = \frac{2^{\alpha+2\nu} \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2} + \nu\right)}{H_n(\alpha+2\nu)} \ . \qquad (\text{II}, 17)$$

From (II,16) and (II,17), we have

$$\Gamma\left(\frac{\alpha-n}{2}+\nu\right)=\frac{(-\pi)\quad H_n(\alpha+2\nu)}{\left(\frac{\alpha-n+2\nu}{2}\right)\sin\left[\pi\left(\frac{\alpha-n+\nu}{2}\right)\right]2^{\alpha+2\nu}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}+\nu\right)}\ . \qquad (II,18)$$

Taking into account (II,13), (II,15) and (II,18), we have

$$\begin{aligned}J_{\frac{\alpha-n}{2}}\left[m(P\pm i0)^{\frac{1}{2}}\right]&=\frac{m^{\frac{\alpha-n}{2}}(P\pm i0)^{-\frac{\alpha-n}{4}}}{2^{\frac{\alpha-n}{2}}}\\ &\cdot\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}m^{2\nu}(P\pm i0)^{\frac{\alpha-n+2\nu}{2}}}{2^{2\nu}\nu!\left(\frac{\alpha-n}{2}+\nu\right)(-\pi)H_n(\alpha+2\nu)}\\ &\cdot\left(\frac{\alpha-n}{2}+\nu\right)\sin\left[\pi\left(\frac{\alpha-n+2\nu}{2}\right)\right]2^{\alpha+2\nu}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}+\nu\right)\ .\end{aligned} \qquad (II,19)$$

By other way, the following formula is valid (cf. [3], p.3, formula (2)):

$$\Gamma\left(\frac{\alpha}{2}+\nu\right)=\frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right)\dots\left(\frac{\alpha}{2}+\nu-1\right)\Gamma\left(\frac{\alpha}{2}\right)\ . \qquad (II,20)$$

Equivalently,

$$\begin{aligned}(-1)^{\nu}\frac{1}{\nu!}\Gamma\left(\frac{\alpha}{2}+\nu\right)&=\frac{(-1)^{\nu}\left(\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}+1\right)\dots\left(\frac{\alpha}{2}+\nu-1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\nu!}\\ &=\frac{\left(-\frac{\alpha}{2}\right)\left(-\frac{\alpha}{2}-1\right)\dots\left[-\left(\frac{\alpha}{2}+\nu-1\right)\right]}{\nu!}\Gamma\left(\frac{\alpha}{2}\right)\ .\end{aligned} \qquad (II,21)$$

Now we put, by definition,

$$(-1)^{\nu}\frac{1}{\nu!}\Gamma\left(\frac{\alpha}{2}+\nu\right)=\binom{-\frac{\alpha}{2}}{\nu}\Gamma\left(\frac{\alpha}{2}\right)\ . \qquad (II,22)$$

By putting (II,22) into (II,19), we get at

$$\begin{aligned}J_{\frac{\alpha-n}{2}}\left[m(P\pm i0)^{\frac{1}{2}}\right]&=(-1)\pi^{\frac{n-2}{2}}m^{\frac{\alpha-n}{2}}\cdot(P\pm i0)^{-\frac{\alpha-n}{4}}2^{\alpha-\left(\frac{\alpha-n}{2}\right)}\Gamma\left(\frac{\alpha}{2}\right)\\ &\cdot\sum_{\nu=0}^{\infty}\binom{-\frac{\alpha}{2}}{\nu}\frac{(P\pm i0)^{\frac{\alpha-n+2\nu}{2}}}{H_n(\alpha+2\nu)}m^{2\nu}\cdot\sin\left[\pi\left(\frac{\alpha-n+2\nu}{2}\right)\right]\ .\end{aligned} \qquad (II,23)$$

with the hypothesis

$$\alpha-n+2\nu=4p+1,\qquad p=0,1,\dots, \qquad (II,24)$$

we can arrive at the following formula

$$J_{\frac{\alpha-n}{2}}\left[m(P\pm i0)^{\frac{1}{2}}\right]=(-1)\pi^{\frac{n-2}{2}}2^{\alpha-\left(\frac{\alpha-n}{2}\right)}\Gamma\left(\frac{\alpha}{2}\right)m^{\frac{\alpha-n}{2}} \cdot \sum_{\nu=0}^{\infty}\binom{-\frac{\alpha}{2}}{\nu}m^{2\nu}\frac{(P\pm i0)^{\frac{\alpha-n+2\nu}{2}}}{H_n(\alpha+2\nu)} . \qquad (\mathrm{II},25)$$

