GEOMETRY ON STATE AND WEIGHT ORBITS

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1 Introduction

Let $\mathcal{A}$ be a unital C* algebra, $G_{\mathcal{A}}$ and $U_{\mathcal{A}}$ the set of invertible and unitary elements of $\mathcal{A}$. If φ is a faithful state we consider the similarity and unitary orbits

$$\mathcal{O}_\varphi = \{\varphi(g^{-1}\ .\ g) : g \in G_{\mathcal{A}}\} \qquad \text{and} \qquad \mathcal{U}_\varphi = \{\varphi(u^*\ .\ u) : u \in U_{\mathcal{A}}\}.$$

Observe that in some cases the orbit $\mathcal{U}_\varphi$ gives all the normal states of the algebra, for example if $\mathcal{A}$ is a type III$_1$ factor with separable predual [CS].

If φ is a faithful, normal and semifinite weight on a von Neumann algebra M, we will also denote by $\mathcal{U}_\varphi$ the set of weights $\{\varphi(u^*\ .\ u) : u \in U_{\mathcal{A}}\}$.

Even if φ is a faithful and normal state of $\mathcal{L}(H)$ (with H an infinite dimensional Hilbert space), the unitary orbit $\mathcal{U}_\varphi$ is not a topological submanifold of $\mathcal{L}(H)^*$ and neither of $\mathcal{L}(H)_*$ (see 2.2.1). Therefore, if we want to give these orbits a submanifold structure, the previous facts justify the use of topologies other than the usual topology of the dual. Moreover, in the case φ is a weight on a von Neumann algebra M, the orbit $\mathcal{U}_\varphi$ is not "included" in any Banach space. We shall introduce representations for these orbits in other Banach spaces in order for them to acquire an homogeneous reductive structure and a submanifold structure. The existence of an homogeneous reductive structure is a restrictive fact. A homogeneous reductive space is a manifold Q and a transitive action of a Banach–Lie group G (in our case the groups $G_{\mathcal{A}}$ and $U_{\mathcal{A}}$) in Q, $L : G \times Q \to Q$, with:

- Homogeneous structure: for each $q \in Q$, the map

$$\pi_q : G \to Q \qquad , \qquad \pi_q(g) = L_g q$$

is a principal fiber bundle with structure group $I_q = \{g \in G : L_g q = q\}$ (called the isotropy group of q).

- Reductive structure: let $\mathcal{G}$ be the Lie algebra of G (in our case $\mathcal{A}$ and $\mathcal{A}_{ah} = \{x \in \mathcal{A} : x^* = -x\}$, respectively). Then for every $q \in Q$, there exists a linear subspace $\mathcal{H}^q$ such that

$$\mathcal{G} = \mathcal{H}^q \oplus \mathcal{I}_q \qquad (\mathcal{I}_q \text{ Lie algebra of } I_q)$$

that is invariant for interior automorphisms of I_q and such that the distribution $q \mapsto \mathcal{H}^q$ is smooth.

The reductive structure induces a linear connection ∇ in the tangent fiber of Q (see [MR] and [V] for details).

In the second section we consider the case $\mathcal{A} = M$ is a von Neumann algebra. If φ is a faithful state, we describe the geometric structure of $\mathcal{O}_\varphi$ and $\mathcal{U}_\varphi$ with a topology different than the usual one, as we discussed above. If φ is a faithful, normal, strictly semifinite weight we imbed $\mathcal{U}_\varphi$ in a set of projections of M_1 ($=< M, e_\varphi >''$, where e_φ is the Jones projection of E_φ) and we study its geometric structure. In each case the geometric invariants are explicitly computed. In both cases the existence of a faithful, normal conditional expectation $E : M \to M^\varphi$ plays a central role.

This property is no longer available in the C* algebra case, making the study of $\mathcal{O}_\varphi$ and $\mathcal{U}_\varphi$ more difficult. In the third section we consider the case $\mathcal{A}$ is a unital C* algebra and we represent the orbits in the tensor products $\mathcal{A} \otimes \mathcal{A}$ and $\mathcal{A}_{ah} \otimes \mathcal{A}_{ah}$ respectively to give them a differential structure. We show explicit formulas for the geometric invariants.

These orbits have been studied in [AV1], [AV2] and [V], and this paper surveys the results discussed in those works. We omit most proofs and the interested reader is refered to the papers cited above.

2 The von Neumann algebra case

2.1 Preliminaries and notations

If φ is a faithful normal semifinite weight and σ_t^φ the modular group of φ, denote by $M^\varphi = \{x \in M : \sigma_t^\varphi(x) = x, \forall t \in \mathbb{R}\}$. A strictly semifinite weight is a semifinite weight of M such that $\varphi|_{M^\varphi}$ is also semifinite on M^φ.

As is standard notation, $\mathcal{N}_\varphi = \{x \in M : \varphi(x^*x) < \infty\}$ and $(H_\varphi, \pi_\varphi, \eta_\varphi)$ denotes the GNS triple for φ , i.e. H_φ is the completion of $\mathcal{N}_\varphi$ to a Hilbert space, $\pi_\varphi : M \to \mathcal{L}(\mathcal{H}_\varphi)$ is the usual $*$-isomorphism, $\eta_\varphi : \mathcal{N}_\varphi \to H_\varphi$ is the canonical imbedding, with

$$< \eta_\varphi(y), \eta_\varphi(z) >= \varphi(z^*y) \quad \text{and} \quad \pi_\varphi(x)\,(\eta_\varphi(y)) = \eta_\varphi(xy) \text{ for } y, z \in \mathcal{N}_\varphi, x \in M$$

Let $E : M \to N$ be a normal conditional expectation and let ψ be a normal state of N. Denote also by ψ the extension of this state to M given by $\psi \circ E$. Let e be the Jones projection of E (and ψ), i.e. the orthogonal projection obtained as the closure of E as an operator on $L^2(M, \psi)$ (with range $L^2(N, \psi)$). As it is also standard, denote by M_1 the algebra generated by M and e in $\mathcal{L}(L^2(M, \psi))$.

After the introduction of Jones paper [Jo] several notions of index for a conditional expectation between von Neumann algebras appeared ([PP], [Ko], [W]). We will follow the terminology of the paper by Baillet, Denizeau and Havet [BDH], where three notions of index are considered. Let $N \subset M$ be a von Neumann subalgebra and $E : M \to N$ a conditional expectation.

