Approximation by ratios of bounded analytic functions.

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Abstract

Let $\mathbb{D}$ be the unit disk and E be a compact subset of the H^∞ maximal ideal space, $M(H^\infty)$. We show that any continuous function f on an open neighborhood $U \subset M(H^\infty)$ of E such that $f|_{U\cap\mathbb{D}} \in H^\infty(U \cap \mathbb{D})$ can be uniformly approximated on E by ratios h/g, where $h, g \in H^\infty$ and g is zero free on E. We also characterize those sets E for which h/g can be replaced by h for every f. Finally, we prove that any inner function u can be written as a rational function of interpolating Blaschke products. It follows that u can be uniformly approximated by ratios of interpolating Blaschke products on any compact set $E \subset M(H^\infty)$ where u is zero free.

Introduction.

Let H^∞ be the uniform algebra of bounded analytic functions on the unit disk $\mathbb{D}$ and let $M(H^\infty)$ be its maximal ideal space. If $E \subset M(H^\infty)$ is a closed set, let $C(E)$ denote the uniform algebra of continuous functions on E. A unital subalgebra A of $C(E)$ is called 'full' if every $a \in A$ that is invertible in $C(E)$ is also invertible in A.

The main purpose of this paper is to study the closed subalgebra and the closed full subalgebra of $C(E)$ generated by $H^\infty|_E$. We denote these algebras by H^∞_E and $R_{H^\infty}(E)$, respectively. In the preliminary section we fix notation and establish the background later required. The main results are in Section 2, where we show that every bounded analytic germ on E is in $R_{H^\infty}(E)$, and we give a

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necessary and sufficient condition on E for the inclusion of all such germs in H^∞_E. This answers affirmatively a question of Bishop as anticipated by Garnett (see [1, p. 10]). All these results turned out to be completely analogous to Runge's rational and polynomial approximation theorems for analytic germs on compact subsets of the complex plane.

In Section 3 we show that every $f \in R_{H^\infty}(E)$ can be approximated on E by ratios a/b, where $a \in H^\infty$ and b is an interpolating Blaschke product. When $f \in R_{H^\infty}(E)^{-1}$ (i.e.: f is invertible in $R_{H^\infty}(E)$) it is also possible to take $a = cd$, with $c \in (H^\infty)^{-1}$ and d an interpolating Blaschke product.

Although with very different techniques, the last section is inspired in Jones's version of the Douglas-Rudin theorem. It provides an approximation theorem for an inner function u on any compact set $E \subset M(H^\infty)$ where u is zero free by ratios of interpolating Blaschke products. It is also proved that u can be written as a rational function of interpolating Blaschke products (without taking limits).

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1 Preliminaries.

If X is a compact Hausdorff space we write $C(X)$ for the uniform algebra of complex valued continuous functions on X. Let A be a (complex) uniform algebra. The maximal ideal space of A is

$$M(A) = \{\varphi : A \to \mathbb{C} : \varphi \text{ is linear, multiplicative, and } \varphi(1) = 1\}$$

endowed with the weak $*$ topology induced by the dual space of A. It is known that $M(A)$ is a compact Hausdorff space, and the Gelfand transform $\hat{}: A \to C(M(A))$, defined by $\hat{a}(\varphi) = \varphi(a)$, allows us to think of A as a closed subalgebra of $C(M(A))$. For a compact subset $E \subset M(A)$ let A_E be the closure in $C(E)$ of the restriction algebra $\hat{A}|_E$ and $R_A(E)$ be the closure in $C(E)$ of

$$\{\hat{a}|_E \;/\; \hat{b}|_E : a, b \in A \text{ with } \hat{b} \text{ zero free on } E\}.$$

Clearly, $R_A(E)$ is the minimal closed full subalgebra of $C(E)$ containing $\hat{A}|_E$. Next we are going to identify their maximal ideal spaces. The A-hull of E is

$$\hat{E} = \{\varphi \in M(A) : |\hat{a}(\varphi)| \le \sup_E |\hat{a}| \text{ for all } a \in A\}$$

and the A-rational hull of E is

$$\tilde{E} = \{\varphi \in M(A) : \hat{a}(\varphi) \in \hat{a}(E) \text{ for all } a \in A\}.$$

We say that E is A-convex if $E = \hat{E}$ and that E is A-rationally convex if $E = \tilde{E}$. The next lemma is a very particular case of Lemmas 28.6 and 29.6 in [15] (see also [15, Lemma 29.4]).

Lemma 1.1 *Let A be a uniform algebra and suppose that $E \subset M(A)$ is a compact set. Then $M(A_E) = \hat{E}$ and $M(R_A(E)) = \tilde{E}$.*

The lemma easily implies that $A_E = A_{\hat{E}}$ and $R_A(E) = R_A(\tilde{E})$. We will study these algebras when $A = H^\infty$ and $E \subset M(H^\infty)$ is a compact set. In [16, Thm. 2.4] it is proved that H^∞ is a separating algebra, meaning that $E = \tilde{E}$ for every compact set $E \subset M(H^\infty)$. Thus Lemma 1.1 implies that $M(R_{H^\infty}(E)) = E$.

In [1] Bishop introduced the algebra of bounded analytic germs for compact subsets of $M(H^\infty)$ as follows. Given a compact set $E \subset M(H^\infty)$ let $H(E)$ be the functions f on E which have a continuous extension F to some open neighborhood U of E such that F is bounded and analytic on $U \cap \mathbb{D}$ (that is, $F \in H^\infty(U \cap \mathbb{D})$). Conversely, if $U \subset M(H^\infty)$ is an open neighborhood of E and $F \in H^\infty(U \cap \mathbb{D})$ then Theorem 3.2 of [16] asserts that for any open set $V \subset M(H^\infty)$ such that $E \subset V \subset \overline{V} \subset U$, the function F extends by continuity to a function $\tilde{F} \in C(\overline{V})$. This means that $\tilde{F}|_E$ is in $H(E)$. Let $H^\infty_{\text{loc}}(E)$ denote the closure in $C(E)$ of $H(E)$. Clearly, $H^\infty_E \subset R_{H^\infty}(E) \subset H^\infty_{\text{loc}}(E)$. We prove in the next section that the last two algebras always coincide and give a necessary and sufficient condition for the equality of the three algebras.