From (II,10) and (II,25), we have

$$W_\alpha(P\pm i0,m)=\sum_{\nu=0}^{\infty}\binom{-\frac{\alpha}{2}}{\nu}m^{2\nu}\frac{(P\pm i0)^{\frac{\alpha-n+2\nu}{2}}}{H_n(\alpha+2\nu)} = \frac{\left[m^{-1}(P\pm i0)^{\frac{1}{2}}\right]^{\frac{\alpha-n}{2}} J_{\frac{\alpha-n}{2}}\left[m(P\pm i0)^{\frac{1}{2}}\right]}{(-1)2^{\frac{-n-\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{n-2}{2}}} . \qquad (\mathrm{II},26)$$

This last formula (II,26) shall be completely justify in the paragraph (IX)

III. The explicit expression of $W_\alpha(P\pm i0,m)$.

The explicit definitory formula of $W_\alpha(P\pm i0,m)$ is just the same that the formula which we arrive at (II,25), that is

$$W_\alpha(P\pm i0,m)=\frac{\left[m^{-1}(P\pm i0)^{\frac{1}{2}}\right]^{\frac{\alpha-n}{2}}}{(-1)\pi^{\frac{n-2}{2}}2^{\frac{n+\alpha}{2}}\Gamma^{-1}\left(\frac{\alpha}{2}\right)}\cdot J_{\frac{\alpha-n}{2}}\left[m(P\pm i0)^{\frac{1}{2}}\right] . \qquad (\mathrm{III},1)$$

IV. $W_{-2k}(P\pm i0,m)=(\Box+m^2)^k\delta$.

From the formal definitory formula of $W_\alpha(P\pm i0,m)$ (cf. (II,10)), we have

$$W_\alpha(P\pm i0,m)=\sum_{\nu=0}^{\infty}\binom{-\frac{\alpha}{2}}{\nu}m^{2\nu}R_{\alpha+2\nu}(P\pm i0) . \qquad (\mathrm{IV},1)$$

We shall prove the following assertion

$$W_{-2k}(P\pm i0,m)=(\Box+m^2)^k\delta, \qquad k=0,1,2,\dots,\dots ; \qquad (\mathrm{IV},2)$$

here $(\Box + m^2)^k$ is the n-dimensional ultrahyperbolic Klein-Gordon operator iterated k-times defined by the formula

$$(\Box + m^2)^2 = \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \ldots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2 \right) , \qquad (\mathrm{IV}, 2')$$

$p + q = n$ and m a real nonnegative number.

In fact, we know that (cf. formula (II,7))

$$R_{-2k}(P \pm i0) = \Box^k \delta , \qquad (\mathrm{IV}, 3)$$

here $\Box^k$ is defined by (II,2).

Putting

$$-2k = \alpha + 2\nu , \qquad (\mathrm{IV}, 4)$$

then

$$k = -\frac{(\alpha + 2\nu)}{2} . \qquad (\mathrm{IV}, 5)$$

Therefore, we have from (IV,1) and (IV,3),

$$\begin{aligned} W_{-2k}(P \pm i0, m) &= \sum_{\nu=0}^{\infty} \binom{k}{\nu} m^{2\nu} \Box^{k-\nu} \delta \\ &= \left(\Box + m^2 \right)^k \delta , \end{aligned} \qquad (\mathrm{IV}, 6)$$