E is said to be of *weak finite index* if there exists a positive real number λ such that $E - \lambda\mathrm{Id}$ is a positive map. In that case put $\mathrm{Ind}_w(E) := \lambda_0^{-1}$ where λ_0 is the supremum of all such λ's (the index is said to be ∞ if no such λ exists).

In such case ([Po], [BDH], [FK]), there exists another positive constant κ such that $E - \kappa\mathrm{Id}$ is completely positive, and there is a family $\{m_i\}_{i \in I} \subset M$ such that

i) $1 = \sum_{i \in I} m_i e m_i^*$, (e the Jones projection associated to E)

ii) $E(m_i^* m_j) = \delta_{ij} p_i$, (p_i projections in N)

iii) and $\sum_{i \in I} m_i m_i^*$ converges ultraweakly in M.

In [BDH] it is shown that the limit of the latter sum belongs to the center of M, and is called the *index of* E, and will be denoted by $\mathrm{Ind}(E)$.

It is said that E has *strongly finite index* if the family $\{m_i\}_{i \in I}$ is finite.

There are conditional expectations that are of finite index and not of strongly finite index [J], [FK]. Nevertheless these two notions of finite index coincide in many cases, for instance if N is a subfactor [BDH].

Given a left action L_g for g in a group G, we denote by isotropy group of an element ξ the subgroup of G formed by elements h that verifiy $L_h(\xi) = \xi$. Let us abbreviate $[x, y] = xy - yx$, for $x, y \in \mathcal{A}$. For definitions and concepts on homogeneous spaces we refer the reader to [MR].

2.2 Geometry on the orbit of a state

Let us introduce a case where the unitary orbits of faithful and normal states are not (topological) submanifolds of the predual (and neither the dual) space of the algebra. This fact justifies the use of other topologies other than the usual norm topology.

Proposition 2.2.1 *Let φ be a faithful, normal state of $\mathcal{L}(H)$ with H infinite dimensional. Then the unitary orbit $\mathcal{U}_\varphi = \{\varphi \circ Ad(u) : u \in \mathcal{U}(\mathcal{L}(H))\}$ is not a topological submanifold of $\mathcal{L}(H)_*$ (and neither of $\mathcal{L}(H)^*$).*

Proof. There exists $a \in \mathcal{L}(H)_+$ with $\mathrm{Tr}(a) = 1$ (Tr the usual trace), such that $\varphi(x) = \mathrm{Tr}(ax)$ for all $x \in \mathcal{L}(H)$. The element a is of the form $a = \sum_{i=1}^\infty \lambda_i p_i$ with $\dim(\mathrm{R}(p_i)) = m_i < \infty$, and therefore $\sum_{i=1}^\infty \lambda_i m_i = 1$. Moreover, the unitary orbit $\mathcal{U}_\varphi \subset \mathcal{L}(H)_*$ with the norm topology identifies with $\mathcal{U}_a = \{u^*au : u \in \mathcal{U}(\mathcal{L}(H))\} \subset \mathcal{T}(H)$ with the trace norm $\| \ \|_1$, where $\mathcal{T}(H)$ denotes the trace class of $\mathcal{L}(H)$.

Let u_n be unitaries in $\mathcal{L}(H)$ such that $u_n^* a u_n$ converge to b in the usual norm of $\mathcal{L}(H)$.

Then it can be proved that

- **i)** b is compact and positive,
- **ii)** $\sigma(b) = \{\lambda_i : i \in \mathbb{N}\} = \sigma(a)$, $b = \sum_{i=1}^\infty \lambda_i q_i$ with $\dim R(q_i) = m_i$, and $\mathrm{Tr}(b) = 1$,
- **iii)** $u_n^* a u_n \to b$ in $\| \ \|_1$.

If $\mathcal{U}_a = \{u^*au : u \in \mathcal{U}(\mathcal{L}(H))\}$ is a submanifold of $(\mathcal{T}(H), \| \ \|_1) = \mathcal{L}(H)_*$ then $\mathcal{U}_a$ is locally closed in $\mathcal{T}(H)$. That is, each point $c \in \mathcal{U}_a$ has a neighbourhood of the form $\{d \in \mathcal{U}_a : \|c - d\|_1 \leq \varepsilon\}$ which is closed in $\mathcal{T}(H)$. But since the action of the unitaries of $\mathcal{L}(H)$ is isometric on $\mathcal{T}(H)$, the number ε can be chosen the same for all c in $\mathcal{U}_a$. This clearly implies that the orbit $\mathcal{U}_a$ is closed in $\mathcal{T}(H)$. Therefore, by the remarks above, $\mathcal{U}_a$ is closed in the usual norm of $\mathcal{L}(H)$. In his remarkable paper [Vo], Voiculescu proved, as a byproduct of his non-commutative Weyl-von Neumann theorem, that this condition - closedness in norm of the unitary orbit of an operator in $\mathcal{L}(H)$ - implies that the operator generates a finite dimensional C*-algebra. In our case, since a is positive, this implies that the spectrum of a is finite. This leads to a contradiction, since a is also compact and has zero kernel. □

Let M be a von Neumann algebra and φ a normal and faithful state. Denote by M^φ the algebra of fixed points of the modular group σ^φ and by $E_\varphi : M \to M^\varphi$

the unique normal and faithful conditional expectation such that $\varphi \circ E_\varphi = \varphi$ (see [T]). It is apparent that the isotropy group of the action $L_g\varphi = \varphi \circ \mathrm{Ad}(g^{-1})$, i.e. the group $I_\varphi = \{g \in G_M : L_g\varphi = \varphi\}$ is just the group G_{M^φ} of invertible elements of M^φ. Indeed: $\varphi(g^{-1}xg) = \varphi(x)$ for all $x \in M$, putting $x = gy$, is equivalent to $\varphi(gy) = \varphi(yg)$ for all $y \in M$.

Analogously the isotropy group for the action of the unitary group $\mathcal{U}_M$ is the unitary group $\mathcal{U}_{M^\varphi}$ of M^φ.

Let us recall a result by Herrero ([AFHV], vol II, Th. 16.3).