2 Runge's theorem for H^∞.

A well known result of Runge states that if E is a compact subset of the complex plane $\mathbb{C}$ then every analytic function on some open neighborhood of E can be uniformly approximated on E by rational functions without poles on E. An easy corollary is that every function as above can be uniformly approximated on E by polynomials if and only if $\mathbb{C} \setminus E$ is connected, or equivalently, if and only if E coincides with its polynomial hull, $\hat{E} = \{z \in \mathbb{C} : |p(z)| \leq \sup_E |p| \text{ for every polynomial } p\}$. Such a compact set E is called polynomially convex. We will see that both results have analogous versions when the polynomials are replaced by H^∞ and E is a compact subset of $M(H^\infty)$. In [1, Lemma 7.1] Bishop proved that $H^\infty|_E = H(E)$ when E is the set of common zeros for a finite family of functions in H^∞, and he asked for a characterization of the compact sets $E \subset M(H^\infty)$ such that $H^\infty_E = H^\infty_{\text{loc}}(E)$. As is the case for Runge's theorem, Garnett suggested that the answer should be $E = \hat{E}$. This fact will be a consequence of our main result, Theorem 2.4.

Some auxiliary lemmas are required; the first one is an estimate obtained by Garnett and Jones in [7, Lemma 3.3] for bounded solutions of a $\overline{\partial}$ equation. For $z, \omega \in \mathbb{D}$ the pseudohyperbolic metric is $\rho(z, \omega) = |(z - \omega)/(1 - \overline{\omega} z)|$.

Lemma 2.1 *Let $\psi \in C^\infty(\mathbb{D})$ such that for some $0 < \alpha < 1$ and $M > 0$, ψ is supported in a pseudohyperbolic α-neighborhood of a Carleson contour Λ and $|\psi(z)| \leq M(1-|z|)^{-1}$. Then there exists $F \in C^\infty(\mathbb{D})$ such that $\overline{\partial} F = \psi$ and $\|F\|_\infty \leq C(\Lambda, \alpha, M)$.*

It will be convenient to fix some notation. We avoid writing the 'hat' for the Gelfand transform unless the contrary is stated. Also, we need to distinguish between sets defined by inequalities on $\mathbb{D}$ or on $M(H^\infty)$. So, if $f \in H^\infty$ and $\alpha > 0$ let us establish the following conventions for the rest of the paper,

$$\{|u| < \alpha\} \stackrel{\text{def}}{=} \{x \in M(H^\infty) : |u(x)| < \alpha\}$$

and

$$\{|u(z)| < \alpha\} \stackrel{\text{def}}{=} \{z \in \mathbb{D} : |u(z)| < \alpha\}.$$

The same conventions will stand for any other inequality or equality.

Lemma 2.2 *Let $E \subset M(H^\infty)$ be a compact set and let $U \subset M(H^\infty)$ be an open neighborhood of E. There exist finitely many functions $u_j \in H^\infty$, with $\|u_j\| \leq 1$ for $1 \leq j \leq N$, such that for some $0 < \alpha < \beta < 1$,*

$$E \subset \bigcap_{j=1}^{N} \{|u_j| > \beta\} \subset \bigcap_{j=1}^{N} \{|u_j| \geq \alpha\} \subset U. \tag{2.1}$$

Proof. Since H^∞ is a separating algebra, for every $x \in M(H^\infty) \setminus U$ there is $v_x \in H^\infty$ with $\|v_x\| \leq 1$, such that $v_x(x) = 0$ and $\inf_E |v_x| > 0$. By the compactness of $M(H^\infty) \setminus U$ there are $v_1, \ldots, v_N \in H^\infty$ zero free on E, with $\|v_j\| \leq 1$ for $1 \leq j \leq N$, so that $M(H^\infty) \setminus U \subset \bigcup_{j=1}^{N} \{|v_j| < (1/4)\inf_E |v_j|\}$. For $1 \leq j \leq N$ put $\alpha_j = (1/4)\inf_E |v_j|$ and $\beta_j = 2\alpha_j$. Therefore $0 < \alpha_j < \beta_j < 1$ and

$$E \subset \bigcap_{1 \leq j \leq N} \{|v_j| > \beta_j\} \subset \bigcap_{1 \leq j \leq N} \{|v_j| \geq \alpha_j\} \subset U. \tag{2.2}$$

Changing the functions v_j if necessary, we can assume that $\alpha_j = \alpha$ and $\beta_j = \beta$ for some $0 < \alpha < \beta < 1$ and all j without affecting the norm restrictions. To do so, we first show by induction on N that there are positive integers n_j such that

$$\max\{\alpha_j^{n_j} : 1 \leq j \leq N\} < \min\{\beta_j^{n_j} : 1 \leq j \leq N\}. \tag{2.3}$$

For $N = 1$ there is nothing to prove. So, for $N > 1$ suppose that there are n'_j $(1 \leq j \leq N-1)$ such that

$$a = \max\{\alpha_j^{n'_j} : 1 \leq j \leq N-1\} < \min\{\beta_j^{n'_j} : 1 \leq j \leq N-1\} = b.$$

If n, m are positive integers such that $\log b / \log \beta_N < n/m < \log a / \log \beta_N$, then $a^m < \beta_N^n < b^m$. Thus (2.3) follows taking $n_j = mn'_j$ for $1 \le j \le N-1$ and $n_N = n$.

Henceforth (2.2) implies (2.1) with $u_j = v_j^{n_j}$, $\alpha = \max\{\alpha_j^{n_j} : 1 \le j \le N\}$ and $\beta = \min\{\beta_j^{n_j} : 1 \le j \le N\}$. $\diamondsuit$

The next lemma uses two well known inequalities for the metric ρ. If $u \in H^\infty$ satisfies $\|u\|_\infty \le 1$, the Schwarz-Pick inequality says that $\rho(u(z_1), u(z_2)) \le \rho(z_1, z_2)$ for all $z_1, z_2 \in \mathbb{D}$. Also, in [6, p. 4] it is shown that

$$\rho(z_1, z_2) \ge \rho(|z_1|, |z_2|) \ge ||z_1| - |z_2|| \quad \text{for all} \quad z_1, z_2 \in \mathbb{D}. \tag{2.4}$$