$k = 0, 1, \ldots .$

V. $\left(\Box + m^2 \right)^k W_{2k}(P \pm i0, m) = \delta$**,** $k = 0, 1, \ldots$**.**

From the formula (II,10) and (II,7), we have

$$\begin{aligned} W_{2k}(P \pm i0, m) &= \sum_{\nu=0}^{\infty} \binom{-k}{\nu} (m^2)^{\nu} \Box^{-k-\nu} \delta \\ &= \left(\Box + m^2 \right)^{-k} \delta . \end{aligned} \qquad (\mathrm{V}, 1)$$

By applying the operator $\left(\Box + m^2 \right)^k$ to both members of (V,1), we get at

$$\left(\Box + m^2 \right)^k W_{2k}(P \pm i0, m) = \left(\Box + m^2 \right)^k \cdot \left(\Box + m^2 \right)^{-k} \delta = \delta . \qquad (\mathrm{V}, 2)$$

Therefore, $W_{2k}(P \pm i0, m)$ is the unique elementary retarded $(P \pm i0)^{\lambda}$-ultrahyperbolic solution of the Klein-Gordon operator, iterated k-times.

Putting $k = 1$, the formula (V,2) says that $W_2(P \pm i0, m)$ is the unique elementary retarded $(P \pm i0)^{\lambda}$-ultrahyperbolic solution of the Klein-Gordon operator.

VI. $W_0(P \pm i0, m) = \delta$.

Putting $k = 0$ in (IV,6), we have

$$W_0(P \pm i0, m) = \delta \ . \tag{VI, 1}$$

VII. $W_\alpha(P \pm i0, m = 0) = R_\alpha(P \pm i0)$.

From the formal definitory formula of $W_\alpha(P \pm i0, m)$ (cf. formula (II,10)), we have

$$W_\alpha(P \pm i0, m) = \binom{-\frac{\alpha}{2}}{0} m^{2.0} R_{\alpha+2.0}(P \pm i0) + \sum_{\nu=1}^{\infty} \binom{-\frac{\alpha}{2}}{\nu} m^{2\nu} R_{\alpha+2\nu}(P \pm i0) \ . \tag{VII, 1}$$

The second sumand of the right-hand member of (VII,1) vanishes for $m = 0$ and then, we have

$$W_\alpha(P \pm i0, m = 0) = R_\alpha(P \pm i0) \ . \tag{VII, 2}$$

We can give another proof of the formula $W_\alpha(P \pm i0, m = 0) = R_\alpha(P \pm i0)$.

We begin by remembering the definitory formula of $W_\alpha(P \pm i0, m)$, (cf. formula (III,1):

$$W_\alpha(P \pm i0, m) = \frac{\left[m^{-2}(P \pm i0)\right]^{\frac{\alpha-n}{4}} J_{\frac{\alpha-n}{2}}\left[m(P \pm i0)^{\frac{1}{2}}\right]}{(-1)2^{-\frac{n-\alpha}{2}} \pi^{\frac{n-2}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \ , \tag{VII, 3}$$

here α is a complex parameter, m a nonnegative real number and n the dimension of the space.

We know that the well-known Bessel function of the first kind is defined by the formula

$$J_{\frac{\alpha-n}{2}}\left\{\left[m^2(P\pm i0)\right]^{\frac{1}{2}}\right\}$$

$$=\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\left\{\frac{\left[m^2(P\pm i0)\right]^{\frac{1}{2}}}{2}\right\}^{\frac{\alpha-n}{2}+2\nu}}{\nu!\Gamma\left(\frac{\alpha-n}{2}+\nu+1\right)}$$

$$=\left\{\frac{\left[m^2(P\pm i0)\right]^{\frac{1}{2}}}{2}\right\}^{\frac{\alpha-n}{2}}\cdot\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\left\{\frac{\left[m^2(P\pm i0)\right]^{\frac{1}{2}}}{2}\right\}^{2\nu}}{\nu!\Gamma\left(\frac{\alpha-n}{2}+\nu+1\right)}$$