Theorem *If $M \subset N$ are Banach algebras such that M is complemented (as a linear subspace) in N, then the quotient map $\pi : G_N \to G_N/G_M$ has continuous local cross sections.*

Note that this is the case of the inclusion $M^\varphi \subset M$, using $\mathrm{Ker}(E_\varphi)$ as supplement for M^φ in M. Therefore, if we consider $\mathcal{O}_\varphi = \{\varphi \circ \mathrm{Ad}(g^{-1}) : g \in G_M\}$ endowed with the quotient topology (i.e. the topology which makes $[g] \longmapsto \varphi \circ \mathrm{Ad}(g^{-1})$; $G_M/G_{M^\varphi} \longrightarrow \mathcal{O}_\varphi$ a homeomorphism), the map $\pi_\varphi : G_M \longrightarrow \mathcal{O}_\varphi$, $\pi_\varphi(g) = \varphi \circ \mathrm{Ad}(g^{-1})$ has continuous local cross sections.

This also enables one to define local charts in order to give $\mathcal{O}_\varphi$ a differentiable structure. It suffices to find a local chart around φ. Since $\mathcal{O}_\varphi$ has the quotient topology, one can find basic neighborhoods of the form $\mathcal{O}(V) = \{\varphi \circ \mathrm{Ad}(g^{-1}) : g \in V\}$, with V an open subset of G_M containing 1. For a suitable $V \subset G_M$ the map

$$\begin{array}{rccl} \Lambda_\varphi : & \mathrm{Ker}\ (E_\varphi) & \longrightarrow & \mathcal{O}(V) \subset \mathcal{O}_\varphi \\ & x & \longmapsto & \varphi \circ \mathrm{Ad}(e^{-x}) \end{array}$$

is a local chart for $\mathcal{O}_\varphi$ (around φ).

For every $\psi \in \mathcal{O}_\varphi$ the maps Λ_ψ together with their correspondents neighborhoods give $\mathcal{O}_\varphi$ an analytic manifold structure.

Moreover, the map

$$\mathcal{O}(V) \longrightarrow G_M \quad ; \quad \varphi \circ \mathrm{Ad}(e^{-x}) \longmapsto e^{-x}$$

is an analytical cross section for π_φ $(\pi_\varphi(g) = \varphi \circ \mathrm{Ad}(g^{-1}))$ on a neighborhood of φ. Analytic cross sections can be obtained around each point of $\mathcal{O}_\varphi$ transfering this one. Therefore we have the following

Theorem 2.2.2 *Let φ be a faithful normal state, then the set $\mathcal{O}_\varphi$ is an analytic homogeneous space under the action of G_M.*

Furthermore $\mathcal{O}_\varphi$ admits a canonical reductive structure: a distribution of supplements $\mathcal{H}^\psi$, $\psi \in \mathcal{O}_\varphi$ for the Lie algebra of the isotropy group I_ψ (=Banach algebra $gM^\varphi g^{-1}$, if $\psi = \varphi \circ \mathrm{Ad}(g^{-1})$) in the Lie algebra of $G_M(=M)$. Namely, $\mathcal{H}^\psi = \mathrm{Ker}(E_\psi)$.

Note that the derivative of π_φ at 1, $d(\pi_\varphi)_1 : M \to (T\mathcal{O}_\varphi)_\varphi$ is the map $x \longmapsto \varphi([\ .\ , x])$. Therefore $\mathrm{Ker}\big(d(\pi_\varphi)_1\big) = M^\varphi$.

Let us call $K_\varphi : \big(T\mathcal{O}_\varphi\big)_\varphi \longrightarrow \mathcal{H}^\varphi \ \subset M$,

$$K_\varphi = \left(d(\pi_\varphi)_1\big|_{\mathcal{H}^\varphi} \right)^{-1}$$

We can complete the reductive structure of $\mathcal{O}_\varphi$ writing

$$\mathcal{H}^\psi_h = h\mathcal{H}^\psi, \text{ for } h \in G_M$$

$$K_\psi = \mathrm{Ad}(g) \circ K_\varphi \circ \mathrm{Ad}(g^{-1}), \text{ if } \psi = \varphi \circ \mathrm{Ad}(g^{-1})$$

The invariance of $\mathcal{H}^\varphi$ under the action of I_φ guarantees that the definition of the operator K_ψ does not depend on the choice of the invertible g in the fiber of ψ.

As it is usual for homogeneous reductive spaces, the covariant derivative, the geodesic curves, curvature and torsion tensors can be explicitely computed (see [MR]).

Let us first introduce the connection defined on $\mathcal{O}_\varphi$. Let $\gamma(t)$ be a smooth curve in $\mathcal{O}_\varphi$, with $\gamma(0) = \varphi$ and $t \in I$ $(0 \in I$). The following

$$\dot\Gamma(t) = K_{\gamma(t)}\,(\dot\gamma(t))\ \ \Gamma(t) \qquad (2.1\ )$$

will be called the *transport equation* for $\gamma(t)$. As in [V] the connection that defines the reductive structure can be computed

$$\left(\nabla_X Y\right)_q = \frac{d}{dt} \left\{ d\left(L_{\Gamma(t)}\right)^{-1}_{\gamma(t)} Y\left(\gamma(t)\right) \right\} \Bigg|_{t=0} \qquad (2.2\ )$$

for X, Y fields in $\mathcal{O}_\varphi$, γ a curve such that $\gamma(0) = \varphi$, $\dot\gamma(0) = X_\varphi$ and Γ the solution of the transport equation 2.1 for γ.

If $\delta : I \to \mathcal{O}_\varphi$ is a geodesic of the connection 2.2 induced by the reductive structure then satisfies the differential equation

$$0 = \ddot\delta(t) + \dot\delta(t) \circ \Big[K_{\delta(t)}\dot\delta(t), (.) \Big]$$

Remark 2.2.3 1) If $\psi \in \mathcal{O}_\varphi$ and $V \in (T\mathcal{O}_\varphi)_\psi$, then the geodesic δ satisfying $\delta(0) = \psi$ and $\delta'(0) = V$ is given by

$$\delta(t) = \psi \circ \mathrm{Ad}\big(e^{-tK_\psi(V)}\big)\ ,\ t \in \mathbb{R}$$

In other words, geodesics are of the form $\psi \circ \mathrm{Ad}(e^{-tx})$, for $x \in \mathrm{Ker}(E_\psi)$.