Lemma 2.3 *Let $u_j \in H^\infty$ with $\|u_j\| \le 1$ $(1 \le j \le N)$ and suppose that $0 < \gamma_1 < \gamma_2 < 1$. Then there exists $\phi \in C^\infty(\mathbb{D})$ with values in $[0,1]$ such that*
(a) *$\phi \equiv 1$ on $\bigcap_{j=1}^N \{|u_j(z)| \ge \gamma_2\}$,*
(b) *$\phi \equiv 0$ on $\bigcup_{j=1}^N \{|u_j(z)| \le \gamma_1\}$,*
(c) *there is a Carleson contour $\Lambda \subset \mathbb{D}$ and $0 < \rho_0 < 1$ such that $\nabla\phi$ is supported in $\{z \in \mathbb{D} : \rho(z, \Lambda) \le \rho_0\}$, and*
(d) *there is a constant $K(\rho_0) > 0$ such that $\sup_{\mathbb{D}} (1 - |z|)|\nabla\phi(z)| \le K(\rho_0)$.*

Proof. Put $\varepsilon = (\gamma_2 - \gamma_1)/4$. The function $\varphi(x) = \min\{|u_1(x)|, \dots, |u_N(x)|\}$ is continuous on $M(H^\infty)$. By a result of Bishop [2, Thm. 1.1] there exists a Carleson contour $\Lambda' \subset \mathbb{D}$ such that for any connected component $\mathcal{O} \subset \mathbb{D} \setminus \Lambda'$,

$$\sup\{|\varphi(z) - \varphi(\omega)| : z, \omega \in \mathcal{O}\} \le \varepsilon, \tag{2.5}$$

and such that the family $\mathcal{L} = \{\mathcal{O} : \mathcal{O} \text{ is a connected component of } \mathbb{D} \setminus \Lambda'\}$ is locally finite. The local finitiness of $\mathcal{L}$ is not expressly stated in Bishop's theorem but it follows from his construction. More precisely, Figure 2.1 in [2] illustrates the fact that for every $z \in \mathbb{D}$ there is an open neighborhood $V \subset \mathbb{D}$ of z such that V meets no more than four sets $\mathcal{O} \in \mathcal{L}$. We say that a component $\mathcal{O}$ is blue if

$$\inf\{\varphi(z) : z \in \mathcal{O}\} \ge (\gamma_1 + \gamma_2)/2. \tag{2.6}$$

Otherwise $\mathcal{O}$ will be called a red component. Consider the region $R \subset \mathbb{D}$ formed by the closure (in $\mathbb{D}$) of $\bigcup\{\mathcal{O} \text{ is a blue component}\}$. The local finiteness of $\mathcal{L}$ easily implies that a point $\omega \in \mathbb{D}$ is in ∂R if and only if there are a blue component $\mathcal{O}_b$ and a red component $\mathcal{O}_r$ such that $\omega \in \overline{\mathcal{O}}_b \cap \overline{\mathcal{O}}_r$. So, $\partial R \subset \Lambda'$. Moreover, we claim that if $\omega \in \partial R$ then
(i) $|u_j(\omega)| \ge (\gamma_1 + \gamma_2)/2$ for all $1 \le j \le N$, and
(ii) $|u_{j_0}(\omega)| \le (\gamma_1 + \gamma_2)/2 + \varepsilon$ for some $1 \le j_0 \le N$ depending on ω.

In fact, since $\omega \in \overline{\mathcal{O}}_b$ then (i) is an immediate consequence of (2.6). On the other hand, since $\mathcal{O}_r$ does not satisfy (2.6) then there is $z_0 \in \mathcal{O}_r$ such that $\varphi(z_0) < (\gamma_1+\gamma_2)/2$. Therefore (2.5) implies that $\varphi(z) \le (\gamma_1+\gamma_2)/2+\varepsilon$ for every $z \in \mathcal{O}_r$. In particular, this inequality holds for $\omega \in \overline{\mathcal{O}}_r$ and then (ii) follows from the definition of φ.

Let $z \in \bigcap\{|u_j(z)| \ge \gamma_2\}$ and $\omega \in \partial R$. By (ii), (2.4) and the Schwarz-Pick inequality

$$\rho(z,\omega) \ge \rho(u_{j_0}(z), u_{j_0}(\omega)) \ge \gamma_2 - [(\gamma_1+\gamma_2)/2+\varepsilon] = (\gamma_2-\gamma_1)/4 = \varepsilon. \quad (2.7)$$

Now let $z \in \bigcup\{|u_j(z)| \le \gamma_1\}$ (so, there is $1 \le j_1 \le N$ such that $|u_{j_1}(z)| \le \gamma_1$) and $\omega \in \partial R$. By (i), (2.4) and the Schwarz-Pick inequality,

$$\rho(\omega, z) \ge \rho(u_{j_1}(\omega), u_{j_1}(z)) \ge (\gamma_1+\gamma_2)/2 - \gamma_1 = (\gamma_2-\gamma_1)/2 = 2\varepsilon. \quad (2.8)$$

Therefore $\bigcap\{|u_j(z)| \ge \gamma_2\} \subset R$, $\bigcup\{|u_j(z)| \le \gamma_1\} \subset I\!\!D \setminus R$, and the pseudohyperbolic distance from any of these sets to ∂R is bounded below by ε. We can now fix $\rho_0 < \varepsilon$ and modify the characteristic function χ_R within a pseudohyperbolic ρ_0-neighborhood of ∂R in order to obtain a function $\phi \in C^\infty(I\!\!D)$ satisfying (d). There is a standard procedure to obtain ϕ from χ_R using a partition of the unity and convolving with suitable smoth functions (see [6, pp. 356-357]). We outline a proof. For every integer $n \ge 0$ consider the circular strips $T_n = \{1-2^{-n} \le |z| < 1-2^{-n-1}\}$ and $\tilde{T}_n = T_{n-1} \cup T_n \cup T_{n+1}$ if $n \ge 1$. We can modify χ_R on each $\tilde{T}_n$ in order to obtain $h_n \in C^\infty(\tilde{T}_n)$ such that $0 \le h_n \le 1$, $h_n(z) = \chi_R(z)$ if $\operatorname{dist}(z, \partial R) > \rho_0/2^{n+2}$, and $|\nabla h_n| \le C(\rho_0)2^n$ for some constant $C(\rho_0) > 0$. Take $g_n \in C^\infty(I\!\!D)$ supported in $\tilde{T}_n$ such that $0 \le g_n \le 1$, $\sum_{n=0}^\infty g_n = 1$ and $|\nabla g_n| \le C2^n$, where $C > 0$ is an absolute constant.