$$=\frac{\left[m(P\pm i0)^{\frac{1}{2}}\right]^{\frac{\alpha-n}{2}}}{2^{\frac{\alpha-n}{2}}}\frac{1}{\Gamma\left(\frac{\alpha-n}{2}+1\right)}+\frac{\left[m(P\pm i0)^{\frac{1}{2}}\right]^{\frac{\alpha-n}{2}}}{2^{\frac{\alpha-n}{2}}}$$

$$\cdot\sum_{\nu=1}^{\infty}(-1)^{\nu}\left\{\frac{\left[m(P\pm i0)^{\frac{1}{2}}\right]}{2}\right\}^{2\nu}\frac{1}{\nu!\Gamma\left(\frac{\alpha-n}{2}+\nu+1\right)}\ . \qquad (\mathrm{VII},4)$$

Putting (VII,4) into (VII,3) and taking into account the formula (I,3;17), p.23, of [6], we get at

$$W_{\alpha}(P\pm i0,m)=\frac{(P\pm i0)^{\frac{\alpha-n}{2}}}{2^{\frac{\alpha-n}{2}}\Gamma\left(\frac{\alpha-n}{2}+1\right)(-1)2^{-\frac{n-\alpha}{2}}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}$$

$$+\frac{(P\pm i0)^{\frac{\alpha-n}{2}}}{(-1)2^{-\frac{n-\alpha}{2}}\pi^{\frac{n-2}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\cdot\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}\left[\frac{m(P\pm i0)^{\frac{1}{2}}}{2}\right]^{2\nu}}{\nu!\Gamma\left(\frac{\alpha-n}{2}+\nu+1\right)}\ . \qquad (\mathrm{VII},5)$$

The second summand of the right-hand member of (VII,5) vanishes for $m=0$ and then, we have,

$$W_{\alpha}(P\pm i0,m=0)=\frac{(P\pm i0)^{\frac{\alpha-n}{2}}}{\pi^{\frac{n}{2}}2^{\frac{\alpha-n}{2}}\Gamma\left(\frac{\alpha-n}{2}+1\right)(-1)2^{-\frac{n-\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\ . \qquad (\mathrm{VII},6)$$

So, taking into account that (cf. [3], p.3, formula (1))

$$\Gamma\left(1+\frac{\alpha-n}{2}\right)=\left(\frac{\alpha-n}{2}\right)\Gamma\left(\frac{\alpha-n}{2}\right)\ , \qquad (\mathrm{VII},7)$$

and (cf. [3], p.3, formula (5))

$$\Gamma\left(\frac{\alpha-n}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)=-\frac{\pi}{\left(\frac{\alpha-n}{2}\right)}\frac{1}{\sin\left[\pi\left(\frac{\alpha-n}{2}\right)\right]}\ ; \qquad \text{(VII, 8)}$$

we arrive (by remembering the condition (II,24)) to

$$W_\alpha(P\pm i0, m=0)=R_\alpha(P\pm i0)\ . \qquad \text{(VII, 9)}$$

VIII. The composition formula $W_\alpha(P\pm i0,m)*W_\beta(P\pm i0,m)=W_{\alpha+\beta}(P\pm i0,m)$**.**

Taking into account the formal definitory formula of $W_\alpha(P\pm i0,m)$, (cf. (II,10)), we have

$$W_\alpha(P\pm i0,m)=\sum_{\nu=0}^{\infty}\binom{-\frac{\alpha}{2}}{\nu}m^{2\nu}R_{\alpha+2\nu}(P\pm i0)\ . \qquad \text{(VIII, 1)}$$

and

$$W_\beta(P\pm i0,m)=\sum_{\nu=0}^{\infty}\binom{-\frac{\beta}{2}}{\nu}m^{2\nu}R_{\beta+2\nu}(P\pm i0)\ . \qquad \text{(VIII, 2)}$$

Also, we have,

$$\mathcal{F}\left[W_\alpha(P\pm i0,m)\right]=\sum_{\nu=0}^{\infty}\binom{-\frac{\alpha}{2}}{\nu}m^{2\nu}\mathcal{F}\left[R_{\alpha+2\nu}(P\pm i0)\right]\ . \qquad \text{(VIII, 3)}$$

This last formula can be justified by the following lines.