2) The torsion and the curvature tensors can also be computed. If V, W and Z are tangent vectors to $\mathcal{O}_\varphi$ at φ and we denote by v, w and z the images of V, W and Z under the map K_φ, then

$$K_\varphi\big(T(V,W)\big) = (I - E_\varphi)([v,w])$$

and

$$K_\varphi\big(R(V,W),Z\big) = (I - E_\varphi)([z, E_\varphi([v,w])])$$

where $[a,b] = ab - ba$, for $a, b \in M$.

Let us denote $\psi^\times(x) := \overline{\psi(x^*)}$, for $\psi \in \mathcal{O}_\varphi$ ($\psi^\times$ is the usual adjoint of ψ) and $S(M)$ for the space of states of M.

It can be proved that $\mathcal{U}_\varphi = \{\varphi \circ \mathrm{Ad}(u^*) : u \in U_M\} = \{\psi \in \mathcal{O}_\varphi : \psi^\times = \psi\} = \mathcal{O}_\varphi \cap S(M)$ and the following

Proposition 2.2.4 *The set $\mathcal{U}_\varphi$ of selfadjoint elements of $\mathcal{O}_\varphi$ is an homogeneous reductive space and a C^∞ submanifold of $\mathcal{O}_\varphi$.*

See [AV1] for details.

As with $\mathcal{O}_\varphi$ we can compute formulas for geodesics, torsion and curvature for $\mathcal{U}_\varphi$.

Remark 2.2.5 If $\psi \in \mathcal{U}_\varphi$ and $V \in (T\mathcal{U}_\varphi)_\psi$, then the geodesic δ satisfying $\delta(0) = \psi$ and $\delta'(0) = V$ is given by

$$\delta(t) = \psi \circ \mathrm{Ad}\big(e^{-tK_\psi(V)}\big)\ ,\ t \in \mathbb{R}$$

Analogously we can compute the torsion and the curvature tensors.

We can construct a C^∞ retraction from $\mathcal{O}_\varphi$ to $\mathcal{U}_\varphi$. By [PR], in view of the existence of $E_\varphi : \mathcal{A} \longrightarrow M^\varphi$ there is a unique descomposition for each $g \in G_M$ in the following way:

$$g = ue^x b$$

with $u \in U_M$, $x = x^* \in \mathrm{Ker}(E_\varphi)$ and $b \in \mathrm{Im}(E_\varphi) = \mathcal{A}^\varphi$ positive and invertible. Therefore we have a natural map $g \longmapsto u$ from G_M into U_M.

Proposition 2.2.6 *The map $R : \mathcal{O}_M \longrightarrow \mathcal{U}_\varphi$ defined by $R(\varphi \circ Ad(g^{-1})) = \varphi \circ Ad(u^*)$, with u as above, is a retraction.*

2.3 Geometry on the orbit of a faithful, normal and strictly semifinite weight

Let φ be a faithful, normal, strictly semifinite weight on a von Neumann algebra M. Let $(H_\varphi, \pi_\varphi, \eta_\varphi)$ be the GNS triple as in 2.1, and $E : M \to N$ a φ invariant conditional expectation onto a von Neumann subalgebra $N \subset M$. Let $e \in L(H_\varphi)$ be the Jones projection associated to E, that is e is the orthogonal projection on $\overline{N \cap \mathcal{N}_\varphi} \subset H_\varphi$, and $M_1 = (M, e)'' \subset L(H_\varphi)$ the basic extension. From now on we shall identify $x \in M$ with its image $\pi_\varphi(x) \in \mathcal{L}(H_\varphi)$.

As we have seen in 2.2.1, the unitary orbits $\mathcal{U}_\varphi = \{\varphi \circ \mathrm{Ad}(u) : u \in U_M\}$ may not be topological submanifolds of M^* or M_*. Moreover, when φ is a weight the orbit is not "included" in any Banach space. Nevertheless, we shall introduce a representation for $\mathcal{U}_\varphi$ which will allow us to present it as a space of projections of the basic extension of M by E_φ. $\mathcal{U}_\varphi$ will be considered with the quotient topology U_M/U_{M^φ}.

If φ is a faithful, normal, strictly semifinite weight, let e_φ be the Jones projection of $E_\varphi : M \to M^\varphi$. The orbit $\mathcal{U}_M(e_\varphi) = \{u^* e_\varphi u : u \in \mathcal{U}(M)\}$ is a C^∞ homogeneous space [AV1]. The continuous map (called the basic representation of $\mathcal{U}_\varphi$)

$$\beta : \mathcal{U}_\varphi \to \mathcal{U}_M(e_\varphi) \subset M_1$$

$$\beta(\varphi \circ \mathrm{Ad}(u)) = u^* e_\varphi u$$

is a bijection that preserves the adjoint and the orbits.

The tangent space $T(\mathcal{U}_M(e_\varphi))_{e_\varphi}$ identifies with the space $\{x e_\varphi - e_\varphi x (= [x, e_\varphi]) : x \in M, x^* = -x\}$.

$\mathcal{U}_M(e_\varphi) \subset M_1$ is a Banach homogeneous space, consisting of projections of M_1. It has a natural connection (see [ALRS]), for example, its geodesics can be easily computed. Namely, the unique geodesic φ_t on $\mathcal{U}_M(e_\varphi)$ with $\varphi_0 = e_\varphi$ and $\frac{d}{dt}\varphi_t\big|_{t=0} = [x, e_\varphi]$ is given by

$$\varphi_t = e^{t x_0} e_\varphi e^{-t x_0}$$

where $x_0 = x - E_\varphi(x)$.