Let $\phi \overset{\text{def}}{=} \sum_{n=0}^\infty g_n h_n$ and fix an arbitrary $z \in T_j$ (for some $j \ge 0$). A straightforward calculation shows that $\phi(z) = \chi_R(z)$ if $\operatorname{dist}(z, \partial R) > \rho_0/2^{j+2}$ and $|\nabla g(z)| \le K2^j$, where K is a positive constant depending only on ρ_0. Condition (d) follows because $2^j(1-|z|) \le 1$. Furthermore, an elementary inequality shows that if $z \in T_j$ and $\operatorname{dist}(z, \partial R) > \rho_0/2^{j+1}$ then $\rho(z, \partial R) > \rho_0$. Thus ϕ coincides with χ_R for points whose pseudohyperbolic distance to ∂R is more than ρ_0. That is, (c) holds for the Carleson contour $\Lambda = \partial R$. Moreover, (a) and (b) are immediate consequences of (2.7) and (2.8), respectively. $\diamondsuit$

Theorem 2.4 *Let $E \subset M(H^\infty)$ be a compact set. Then $H^\infty_{\text{loc}}(E) = R_{H^\infty}(E)$.*

Proof. By the density of $H(E)$ in $H^\infty_{\text{loc}}(E)$ it suffices to show that $H(E) \subset R_{H^\infty}(E)$. Let $U \subset M(H^\infty)$ be an open neighborhood of E and $f \in H^\infty(U \cap I\!\!D)$. By Lemma 2.2 there are $u_j \in H^\infty$, with $\|u_j\| \le 1$ for $1 \le j \le N$, and $0 < \alpha < \beta < 1$ such that the chain of inclusions (2.1) holds. For $k = 1, 2, 3, 4$ put $\gamma_k = \alpha + k(\beta - \alpha)/5$. So,

$$\alpha < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < \beta. \tag{2.9}$$

Let ϕ be the function provided by Lemma 2.3 corresponding to u_j $(1 \le j \le N)$ and the parameters γ_1 and γ_2. By (2.1) and (2.9),

$$I\!\!D \setminus U \subset \bigcup_{j=1}^{N} \{|u_j(z)| < \alpha\} \subset \bigcup_{j=1}^{N} \{|u_j(z)| < \gamma_1\}.$$

Hence, by condition (b) of Lemma 2.3 $\phi \equiv 0$ on an open neighborhood of $I\!\!D \setminus U$, implying that ϕf is a well defined bounded function on the whole disk, which in addition belongs to $C^\infty(I\!\!D)$.

It is clear from (2.9) that the compact sets $X = \bigcap_{j=1}^{N} \{|u_j| \ge \gamma_4\}$ and $Y = \bigcup_{j=1}^{N} \{|u_j| \le \gamma_3\}$ are disjoint.

Consider the algebra $R_{H^\infty}(X \cup Y)$. Since X is a closed-open subset of $X \cup Y = M(R_{H^\infty}(X \cup Y))$ then Shilov's idempotent theorem asserts that $R_{H^\infty}(X \cup Y)$ contains the characteristic function of X. Therefore, for a fixed $0 < \eta < 1/4$ there exist $g, h \in H^\infty$, with g zero free on $X \cup Y$, such that

$$|h/g| < \eta \quad \text{on} \quad Y \qquad \text{and} \qquad |h/g - 1| < \eta \quad \text{on} \quad X. \tag{2.10}$$

In particular, h is zero free on X. On the other hand, by Lemma 2.3 (a),

$$\operatorname{supp} \overline{\partial} \phi \subset \bigcup_{j=1}^{N} \{|u_j(z)| \le \gamma_2\} \subset \bigcup_{j=1}^{N} \{|u_j(z)| \le \gamma_3\} = Y \cap I\!\!D.$$

Consequently, $|h/g| < \eta$ on $\operatorname{supp} \overline{\partial}\phi$. Now Lemmas 2.3 and 2.1 tell us that for every positive integer n, the differential equation

$$\overline{\partial} a_n = (h/g)^n f \overline{\partial} \phi$$

has a solution $a_n \in C^\infty(I\!\!D)$ satisfying the uniform estimate

$$\sup_{I\!\!D} |a_n| \le C(\phi) \sup_{U} |f| \, \eta^n = C' \eta^n, \tag{2.11}$$

where $C(\phi)$ is a constant depending on ϕ. Thus, the functions $F_n = f\phi h^n - g^n a_n$ are in H^∞ for every $n \ge 1$. Let $z \in X \cap I\!\!D$. Since ϕ satisfies (a) of Lemma 2.3 then (2.9) implies that $\phi(z) = 1$, and (2.10) implies that $|g(z)/h(z)| < (1 - \eta)^{-1}$. These facts and (2.11) give on $X \cap I\!\!D$

$$|F_n/h^n - f| = |f\phi - (g/h)^n a_n - f| = |(g/h)^n a_n| < C' \frac{\eta^n}{(1 - \eta)^n},$$

which tends to zero as $n \to \infty$. That is, the sequence $F_n/h^n \in R_{H^\infty}(X)$ tends uniformly to f on the set $X \cap \mathbb{D}$. By (2.1) and (2.9) X is a neighborhood of E. Hence the corona theorem implies that f is the uniform limit on E of F_n/h^n. $\diamondsuit$

Lemma 2.5 *Let $K_1 \subset K \subset M(H^\infty)$ be compact sets. Consider the uniform algebra A defined as the closure in $C(K_1)$ of the restriction algebra $R_{H^\infty}(K)|_{K_1}$. Then $A = R_{H^\infty}(K_1)$ if and only if K_1 is $R_{H^\infty}(K)$-convex.*

Proof. By Lemma 1.1 the maximal ideal space $M(A)$ coincides with the $R_{H^\infty}(K)$-hull of K_1.