Let $f(z,\lambda)$, $z\in\mathbb{C}$, be an entire function of the variables z,λ:

$$f(z,\lambda)=\sum_{\nu=0}^{\infty}a_\nu(\lambda)^2z^\nu\ . \qquad \text{(VIII, 4)}$$

Let us consider the family of distributions of the form (cf. [2], p.285):

$$\begin{aligned}T(P\pm i0,\lambda)&=(P\pm i0)^\lambda f(P\pm i0,\lambda)\\&=(P\pm i0)^\lambda\sum_{\nu=0}^{\infty}a_\nu(\lambda)(P\pm i0)^\lambda\ .\end{aligned} \qquad \text{(VIII, 5)}$$

Our purpose is to evaluate the Fourier transform of $T(P \pm i0, \lambda)$ in the sense of Gelfand:

$$\{T(P \pm i0, \lambda)\}^{\wedge} = \left\{(P \pm i0)^{\lambda} \sum_{\nu=0}^{\infty} a_{\nu}(\lambda) \cdot (P \pm i0)^{\nu}\right\}^{\wedge} . \tag{VIII,6}$$

We have to show that

$$\left\{(P \pm i0)^{\lambda} \sum_{\nu=0}^{\infty} a_{\nu}(\lambda)(P \pm i0)^{\nu}\right\}^{\wedge} = \sum_{\nu=0}^{\infty} a_{\nu}(\lambda) \left\{(P \pm i0)^{\lambda+\nu}\right\}^{\wedge} . \tag{VIII,7}$$

Let us suppose, provisionally that $\mathrm{Re}\,\lambda > -1$; then, the terms of the sequence $(n = 0, 1, \ldots)$

$$\{g_n\} = \left\{(P \pm i0)^{\lambda} \sum_{\nu=0}^{n} a_{\nu}(\lambda)(P \pm i0)^{\nu}\right\} \tag{VIII,8}$$

are locally integrable functions. Since, by hypothesis, $f(z, \lambda)$ is an entire function, we conclude that the sequence (VIII,8) converges uniformly in every compact $K \subset I\!R^n$.

Therefore, by [7], Theorem XVI, p.76, the sequence $\{g_n\}$ is convergent in D' , and, by the continuity of the Fourier transform, we conclude that the equation (VIII,7) is valid when $\mathrm{Re}\,\lambda > -1$.

Then, we have, taking into account the formula (II,3) that

$$\mathcal{F}\left[W_{\alpha}(P \pm i0, m)\right] = \sum_{\nu=0}^{\alpha} \binom{-\frac{\alpha}{2}}{\nu} m^{2\nu}(Q \mp i0)^{-\frac{\alpha+2\nu}{2}} , \tag{VIII,9}$$

and also, we have

$$\mathcal{F}\left[W_{\beta}(P \pm i0, m)\right] = \sum_{p=0}^{\infty} \binom{-\frac{\beta}{2}}{p} m^{2p}(Q \mp i0)^{-\frac{\beta+2p}{\nu}} \tag{VIII,10}$$

and

$$\mathcal{F}\left[W_{\alpha+\beta}(P \pm i0, m)\right] = \sum_{r=0}^{\infty} \binom{-\frac{\alpha+\beta}{2}}{r} m^{2r}(Q \mp i0)^{-\frac{\alpha+\beta+2r}{2}} . \tag{VIII,11}$$