Remark 2.3.1 **a)** One has the following commutative diagram

$$\begin{array}{ccc} \mathcal{U}(M) & \xrightarrow{\Pi_\varphi} & \mathcal{U}_\varphi \\ & \searrow \Pi_{e_\varphi} & \uparrow \beta \\ & & \mathcal{U}_M(e_\varphi) \end{array}$$

where $\Pi_\varphi(u) = \varphi \circ \mathrm{Ad}(u)$ and $\Pi_{e_\varphi}(u) = u^* e_\varphi u$, for $u \in \mathcal{U}(M)$. The map Π_{e_φ} has local cross sections, namely (near e_φ):

$$s(u^* e_\varphi u) = (E_\varphi(u) E_\varphi(u^*))^{-1/2} E_\varphi(u) u\,,$$

(i.e. the unitary part of $E_\varphi(u)u$ in its polar decomposition) defined in $\{u^* e_\varphi u : \|u^* e_\varphi u - e_\varphi\| < 1\}$, which takes values in U_M and satisfies

$$\Pi_{e_\varphi} \circ s(u^* e_\varphi u) = u^* e_\varphi u \ \text{ and } s(e_\varphi) = 1.$$

Note that if $\|u^* e_\varphi u - e_\varphi\| < 1$ then also $\|e_\varphi u^* e_\varphi u e_\varphi - e_\varphi\|$ and $\|e_\varphi - e_\varphi u e_\varphi u^* e_\varphi\|$ are strictly less than 1, which implies (using the properties of the basic extension) that $\|E_\varphi(u^*)E_\varphi(u) - 1\| < 1$ and $\|1 - E_\varphi(u)E_\varphi(u^*)\| < 1$, and therefore s is well defined.

b) If M is endowed with the left M^φ-Hilbert module norm induced by E_φ, i.e. $\|x\|_{E_\varphi} = \|E_\varphi(x^*x)\|^{1/2}$, then s is continuous. This is a straightforward verification, using the fact noted in a), that if $u^* e_\varphi u$ is close to e_φ then $E_\varphi(u^*)E_\varphi(u)$ and $E_\varphi(u)E_\varphi(u^*)$ are close to 1. Now, if Π_{e_φ} has continuous local cross section, by means of the diagram above it is easy to see that the basic representation β is a homeomorphism. Therefore $\mathcal{U}_\varphi$ with the quotient topology $(U_M, \| \ \|_{E_\varphi})/U_{M^\varphi}$ can be regarded as a manifold of projections of M_1.

c) It is known that the equivalence in M of the usual norm with the Hilbert module norm is equivalent to the finite index condition. Therefore if the index of E_φ is finite, then $\mathcal{U}_\varphi$ is homeomorphic to $\mathcal{U}_M(e_\varphi)$, with the usual norms. We do not know if β is (norm) continuous in general. In other words, if the Hilbert module norm and the usual norm of M can induce the same quotient topology in U_M/U_{M^φ} in cases other than the finite index situation.

If $\mathrm{Ind}(E_\varphi) < \infty$ one can do more. Let us recall the following result from [AS2].

Theorem 2.3.2 *Let $N \subset M$ be a von Neumann algebra, $E : M \to N$ a normal and faithful conditional expectation and e and M_1 as before. Then, the following statements are equivalent*

1) *The weak index of E is finite.*

2) *$\mathcal{S}_M(e) = \{geg^{-1} : g$ invertible in $M\}$ is an analytic homogeneous Banach space under the action of the invertible elements of M and an analytic submanifold of M_1.*

3) *$\mathcal{U}_M(e) = \{ueu^* : u \in \mathcal{U}(M)\}$ is a C^∞ homogeneous Banach space under the action of the unitary elements of M and a C^∞ submanifold of M_1.*

If $\mathrm{Ind}(E_\varphi) < \infty$ one has to consider semifinite algebras (see [J], [AV2]), and using the following

Theorem 2.3.3 *Let M be a semifinite von Neumann algebra with finite dimensional center, φ a faithful normal strictly semifinite weight and $E_\varphi : M \to M^\varphi$ the unique normal and faithful conditional expectation that leaves φ invariant. Let τ be a tracial weight in M and h affiliated to $Z(M^\varphi)$ (the center of M^φ) such that $\varphi = \tau(h.)$ (see [PT]). The following statements are equivalent:*

i) *E_φ has weakly finite index.*

ii) *$\sigma(h)$ is finite.*

(from [AV2]) we can state

Corollary 2.3.4 *Let M be a von Neumann algebra with finite dimensional center and φ a faithful, normal and strictly semifinite weight. The following statements are equivalent:*

1) *E_φ has weakly finite index.*

2) *M is semifinite and $\sigma(h)$ is finite, if h is the Radon–Nikodym derivative of the weight φ respect to a faithful, normal and semifinite trace on M.*

3) *$\mathcal{U}_M(e_\varphi) = \{ue_\varphi u^* : u \in \mathcal{U}(M)\}$ is a C^∞ Banach homogeneous space under the action of the unitaries of M and a C^∞ submanifold of $M_1 = (M, e)''$.*

If the weak index of E_φ is infinite, $\mathcal{U}_M(e_\varphi)$ and $\mathcal{S}_M(e_\varphi)$ are manifolds of projections of M_1, but not submanifolds of M_1.

Remark 2.3.5 As we noted before, these conditions allow us to give a differential structure to $\mathcal{U}_\varphi$ the orbit of a faithful, normal and strictly semifinite weight φ, through the model

$$\begin{array}{rcl} \mathcal{U}_\varphi & \leftrightarrow & \mathcal{U}_M(e_\varphi) \\ \varphi \circ \mathrm{Ad}(u) & \mapsto & u^* e_\varphi u. \end{array}$$

Note that if E_φ has finite index, then with the notations above, $M^\varphi = \{h\}' \cap M$. Therefore, $(M^\varphi)' \cap M \subset M^\varphi$. Indeed, since if $p_1, ..., p_n$ are the minimal spectral projections of h, then $(M^\varphi)'$ consists of the elements of M wich have "diagonal" matrices with respect to $p_1, ..., p_n$.

Recall again from [AS2], that if $E : M \to N$ has finite index and $N' \cap M \subset N$, then the mapping

$$\mathcal{U}_M(e) = \{u^* e u : u \in U_M\} \longrightarrow \mathcal{O}_E = \{\mathrm{Ad}(u) \circ E \circ \mathrm{Ad}(u^*) : u \in U_M\}$$

is a covering map with fibre homeomorphic to $n(E)/U_N$, where $n(E) = \{u \in \mathcal{U}(M) : \mathrm{Ad}(u)(N) \subset N\}$.