If $A = R_{H^\infty}(K_1)$ then $K_1 = M(A)$, and consequently K_1 is $R_{H^\infty}(K)$-convex. On the other hand, if K_1 is $R_{H^\infty}(K)$-convex then $K_1 = M(A)$. It is clear that $A \subset R_{H^\infty}(K_1)$, so we must prove the other inclusion. For any $f \in R_{H^\infty}(K_1)$ and $\varepsilon > 0$ there are $h, g \in H^\infty$ such that g is zero free on K_1 and $\sup_{K_1} |f - h/g| < \varepsilon$. So, it will be enough to prove that $h|_{K_1}$ and $(g|_{K_1})^{-1}$ belong to A. Since $H^\infty|_{K_1} \subset R_{H^\infty}(K)|_{K_1} \subset A$ then $h|_{K_1}, g|_{K_1} \in A$. In addition, since g never vanishes on $K_1 = M(A)$ then $g|_{K_1} \in A^{-1}$. $\diamondsuit$

Corollary 2.6 *Let $E \subset M(H^\infty)$ be a compact set. Then $H^\infty_{\mathrm{loc}}(E) = H^\infty_E$ if and only if E is H^∞-convex.*

Proof. Apply Lemma 2.5 to $K = M(H^\infty)$ and $K_1 = E$. In this case $R_{H^\infty}(K) = H^\infty$ and $A = H^\infty_E$. Hence, the lemma says that $H^\infty_E = R_{H^\infty}(E)$ if and only if E is H^∞-convex. The corollary follows from Theorem 2.4. $\diamondsuit$

Some additional properties of the compact set $E \subset M(H^\infty)$ translate into particular features of the algebra $R_{H^\infty}(E)$. Many classical algebras associated to H^∞ appear in this way. We give an account of several of these algebras.

Let $S(H^\infty)$ denote the Shilov boundary of H^∞. For $x \in M(H^\infty)$ let $\operatorname{supp} \mu_x \subset S(H^\infty)$ be the support of the unique representing measure μ_x of x. Using that $\operatorname{supp} \mu_x$ is an intersection of peak sets for H^∞ [10, p. 207], it is easy to prove that $H^\infty|_{\operatorname{supp} \mu_x}$ is a uniform algebra (see [5, p. 65]). Therefore $H^\infty|_{\operatorname{supp} \mu_x} = H^\infty_{\operatorname{supp} \mu_x}$ and its maximal ideal space is $(\operatorname{supp} \mu_x)\hat{}$, the H^∞-hull of $\operatorname{supp} \mu_x$.

If $E \subset M(H^\infty)$ is a compact set with the properties

(a) $(\operatorname{supp} \mu_x)\hat{} \subset E$ whenever $x \in E$, and

(b) $S(H^\infty) \subset E$,

then $R_{H^\infty}(E)$ is a Douglas algebra, and every Douglas algebra has this form (see [6, IX]). When E satisfies property (a) regardless of whether it satisfies (b) or not, the algebra $R_{H^\infty}(E)$ shares many of the features of Douglas algebras. Indeed, suppose that $f \in R_{H^\infty}(E)$ and $x \in E$. By Corollary 2.6 and the above comments, $f|_{\operatorname{supp} \mu_x} \in H^\infty|_{\operatorname{supp} \mu_x}$. Then $|f(x)| \leq \sup\{|f(y)| : y \in \operatorname{supp} \mu_x\}$, implying that the Shilov bounday of $R_{H^\infty}(E)$ is contained in $E \cap S(H^\infty)$. The other inclusion also holds, because $R_{H^\infty}(E) \supset H^\infty|_E$ and every point in $S(H^\infty)$ is a strong boundary point for H^∞ (see [15, Thm. 7.18]).

Hence, we can think of $R_{H^\infty}(E)$ as a closed subalgebra of $C(E \cap S(H^\infty))$ that contains $H^\infty|_{E\cap S(H^\infty)}$. It follows that $R_{H^\infty}(E)$ can be identified with the restriction to $E \cap S(H^\infty)$ of the Douglas algebra

$$\{f \in C(S(H^\infty)) : f|_{E\cap S(H^\infty)} \in R_{H^\infty}(E)|_{E\cap S(H^\infty)}\}.$$

The fiber algebras appear in this way, where for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the fiber of λ is $M_\lambda = \{\varphi \in M(H^\infty) : \hat{z}(\varphi) = \lambda\}$ and the fiber algebra of λ is $H^\infty|_{M_\lambda}$. Here $\hat{z}$ denotes the Gelfand transform of the identity function. These algebras have been studied by Hoffman in [10]. He proved that $H^\infty|_{M_\lambda}$ is closed in $C(M_\lambda)$, and since M_λ is H^∞-convex then $H^\infty|_{M_\lambda} = H^\infty_{M_\lambda} = R_{H^\infty}(M_\lambda)$.

Let Q be the closure in $M(H^\infty)$ of a nontrivial Gleason part. In [9] Gorkin proved that Q is the zero set of the ideal $\{f \in H^\infty : f|_Q \equiv 0\}$ and in [18] the author proved that $H^\infty|_Q$ is closed in $C(Q)$. Therefore $H^\infty|_Q = H^\infty_Q$ and $M(H^\infty_Q) = Q$.

3 The algebra $R_{H^\infty}(E)$.

In this section we study the structure of the algebra $R_{H^\infty}(E)$. Let $\Gamma \subset M(H^\infty)$ denote the set of trivial Gleason parts for H^∞. We know from Hoffman's theory (see [11]) that a point $x \in M(H^\infty)$ is in Γ if and only if it is not a zero of an interpolating Blaschke product, or what is the same, it is not in the closure of an interpolating sequence. The set Γ is closed and totally disconnected (see [11, p. 88] and [17, Thm. 3.4]).

Let $\mathcal{F} \in H^\infty$ denote the set of finite products of interpolating Blaschke products, where by interpolating we also mean finite Blaschke products with simple zeros and unimodular constants. It is easy to see that the interpolating Blaschke products form a dense subset of $\mathcal{F}$ (see for instance [14, Lemma 1]).

A well known theorem of Guillory, Izuchi and Sarason [8] asserts that a function $f \in H^\infty$ is zero free on Γ if and only if $f = gb$, where $g \in (H^\infty)^{-1}$ and $b \in \mathcal{F}$. We refer to this results as the G-I-S theorem. It is easy to prove that $S(H^\infty) \subset \Gamma$, and though $S(H^\infty)$ is extremely disconnected (i.e.: the closure of an open subset is open), recently Ishii and Izuchi [12] proved that Γ does not have this property. We are not going to use this fact here.