From (VIII,9) and (VIII,10), we have

$$\begin{aligned}&\mathcal{F}\left[W_\alpha(P \pm i0, m)\right] \cdot \mathcal{F}\left[W_\beta(P \pm i0, m)\right] \\ =&\sum_{\nu=0}^{\infty}\binom{-\frac{\alpha}{2}}{\nu} m^{2\nu}(Q \mp i0)^{-\frac{\alpha+2\nu}{2}} \cdot \sum_{p=0}^{\infty}\binom{-\frac{\beta}{2}}{p} m^{2p}(Q \mp i0)^{-\frac{\beta+2p}{2}} \ .\end{aligned} \qquad (\mathrm{VIII}, 12)$$

Taking into account the Theorem 2, page 23, of [6], we arrive at

$$\begin{aligned}&\mathcal{F}[W_\alpha] \cdot \mathcal{F}[W_\beta] \\ =&\sum_{\nu=0}^{\infty}\binom{-\frac{\alpha}{2}}{\nu} m^{2\nu} \cdot \sum_{p=0}^{\infty}\binom{-\frac{\beta}{2}}{p} m^{2p} \cdot (Q \mp i0)^{-\frac{\alpha+\beta+2\nu+2p}{2}} \ .\end{aligned} \qquad (\mathrm{VIII}, 13)$$

Putting

$$2\nu + 2p = 2r \ , \qquad (\mathrm{VIII}, 14)$$

so

$$\nu + p = r \ , \qquad (\mathrm{VIII}, 15)$$

then

$$p = r - p \ , \qquad (\mathrm{VIII}, 16)$$

we have

$$\begin{aligned}&\mathcal{F}[W_\alpha] \cdot [W_\beta] \\ =&\sum_{\nu=0}^{\infty}\binom{-\frac{\alpha}{2}}{\nu} m^{2\nu} \sum_{r=\nu}^{\infty}\binom{-\frac{\beta}{2}}{r-\nu} m^{2(r-\nu)}(Q \mp i0)^{-\frac{\alpha+\beta+2r}{2}} \\ =&\sum_{\nu=0}^{\infty}\binom{-\frac{\alpha}{2}}{\nu} \sum_{r=\nu}^{\infty}\binom{-\frac{\beta}{2}}{r-\nu} m^{2r}(Q \mp i0)^{-\frac{\alpha+\beta+2r}{2}} \\ =&\sum_{s=0}^{\infty}\binom{-\frac{\alpha+\beta}{2}}{s} m^{2s}(Q \mp i0)^{-\frac{\alpha+\beta+2s}{2}} \\ =&\mathcal{F}\left[W_{\alpha+\beta}\right] \ .\end{aligned} \qquad (\mathrm{VIII}, 17)$$

Finally, we get at

$$\mathcal{F}[W_\alpha] \cdot \mathcal{F}[W_\beta] = \mathcal{F}\left[W_{\alpha+\beta}\right] \ . \qquad (\mathrm{VIII}, 18)$$

Note.

Taking into account that $(P \pm i0)^\lambda = P_+^\lambda + e^{i\pi\lambda} P_-^\lambda$ and [6], p.177, the convolution $W_\alpha(P \pm i0, m) * W_\beta(P \pm i0, m)$ is possible.

Therefore, the theorem of interchange between the convolution and the product is valid for integrals of Fourier so, we have

$$\mathcal{F}\left[W_\alpha * W_\beta\right] = \mathcal{F}[W_\alpha] \cdot \mathcal{F}[W_\beta] \ . \tag{VIII, 19}$$

Finally, from (VIII,18) and (VIII,19), and in virtue of the theorem of the identity, we arrive at

$$W_\alpha(P \pm i0, m) * W_\beta(P \pm i0, m) = W_{\alpha+\beta}(P \pm i0, m) \ . \tag{VIII, 20}$$

IX. Justification of the Marcel Riesz symbolic method.

We have from (II,1) that

$$R_{2\alpha+2\nu}(P \pm i0) = \frac{(P \pm i0)^{\frac{2\alpha+2\nu-n}{2}}}{H(2\alpha+2\nu)} \ , \tag{IX, 1}$$

here

$$H(2\alpha+2\nu) = \frac{2^{2\alpha+2\nu} \pi^{\frac{n}{2}} \Gamma\left(\frac{2\alpha+2\nu}{2}\right)}{\Gamma\left(\frac{n-\alpha-2\nu}{2}\right)} \ . \tag{IX, 2}$$

We introduce the differential operator $(\Box - \lambda)$, where $\Box$ is the n-dimensional ultrahyperbolic kernel and $\lambda \in \mathbb{C}$.