In our case, we obtain that if φ is a weight on M satisfying the equivalent conditions of 2.3.4, then the map

$$\mathcal{U}_\varphi \longrightarrow \mathcal{O}_{E_\varphi} = \{\mathrm{Ad}(u) \circ E_\varphi \circ \mathrm{Ad}(u^*) : u \in U_M\}$$

is a covering map with fibre (homeomorphic to) the group $n(E_\varphi)/U_{M^\varphi}$. Moreover, since $Z(M)$ is finite dimensional, it can be shown that this group is finite (see [ArS]).

3 The C* algebra case

3.1 Preliminaries and notation

Let now $\mathcal{A}$ be a C* algebra and φ a faithful state. We will call H_φ, or simply H, the Hilbert space $\mathcal{L}(L^2(\mathcal{A}, \varphi))$ which is the closure of $\mathcal{A}$ under the scalar product $< a, b >_\varphi = \varphi(b^* a)$, for a , $b \in \mathcal{A}$. Let $\overline{H}$ be the conjugate Hilbert space of H and $\overline{H} \otimes H$ the tensor product. Let $\mathcal{A} \otimes \mathcal{A} \subset \mathcal{L}(\overline{H} \otimes H)$ be the C* algebra generated by the elements $\sum_{1 \le i \le n} a_i \otimes b_i$, with a_i , $b_i \in \mathcal{A}$, acting by left multiplication on $\overline{H} \otimes H$.

Along the paper we are going to use the following norms:

$$\|h\|^2_{H_\varphi} = \|h\|^2_H = < h, h >_\varphi = \varphi(h^* h) \text{ , for } h \in H_\varphi = H$$

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|^2_{\overline{H}\otimes H} = < \sum_{i=1}^{n} x_i \otimes y_i, \sum_{i=1}^{n} x_i \otimes y_i >_{\overline{H}\otimes H} = \sum_{1\leq i,j\leq n} \overline{< x_i, x_j >} < y_i, y_j >$$

$$\left\| \sum_{k=1}^{m} a_k \otimes b_k \right\|^2_{\sup} = \sup_{\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\| = 1} \left\| \sum_{\substack{1\leq k\leq m \\ 1\leq i\leq n}} a_k x_i \otimes b_k y_i \right\|_{\overline{H}\otimes H}$$

for $\sum_{i=1}^{n} x_i \otimes y_i \in \overline{H} \otimes H$ and $\sum_{k=1}^{m} a_k \otimes b_k \in \mathcal{A} \otimes \mathcal{A}$.

3.1.1 The imbedding of $\mathcal{S}_{\mathcal{A}}(e_\varphi)$ in $\mathcal{A} \otimes \mathcal{A}$.

Let $\mathcal{A}$ be a unital C^* algebra and φ a faithful state on $\mathcal{A}$.

Denote by $\varphi_1 : \mathcal{A} \to \mathbb{C}.1$ the map $\varphi_1(a) = \varphi(a).1$, which is clearly a conditional expectation. Let e_φ be the Jones projection associated to φ_1, which is the orthogonal projection from H_φ to $\mathbb{C}.1$ and verifies $e_\varphi(a) = \varphi_1(a)$, $a \in \mathcal{A} \subset H_\varphi$.

Let $\mathcal{S}_{\mathcal{A}}(e_\varphi) = \{g e_\varphi g^{-1} : g \in G_{\mathcal{A}}\} \subset \mathcal{L}(H_\varphi)$ be the similarity orbit of e_φ under the action of $G_{\mathcal{A}}$ (group of invertible elements of $\mathcal{A}$). The isotropy group of the action is simply the group of non zero scalars $\mathbb{C}^\times$.

Andruchow and Stojanoff proved in [AS] that the orbit of the Jones projection of a conditional expectation E is a submanifold of $\mathcal{L}(L^2(\mathcal{A}, \phi))$, for ϕ a faithful normal state compatible with E, iff E has finite index. In our case (where $E = \varphi_1$, $\phi = \varphi$) φ_1 will have finite index only if $\mathcal{A}$ is finite dimensional. We are going to imbed $\mathcal{S}_{\mathcal{A}}(e_\varphi)$ in another Banach space in order that it acquires a submanifold structure.

Observe that the element $g e_\varphi g^{-1} \in \mathcal{S}_{\mathcal{A}}(e_\varphi)$ can be regarded as

$$g e_\varphi g^{-1}(x) = g e_\varphi(g^{-1}x) = g\varphi(g^{-1}x) = < x, (g^{-1})^* > g$$

for every $x \in H$. The element $g e_\varphi g^{-1}$ is therefore in correspondence with the element $g \otimes (g^{-1})^* \in \mathcal{A} \otimes \mathcal{A}$ via the natural isometrical bijection which carries $< \ , x > y \in L(H)$ to $y \otimes x \in \overline{H} \otimes H$. The orbit $\mathcal{S}_{\mathcal{A}}(e_\varphi)$ can be imbedded in $\mathcal{A} \otimes \mathcal{A}$

$$g e_\varphi g^{-1} \longrightarrow g \otimes (g^{-1})^*$$

The projector e_φ is carried into the element $1 \otimes 1 \in \mathcal{A} \otimes \mathcal{A}$, and its orbit into the orbit of $1 \otimes 1$ under the left action

$$L_g(1 \otimes 1) = g \otimes (g^{-1})^*$$

The map $\tau : \mathcal{S}_{\mathcal{A}}(e_\varphi) \to \mathcal{A} \otimes \mathcal{A}$ given by $\tau(g e_\varphi g^{-1}) = g \otimes (g^{-1})^*$ is well defined and continuous. We have the following diagram

$$\begin{array}{ccccc} G_{\mathcal{A}} & \longrightarrow & \mathcal{S}_{\mathcal{A}}(e_\varphi) & \longleftrightarrow & \mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1 \otimes 1) \\ & \searrow & \Big\downarrow & & \\ & & O_\varphi & & \end{array}$$

$\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1 \otimes 1) = \{g \otimes (g^{-1})^* : g \in G_{\mathcal{A}}\} \subset \mathcal{A} \otimes \mathcal{A}$ is a natural model to study the geometry of $\mathcal{S}_{\mathcal{A}}(e_\varphi)$.

3.1.2 Differentiable structure of $\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1 \otimes 1)$.

We will see that $\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1 \otimes 1)$ is an homogeneous reductive space and a real analytic submanifold of $\mathcal{A} \otimes \mathcal{A}$. For general facts concerning homogeneous reductive sumanifolds see [KN], or [MR] which deals specifically the infinite dimensional case.