Lemma 3.1 *Let $E \subset M(H^\infty)$ be a compact set. Then E is $R_{H^\infty}(\Gamma \cup E)$-convex.*

Proof. Let $x \in \Gamma \setminus E$. We must show that there is $f \in R_{H^\infty}(\Gamma \cup E)$ such that $|f(x)| > \sup_E |f|$.

As said before, every point in $S(H^\infty)$ is a strong boundary point for H^∞. So, if $x \in S(H^\infty)$ there is $f \in H^\infty$ such that $|f(x)| > \sup_E |f|$. If $x \in \Gamma \setminus S(H^\infty)$ then by [16, p. 243] there is an interpolating Blaschke product b such that

$$|b(x)| \leq 1/4 \quad \text{and} \quad \inf_E |b| \geq 1/2. \tag{3.1}$$

The last inequality implies that b is zero free on E, and since b is interpolating, it is also zero free on Γ. Therefore $f = 1/b \in R_{H^\infty}(\Gamma \cup E)$ and by (3.1), $|f(x)| \geq 4 > 2 \geq \sup_E |f|$. ♢

Our next lemma is proved in [16, p. 248] for H^∞, but the proof there works for an arbitrary (complex) uniform algebra. A wide generalization is implicit in [3]. The following proof is due to R. Rupp and it was provided to me by the referee.

Let us denote $\|f\|_E \stackrel{\text{def}}{=} \sup_E |f|$ for $f \in C(E)$, and $\mathbb{C}_* \stackrel{\text{def}}{=} \mathbb{C} \setminus \{0\}$. It is well known that if A is a uniform algebra then $\exp A \stackrel{\text{def}}{=} \{e^a : a \in A\}$ is the connected component of the unit in the topological group A^{-1}. In addition, a famous theorem of Arens and Royden [5, pp. 88-91] states that

$$C(M(A))^{-1} = A^{-1}.\exp C(M(A)) \quad \text{and} \quad \exp C(M(A)) \cap A^{-1} = \exp A^{-1}.$$

Lemma 3.2 *Let A be a uniform algebra and $E \subset M(A)$ be a compact A-convex set. Suppose that $f \in A_E^{-1}$. Then the following statements are equivalent.*
(1) *Given $\varepsilon > 0$ there is $h \in A^{-1}$ such that $\|h - f\|_E < \varepsilon$.*
(2) *There is a continuous map $F : M(A) \to \mathbb{C}_*$ such that $F|_E = f$.*

Proof. We recall that the A-convexity of E implies that $E = M(A_E)$. Suppose that (1) holds and choose $\varepsilon = \|f^{-1}\|_E^{-1}$. Then $\|hf^{-1} - 1\|_E < \varepsilon \|f^{-1}\|_E < 1$, implying that there is $k \in C(E)$ such that $hf^{-1} = e^k$ on E. If $K : M(A) \to \mathbb{C}$ is any continuous extension of k then $F = he^{-K}$ satisfies (2).

Now suppose that (2) holds. By the Arens-Royden theorem there are $G \in A^{-1}$ and $H \in C(M(A))$ such that $F = Ge^H$. Restricting this equality to E we obtain

$$e^H|_E = fG|_E^{-1} \in A_E^{-1} \cap \exp C(E).$$

A new application of the Arens-Royden theorem gives a function $k \in A_E$ such that $fG|_E^{-1} = e^k$. Therefore, $f = e^k G|_E$ and (1) follows by taking $K \in A$ such that $\|k - K\|_E$ is small enough and $h = e^K G$. ♢

Theorem 3.3 *Let $E \subset M(H^\infty)$ be a compact set and let $\varepsilon > 0$. Then*
(I) *For every $f \in R_{H^\infty}(E)$ there is $h \in H^\infty$ and an interpolating Blaschke product b without zeros on E such that*

$$\|f - h/b\|_E < \varepsilon.$$

(II) *For every* $f \in R_{H^\infty}(E)^{-1}$ *there are two interpolating Blaschke products* b_1 *and* b_2 *without zeros on* E*, and* $g \in (H^\infty)^{-1}$ *such that*

$$\|f - gb_1/b_2\|_E < \varepsilon.$$

Proof. By Lemmas 2.5 and 3.1 the restriction of $R_{H^\infty}(\Gamma \cup E)$ to E is dense in $R_{H^\infty}(E)$. Thus, if $f \in R_{H^\infty}(E)$ there are $f_1, f_2 \in H^\infty$ with f_2 zero free on $\Gamma \cup E$, such that

$$\|f - f_1/f_2\|_E < \varepsilon \tag{3.2}$$

The G-I-S theorem now says that $f_2 = ab$, where $a \in (H^\infty)^{-1}$ and $b \in \mathcal{F}$ is zero free on E. Furthermore, by the density of the interpolating Blaschke products in $\mathcal{F}$ we can assume that b is interpolating. Hence (I) holds with $h = a^{-1}f_1$ and $b = a^{-1}f_2$.

Now suppose that $f \in R_{H^\infty}(E)^{-1}$. If we write $A = R_{H^\infty}(\Gamma \cup E)$ then Lemmas 3.1 and 2.5 say that E is A-convex and $A_E = R_{H^\infty}(E)$. So, we are entitled to use Lemma 3.2.

Since $f(E) \subset \mathbb{C}_*$ and E is a compact set then there is an open neighborhood U of E (in $\Gamma \cup E$) and a continuous function $f_0 : U \rightarrow \mathbb{C}_*$ that extends f. Using that Γ is totally disconnected it is easy to see that E has a basis of closed-open neighborhoods in the topological space $\Gamma \cup E$. Then, by restricting f_0 to a smaller neighborhood of E if necessary, we can assume that U is closed-open. Consequently, the function $F : \Gamma \cup E \rightarrow \mathbb{C}_*$ defined by

$$F = \begin{cases} f_0 & \text{on } U \\ 1 & \text{on } (\Gamma \cup E) \setminus U \end{cases}$$

is continuous on $\Gamma \cup E$ and extends f. Henceforth, Lemma 3.2 says that f can be approximated on E by functions of $R_{H^\infty}(\Gamma \cup E)^{-1}$. That is, in (3.2) we can assume that not only f_2 but also f_1 is zero free on Γ. As before, (II) now follows from the G-I-S theorem and the density of the interpolating Blaschke products in $\mathcal{F}$. ♢

4 The Douglas-Rudin-Jones theorem.

In [4] Douglas and Rudin proved that for every unimodular function u in $L^\infty = L^\infty(\partial \mathbb{D})$ and any $\varepsilon > 0$ there are inner functions b_1, b_2 such that

$$\|u - b_1/b_2\|_\infty < \varepsilon.$$

Later Jones gave a constructive proof of this fact, where in addition b_1 and b_2 are interpolating Blaschke products (see [13]). This result is interesting even when u

is an inner function. The zero sequence $\{z_n\}$ of any of the Blaschke products b_j in Jones's proof satisfies the important estimate

$$\inf_n \prod_{k \neq n} \left| \frac{z_n - z_k}{1 - \overline{z}_k z_n} \right| > \delta(\varepsilon) > 0,$$

where $\delta(\varepsilon)$ depends only on ε. Except for this estimate we can give a better result when u is inner.