We shall consider the distributions

$$\{\Box - \lambda\delta\} = \sum_{\nu=0} \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-\nu-1)}{\nu!} (-1)^\nu \, \Box^{-\alpha-\nu} \delta \ . \tag{IX, 3}$$

The formula (IX,3) is the formal development of the left-hand member by the binomial Newton formula.

Remembering the formula (II,7) we shall put, by definition,

$$\Box^{(-\alpha-\nu)} \delta = R_{2(\alpha+\nu)}(P \pm i0) \ . \tag{IX, 4}$$

The general term a_ν of the series (IX,3) is

$$a_\nu = \frac{\alpha(\alpha+1)\cdots(\alpha+\nu-1)}{\nu!}\frac{(P\pm i0)^\nu \Gamma\left(\frac{\alpha-2\alpha-2\nu}{2}\right)(P\pm i0)^{\frac{2\alpha-n}{2}}}{2^{2\alpha+2\nu}\pi^{\frac{n}{2}}\Gamma(\alpha+\nu)} \ . \qquad (\mathrm{IX},5)$$

Equivalently, from (II,20),

$$a_\nu = \frac{(P\pm i0)^{\frac{2\alpha-n}{2}}\lambda^\nu (P\pm i0)^\nu \Gamma\left(\frac{n-2\alpha-2\nu}{2}\right)}{\pi^{\frac{n}{2}}\nu! 2^{2\alpha}2^{2\nu}\Gamma(\alpha)} \ . \qquad (\mathrm{IX},6)$$

Also, we know, from (II,14)

$$\Gamma\left(\frac{n-2\alpha-2\nu}{2}\right)\cdot\Gamma\left(\frac{2\nu+2\alpha-n}{2}\right) = \frac{-\pi}{\left(\frac{2\alpha+2\nu-n}{2}\right)\sin\left[\pi\left(\frac{2\nu+2\alpha-n}{2}\right)\right]} \ . \qquad (\mathrm{IX},7)$$

Therefore,

$$\Gamma\left(\frac{n-2\alpha-2\nu}{2}\right) = \frac{-\pi}{\left(\frac{2\nu+2\alpha-n}{2}\right)\sin\left[\pi\left(\frac{2\nu+2\alpha-n}{2}\right)\right]\Gamma\left(\frac{2\nu+2\alpha-n}{2}\right)} \ . \qquad (\mathrm{IX},8)$$

Putting (IX,8) into (IX,6), we have

$$a_\nu = \frac{(P\pm i0)^{\frac{2\alpha-n}{2}}\left[\lambda(P\pm i0)\right]^\nu(-\pi)}{\pi^{\frac{n}{2}}\Gamma(\alpha)2^{2\alpha}\nu! 2^{2\nu}\ \left(\frac{2\nu+2\alpha-n}{2}\right)\sin\left[\pi\left(\frac{2\nu+2\alpha-n}{2}\right)\right]\Gamma\left(\frac{2\nu+2\alpha-n}{2}\right)} \ . \qquad (\mathrm{IX},9)$$

Also, taking into account (II,13), we arrive at

$$a_\nu = \frac{(P\pm i0)^{\frac{2\alpha-n}{2}}\left[\lambda(P\pm i0)\right]^\nu(-\pi)}{\pi^{\frac{n}{2}}\Gamma(\alpha)2^{2\alpha}\nu! 2^{2\nu}\Gamma\left(\frac{2\nu+2\alpha-n}{2}+1\right)\sin\left[\pi\left(\frac{2\nu+2\alpha-n}{2}\right)\right]} \ . \qquad (\mathrm{IX},10)$$

or, equivalently,

$$a_\nu = \frac{(P\pm i0)^{\frac{2\alpha-n}{2}}\left[\lambda(P\pm i0)\right]^\nu}{\pi^{\frac{n}{2}}\Gamma(\alpha)2^{2\alpha}(\nu!)^2\sin\left[\pi\left(\frac{2\nu+2\alpha-n}{2}\right)\right]\frac{\Gamma\left(\frac{2\nu+2\alpha-n}{2}+1\right)}{\Gamma(\nu+1)}2^{2\nu}} \ . \qquad (\mathrm{IX},11)$$