For $g \in G_{\mathcal{A}}$, let $L_g : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ given by

$$L_g\left(\sum_{1 \le i \le n} a_i \otimes b_i\right) = \sum_{1 \le i \le n} g a_i \otimes (g^{-1})^* b_i.$$

This map defines an action of $G_{\mathcal{A}}$ on $\mathcal{A} \otimes \mathcal{A}$ with isotropy group $I_{g\otimes(g^{-1})^*} = \{\lambda.1 : \lambda \in \mathbb{C} \text{ and } \lambda \neq 0\}$.

This action induces the map

$$\begin{array}{rcl} \Pi_{1\otimes 1} : G_{\mathcal{A}} & \longrightarrow & \mathcal{A} \otimes \mathcal{A} \\ g & \longmapsto & L_g(1 \otimes 1) = g \otimes (g^{-1})^* \end{array}$$

Clearly $\Pi_{1\otimes 1}$ is (real) analytic and its image is $\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1 \otimes 1)$.

The following result is an application of the implicit function theorem, and it can be found in [R].

Theorem 3.1.1 *Suppose that $G_{\mathcal{A}}$ acts on a Banach space X, let $x_0 \in X$ such that the map $\pi_{x_0} : G_{\mathcal{A}} \longrightarrow G_{\mathcal{A}}.x_0$ (=orbit of x_0) $\subset X$ is analytic and verifies:*

1) π_{x_0} is open.

2) Ker *$d(\pi_{x_0})_1$ is complemented in $\mathcal{A}$.*

3) The range of $d(\pi_{x_0})_1$ is complemented in X.

Then $G_{\mathcal{A}}.x_0$ is an analytic submanifold of X and an homogeneous space under the action of $G_{\mathcal{A}}$.

Using this theorem it can be proved([AV1]) the following

Corollary 3.1.2 *$\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$ is a C^∞ submanifold of $\mathcal{A}\otimes\mathcal{A}$ and an homogeneous space under the action of $G_\mathcal{A}$.*

The same result is true for the unitary orbit $\mathcal{U}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1) = \{u\otimes u : \ u \in \mathcal{U}_\mathcal{A}\}$ under the action $L_u(1\otimes 1) = u\otimes u$, replacing $\mathcal{A}$ by $\mathcal{A}_{ah} = \{x \in \mathcal{A} : x^* = -x\}$.

Let us exhibit the reductive structure of $\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$. Recall that the isotropy group for the action L_h on $\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$, $L_h(g\otimes(g^{-1})^*) = hg\otimes(h^{-1})^*(g^{-1})^*$, is $I_{g\otimes(g^{-1})^*} = \{\lambda.1 : \lambda \in \mathbb{C} \text{ and } \lambda \neq 0\}$, for all $g \in G_\mathcal{A}$. Therefore a supplement for the Lie algebra of $I_{g\otimes(g^{-1})^*}$ (that is $\{\lambda.1 : \lambda \in \mathbb{C}\}$) in the Lie algebra of $G_\mathcal{A}$ $(= \mathcal{A})$ is $\mathcal{H}^{g\otimes(g^{-1})^*} = \operatorname{Ker}(\varphi \circ \operatorname{Ad}(g^{-1})) = g\mathcal{H}^{1\otimes 1}g^{-1}$. It is apparent that $\mathcal{H}^{g\otimes(g^{-1})^*}$ is invariant under the action of the isotropy group. Let us call

$$K_{1\otimes 1} : T\big(\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)\big)_{1\otimes 1} \longrightarrow \mathcal{H}^{1\otimes 1} \subset \mathcal{A}$$

$$K_{1\otimes 1}(v\otimes 1 - 1\otimes v^*) = v \ \text{ with } v \in \operatorname{Ker}(\varphi)$$

We can complete the reductive structure of $\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$ writing

$$\mathcal{H}_h^{g\otimes(g^{-1})^*} = h.\mathcal{H}^{g\otimes(g^{-1})^*}$$

and

$$K_{g\otimes(g^{-1})^*} = \operatorname{Ad}(g)\circ K_{1\otimes 1}\circ L_{g^{-1}} : T\big(\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)\big)_{g\otimes(g^{-1})^*} \longrightarrow \mathcal{H}^{g\otimes(g^{-1})^*}$$

then

$$K_{g\otimes(g^{-1})^*}(gv\otimes(g^{-1})^* - g\otimes(g^{-1})^*v^*) = g\left(v - \varphi(g^{-1}vg)\ 1\right)g^{-1},$$

with $v \in \operatorname{Ad}(g)\operatorname{Ker}(\varphi\circ\operatorname{Ad}(g^{-1}))$.

As we have seen in the von Neumann case, this structure allows us to construct a linear connection. We will show the equations that define the geodesics and the formulas for the geodesics corresponding to that connection.

Remark 3.1.3 The geodesics of $\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$ for the connection defined by the reductive structure verify the equation

$$\ddot\gamma + M_{(1\otimes K_\gamma\dot\gamma - K_\gamma\dot\gamma\otimes 1)}\dot\gamma = 0$$

where $M_{(1\otimes x - x\otimes 1)}(y\otimes z) = y\otimes xz - xy\otimes z$ for $x, z \in \mathcal{A}$.

Remark 3.1.4 The geodesic γ that verifies $\gamma(0) = g\otimes(g^{-1})^*$ and $\dot\gamma(0) = gv\otimes(g^{-1})^* - g\otimes(g^{-1})^*v^* \in T\left(\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)\right)_{g\otimes(g^{-1})^*}$ $(v \in \mathcal{A})$ is given by

$$\gamma(t) = \exp\left(tg\left(v - \varphi(g^{-1}vg)1\right)g^{-1}\right)g\otimes\left(\exp\left(tg\left(v - \varphi(g^{-1}vg)1\right)g^{-1}\right)\right)^{*-1}g^{*-1}$$

The curvature and torsion tensors can be computed as usual.