Theorem 4.1 *Let u be an inner function. Given $0 < \alpha < 1$ and $0 < \eta < 1/4$ there are two interpolating Blaschke products a_1, a_2 zero free on $\{|u| \geq \alpha\}$, depending on u, α and η, such that for every*

$$\omega \in \Omega = \left\{ z \in \mathbb{D} \colon \frac{3}{4}\eta\alpha < |z| < (1 - \frac{\eta}{2})\frac{\alpha}{4} \right\}$$

there exist $a_3, a_4 \in \mathcal{F}$ zero free on $\{|u| \geq \alpha\}$, depending on u, ω, a_1 and a_2, such that

$$u(z) = \frac{a_2}{a_1}(z) \left(\frac{a_3(z) + \omega a_4(z)}{a_4(z) + \overline{\omega} a_3(z)} \right) \quad \text{for all } z \in \mathbb{D}.$$

In addition, the function $a_2/(a_4 + \overline{\omega} a_3)$ is invertible in H^∞. Furthermore, if $x \in M(H^\infty)$ is such that $|u(x)| \geq \alpha$ then

$$\left| u(x) - \frac{a_2(x) a_3(x)}{a_1(x) a_4(x)} \right| < 20\, |\omega|. \tag{4.1}$$

Proof. Consider the algebra $R_{H^\infty}(E)$, where $E = \Gamma \cup \{|u| \geq \alpha\}$. Since Γ is totally disconnected, there is a relatively closed-open set $V \subset \Gamma$ such that

$$\{x \in \Gamma : |u(x)| \leq \alpha/4\} \subset V \subset \{x \in \Gamma : |u(x)| < \alpha/2\}. \tag{4.2}$$

It is easy to see that V is also closed-open in the topological space E. By Shilov's idempotent theorem the function

$$\chi = \begin{cases} \eta & \text{on } V \\ 1 & \text{on } E \setminus V \end{cases} \tag{4.3}$$

is in $R_{H^\infty}(E)$, and hence in $R_{H^\infty}(E)^{-1}$. Then Theorem 3.3 (II) says that there are interpolating Blaschke products a_1, a_2 zero free on E and $f \in (H^\infty)^{-1}$ such that $\|\chi - f a_1/a_2\|_E < \eta/8$. Since $S(H^\infty) \subset E \setminus V$ (because $|u| \equiv 1$ on $S(H^\infty)$), then $\chi \equiv 1$ on $S(H^\infty)$ and

$$|1 - |f|| = |\chi - |f a_1/a_2|| \leq |\chi - f a_1/a_2| < \eta/8 \ \text{ on } \ S(H^\infty).$$

Furthermore, since $f \in (H^\infty)^{-1}$ then $|1-|f|| < \eta/8$ on the whole space $M(H^\infty)$. In particular, $\|f^{-1}\| < 2$. Therefore for $x \in E$,

$$\begin{aligned}
\left|\chi(x) - \left|\frac{a_1}{a_2}(x)\right|\right| &= |f^{-1}(x)| \left||f(x)\chi(x)| - \left|f(x)\frac{a_1}{a_2}(x)\right|\right| \\
&\leq 2\left(||f(x)\chi(x)| - \chi(x)| + \left|\chi(x) - \left|f(x)\frac{a_1}{a_2}(x)\right|\right|\right) \\
&\leq 2\left(||f(x)| - 1| + \left|\chi(x) - f(x)\frac{a_1}{a_2}(x)\right|\right) \\
&< 2\left(\frac{\eta}{8} + \frac{\eta}{8}\right) = \frac{\eta}{2}.
\end{aligned}$$

That is, if $x \in E$ then

$$\chi(x) - \eta/2 \leq |a_1(x)/a_2(x)| \leq \chi(x) + \eta/2. \tag{4.4}$$

From this inequality, (4.3) and (4.2) we get

$$|ua_1/a_2| \leq (3/2)\eta|u| \leq (3/4)\eta\alpha \quad \text{on } V$$

and

$$|ua_1/a_2| \geq (1-\eta/2)|u| \geq (1-\eta/2)(\alpha/4) \quad \text{on } E \setminus V.$$

Since $\eta < 1/4$ then $(3/4)\eta < (1-\eta/2)(1/4)$. Therefore the function ua_1/a_2 never takes on E any value in the set

$$\Omega = \{\omega \in \mathbb{D} : \ (3/4)\eta\alpha < |\omega| < (1-\eta/2)(\alpha/4)\}.$$

Let $\omega \in \Omega$. Then $ua_1/a_2 - \omega$ never vanishes on E. Also, for $x \in E$ (4.4) gives

$$|\overline{\omega}u(x)a_1(x)/a_2(x)| \leq |\omega|(1+\eta/2) < (\alpha/4)[1-(\eta/2)^2] < 1/4.$$

Consequently,

$$|1 - \overline{\omega}u(x)a_1(x)/a_2(x)| > 1/2 \quad \text{for} \quad x \in E. \tag{4.5}$$

Since a_2 is zero free on E then the H^∞ functions $ua_1 - \omega a_2$ and $a_2 - \overline{\omega}ua_1$ never vanish on E. In particular, they are zero free on Γ, which by the G-I-S theorem means that $ua_1 - \omega a_2 = ga_3$ and $a_2 - \overline{\omega}ua_1 = ha_4$, for some $g, h \in (H^\infty)^{-1}$ and $a_3, a_4 \in \mathcal{F}$ zero free on E. Moreover, since inner functions have modulus 1 on the Shilov boundary of H^∞, then on $S(H^\infty)$,

$$|g| = |ga_3| = |ua_1 - \omega a_2| = |a_2 - \overline{\omega}ua_1| = |ha_4| = |h|.$$