If we demand the following condition:

$$2\alpha+2\nu-n = 4r+1 \ , \qquad r = 0,1,\cdots \ ; \qquad (\mathrm{IX},12)$$

we can write

$$a_\nu \leq \frac{(P\pm i0)^{\frac{2\alpha-n}{2}}[\lambda(P\pm i0)]^\nu}{\pi^{\frac{n}{2}}\Gamma(\alpha)2^{2\alpha}2^{2\nu}\nu!\frac{\Gamma(\alpha+\nu-\frac{n}{2}+1)}{\Gamma(\nu+1)}} \; . \qquad \text{(IX, 13)}$$

The factor

$$\frac{\Gamma\left(\nu+\alpha-\frac{n}{2}+1\right)}{\Gamma(\nu+1)} \xrightarrow[\nu\to\infty]{} 1 \; ; \qquad \text{(IX, 14)}$$

then the series (IX,3) converges as $e^{\lambda\frac{(P\pm i0)}{4}}$.

We conclude that the series (IX,3) converges in every compact $K \subset I\!R^n$. Therefore, by [7], p.76, Theorem XVI, the series (IX,3) is convergent in D' .

Finally, putting $-\lambda = m^2$ we have completely justify the result (II,10) and so all the other conclusions which we have here proved.

X. Notes.

1) We observe that $R_\alpha(P\pm i0)$ ((II,1)) has all their properties valid if and only if $\alpha \neq n+2\ell$, $\ell = 0,1,\cdots$, but, when $\alpha = n+2\ell$, we can consider its finite part (Pf) (cf. [8]). Therefore, the following proposition is valid (cf. [6], pp. 42-44) "The distributional functions of $Pf \;\; R_{2k}(P\pm i0)$ are elementary solutions, for all values of k of the operator $\Box^k$:

$$\Box^k\{Pf \; R_{2k}(P\pm i0)\} = \delta \; . \text{"}$$

2) It is useful to state an equivalent definition of the distributions $(P\pm i0)^\lambda$ (cf.(I,2)). In this definition appear the distributions

$$P_+^\lambda = \begin{cases} P^\lambda & \text{if } P \geq 0, \\ 0 & \text{if } P < 0; \end{cases}$$

$$P_-^\lambda = \begin{cases} 0 & \text{if } P > 0, \\ (-P)^\lambda & \text{if } P \leq 0. \end{cases}$$

We can prove, without difficulty, that the following formula is valid ([2], p.276, formulae (2) and (2'))

$$(P+i0)^\lambda = P_+^\lambda + e^{i\pi\lambda}P_-^\lambda \; ,$$

$$(P - i0)^\lambda = P_+^\lambda + e^{i\pi\lambda} P_-^\lambda \ .$$

From these formulas, we conclude inmediately that

$$(P + i0)^\lambda = (P - i0)^\lambda = P^\lambda \ ,$$

when $\lambda = k =$ positive integer.

Therefore, we can note that $W_\alpha(P \pm i0, m) = W_\alpha(u, m)$ when $\lambda = k =$ positive integer and $u = x_1^2 + \cdots + x_\mu^2 - x_{\mu+1}^2 - \cdots - x_{\mu+\nu}^2$, $\mu + \nu = n$, n is the dimension of the space, α is a complex parameter and m a real nonnegative number. Finally, we observe that $W_\alpha(u, m)$ has the similar properties of $W_\alpha(P \pm i0, m)$ (cf. [9]).

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