Thus $g = \lambda h$, where λ is some unimodular constant. There is no loss of generality if we assume $\lambda = 1$ (replacing h by λh and a_4 by $\overline{\lambda} a_4$ if necessary). Thus $g = h$ and

$$h(a_4 + \overline{\omega} a_3) = (a_2 - \overline{\omega} u a_1) + \overline{\omega}(u a_1 - \omega a_2) = a_2(1 - |\omega|^2),$$

implying that $a_4 + \overline{\omega} a_3$ is zero free on E and $a_2/(a_4 + \overline{\omega} a_3) = h/(1 - |\omega|^2) \in (H^\infty)^{-1}$. In addition, on E we have

$$\frac{a_3}{a_4} = \frac{u a_1 - \omega a_2}{a_2 - \overline{\omega} u a_1},$$

and then

$$u = \frac{a_2}{a_1}\left(\frac{a_3 + \omega a_4}{a_4 + \overline{\omega} a_3}\right) \quad \text{on } E. \tag{4.6}$$

But since $E \supset S(H^\infty)$ then $u a_1(a_4 + \overline{\omega} a_3) = a_2(a_3 + \omega a_4)$ on $M(H^\infty)$, meaning that (4.6) holds on $\mathbb{D}$.

In order to prove (4.1) write $\tau_{z_0}(z) = (z + z_0)/(1 + \overline{z_0} z)$ for $z, z_0 \in \mathbb{D}$. Then (4.6) reads as $u = (a_2/a_1)\tau_\omega(a_3/a_4)$. By (4.2), (4.3) and (4.4),

$$|a_2(x)/a_1(x)| \le (1 - \eta/2)^{-1} \quad \text{if} \quad |u(x)| \ge \alpha.$$

Consequently, on $\{|u| \ge \alpha\} \subset E$ we have

$$\begin{aligned}
\left|u - \frac{a_2 a_3}{a_1 a_4}\right| &= \left|\frac{a_2}{a_1}\tau_\omega\left(\frac{a_3}{a_4}\right) - \frac{a_2 a_3}{a_1 a_4}\right| = \left|\frac{a_2}{a_1}\right|\left|\tau_\omega\left(\frac{a_3}{a_4}\right) - \frac{a_3}{a_4}\right| \\
&\le (1 - \eta/2)^{-1}\left|\tau_\omega\left(\frac{a_3}{a_4}\right) - \frac{a_3}{a_4}\right| \le 2\left|\frac{u a_1}{a_2} - \tau_{-\omega}\left(\frac{u a_1}{a_2}\right)\right| \\
&= 2|\omega|\left|\frac{1 - (\overline{\omega}/\omega)(u a_1/a_2)^2}{1 - \overline{\omega}(u a_1/a_2)}\right| \overset{\text{by (4.5)}}{\le} 4|\omega|\,(1 + |(u a_1/a_2)|^2) \\
&\overset{\text{by (4.4)}}{\le} 4|\omega|\,[1 + (1 + \eta/2)^2] < 20|\omega|.
\end{aligned}$$

This proves the theorem. $\diamondsuit$

Corollary 4.2 *Let u be an inner function and $0 < \alpha < 1$. Given $\varepsilon > 0$ there are two interpolating Blaschke products b_1, b_2 zero free on $\{|u| \ge \alpha\}$ such that*

$$\sup\left\{\left|u(x) - \frac{b_1(x)}{b_2(x)}\right| : \ |u(x)| \ge \alpha\right\} < \varepsilon.$$

Proof. We can assume $\varepsilon < 1$. Retaining the notation of the theorem, take $\eta = \varepsilon/20$. Then any point ω such that $|\omega| = (\varepsilon/20)\alpha$ is in Ω. By Theorem 4.1 there are $a_j \in \mathcal{F}$ $(1 \le j \le 4)$ zero free on $\{|u| \ge \alpha\}$ such that (4.1) holds for $20|\omega| = \varepsilon\alpha < \varepsilon$. The corollary follows approximating a_2a_3 and a_1a_4 by suitable interpolating Blaschke products b_1 and b_2, respectively. $\diamondsuit$

Let u be an inner function and $x \in M(H^\infty)$. If u is zero free on $M(x) \stackrel{\text{def}}{=} M(H^\infty|_{\operatorname{supp}\mu_x})$ then u is invertible in $H^\infty|_{\operatorname{supp}\mu_x}$. Consequently $|u|_{M(x)}|$ attains its maximum and its minimum on $\operatorname{supp}\mu_x$. So, $|u(x)| = 1$ and since μ_x is a probability measure then $u \equiv e^{i\alpha} = \text{const.}$ on $\operatorname{supp}\mu_x$, and hence on $M(x)$.

Corollary 4.3 *Let u be an inner function. Then there exist $b_1, b_2 \in \mathcal{F}$ and $f \in (H^\infty)^{-1}$ such that on the set $\{|u| = 1\}$,*

$$|b_1| = |b_2| = 1 \quad \textit{and} \quad u = (f/\overline{f})b_1\overline{b}_2.$$

Proof. For fixed values of α, η and ω as in Theorem 4.1 consider the functions $a_j \in \mathcal{F}$ $(1 \le j \le 4)$ given by the theorem. Since the Blaschke products a_j are zero free on $\{|u| \ge \alpha\} \supset \{|u| = 1\}$, the comment preceding the corollary says that $|a_j| \equiv 1$ on $\{|u| = 1\}$. On $\{|u| = 1\}$ we then have $1/a_j = \overline{a}_j$, and by the theorem

$$u = \frac{a_2}{a_1}\,\frac{a_3 + \omega a_4}{a_4 + \overline{\omega} a_3} = \left(\frac{a_2}{a_4 + \overline{\omega} a_3}\right)\left(\frac{\overline{a}_4 + \omega \overline{a}_3}{\overline{a}_2}\right) a_3 a_4 \overline{a}_2 \overline{a}_1 = f(1/\overline{f}) b_1 \overline{b}_2,$$

where $b_1 = a_3a_4 \in \mathcal{F}$, $b_2 = a_2a_1 \in \mathcal{F}$ and $f = a_2/(a_4 + \overline{\omega}a_3) \in (H^\infty)^{-1}$. $\diamondsuit$

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