3.2 A natural involution in $\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$.

The usual adjoint of the operators $< \, , x > y \in \mathcal{L}(H)$ (that were associated with the elements $y\otimes x \in \mathcal{A}\otimes\mathcal{A}$), given by $(< \, , x > y)^* =< \, , y > x$, suggests which involution we should use here

$$\times : \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A} \ , \ \Big(\sum_{1\le i\le n} a_i\otimes b_i\Big)^{\times} = \sum_{1\le i\le n} b_i\otimes a_i .$$

Note that $\times$ satisfies

$$\big(\Pi_{1\otimes 1}(g)\big)^{\times} = \Pi_{(1\otimes 1)^*}\big((g^{-1})^*\big).$$

It can be proved that $\mathcal{U}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$ are the "selfadjoint" elements of $\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$, that is $\mathcal{U}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1) = \{\xi \in \mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1) : \xi^{\times} = \xi\}$. Also the following can be found in [AV1]:

Proposition 3.2.1 *$\mathcal{U}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$ is a C^∞ submanifold of $\mathcal{A}_{ah}\otimes\mathcal{A}_{ah}$.*

We can construct a C^∞ retraction from $\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$ to $\mathcal{U}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$. Put $\mu(g) =$ "the unitary part of the polar decomposition of g", $g = \mu(g)(g^*g)^{1/2}$ (i.e. $\mu(g) = g(g^*g)^{-1/2}$). Then we have a natural map $g\otimes(g^{-1})^* \longmapsto \mu(g)\otimes\mu(g)$ from $\mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$ into $\mathcal{U}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$.

Proposition 3.2.2 *The map $r : \mathcal{S}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1) \to \mathcal{U}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$, defined by $r(g\otimes (g^{-1})^*) = \mu(g)\otimes\mu(g)$, is a C^∞ retraction.*

Observe that the Lie algebra of the isotropy group $I_{u\otimes u}$ is $L = \{\lambda.1 : \lambda \in \mathbb{C} \text{ and } \overline{\lambda} = -\lambda\}$, for all $u\otimes u \in \mathcal{U}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$. Therefore the distribution of supplements $u \longmapsto \mathcal{H}^{u\otimes u} = \operatorname{Ker}\big(\varphi\circ \operatorname{Ad}(u^*)\big)\cap\mathcal{A}_{ah}$ of L in the Lie algebra of $U_{\mathcal{A}}$ $(=\mathcal{A}_{ah})$ defines a connection in the homogeneous space $\mathcal{U}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$. It is apparent that $\mathcal{H}^{u\otimes u}$ is invariant under the action of the isotropy group. In this case

$$K_{u\otimes u}(uv\otimes u - u\otimes uv^*) = uvu^*, \text{ with } v\in \operatorname{Ker}(\varphi\circ\operatorname{Ad}(u^*))$$

One can compute formulas for geodesics, torsion and curvature of this connection.

Remark 3.2.3 If $u\otimes u \in \mathcal{U}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)$ and $V \in T\big(\mathcal{U}_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)\big)_{u\otimes u}$, then the geodesic δ satisfying $\delta(0) = u\otimes u$ and $\delta'(0) = V$ is given by

$$\delta(t) = \big(e^{-tK_{u\otimes u}(V)}\big)u\otimes\big(e^{-tK_{u\otimes u}(V)}\big)u \ , \ t\in\mathbb{R}$$

Analogously the torsion and the curvature tensors can be computed.

References

[AFHV] Apostol, C., Fialkow, L.A., Herrero, D.A. and Voiculescu, D., Approximation of Hilbert space operators, Vol 2, Pitman, Boston, 1984.

[ALRS] E. Andruchow, A. Larotonda, L. Recht and D. Stojanoff. Infinite dimensional homogeneous reductive spaces and finite index conditional expectations, preprint 1994.

[AS1] E. Andruchow and D. Stojanoff, Conditional expectations of weak finite index, preprint.

[AS2] E. Andruchow and D. Stojanoff, Geometry of conditional expectations and finite index, International J. Math. Vol. 5, No. 2 (1994), 169-178.

[ArS] M. Argerami and D. Stojanoff, A discrete group associated to a finite index conditional expectation, preprint.

[AV1] E. Andruchow and A. Varela, Geometry and the Jones projection of a state, Integr. Equat. Oper. Th., Vol. 25 (1996), 129-146.

[AV2] Weight centralizer expectations with finite index, E. Andruchow and A. Varela, preprint.

[BDH] M. Baillet, Y. Denizeau and J.-F. Havet, Indice d'une esperance conditionelle, Compositio Math. 66 (1988), 199-236.

[C] A. Connes, Une classification des facteurs de type III, Ann. Sci. Ecole Norm. Sup., 6 (1973), 133-252.

[CS] A. Connes, E. Størmer, Homogeneity of the state space of factors of type III_1, J. Funct. Anal. 28 (1978), 187-196.

[Co] F. Combes, Poids associes a une algebre hilbertienne a gauche, Comp. Math., vol. 23, fasc. 1 (1971), 4-77.

[FK] M. Frank and E. Kirchberg, On conditional expectations of finite index, preprint (1996).

[J] P. Jolissaint, Indice d'esperances conditionelles et algebres de von Neumann finies, Math. Scand., 68 (1991), 221-246.

[Jo] V.F.R. Jones, Index for subfactors, Invent. Math. 72 (1983), 1-25.

[KN] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Vol 2, Interscience Publ., NY, 1969.

[Ko] H. Kosaki, Extension of Jones' index theory to arbitrary factors, J. Funct. Anal. 66 (1986), 123-140.

[LR] A. Larotonda, L. Recht, La órbita de una esperanza condicional como espacio homogéneo reductivo regular, Impresiones previas del Depto. de Mat., FCEyN-UBA, No. 76 (1993).

[MR] L. Mata Lorenzo, L. Recht, Infinite dimensional homogeneous reductive spaces, Reporte 91-11, U.S.B., (1991).

[PR] H. Porta, L. Recht, Conditional expectations and operator decomposition (Preprint), 1992.

[PP] M. Pimsner and S. Popa, Entropy and index for subfactors, Ann. Sci. Ecole Norm. Sup. 19 (1986), 57-106.

[Po] S. Popa, Classification of subfactors and their endomorphisms, CBMS 86, AMS, 1995.

[PT] G.K. Pedersen and M. Takesaki, The Radon-Nikodym theorem for von Neumann algebras, Acta Math. 130 (1973), 53-87.

[R] I. Raeburn, The relation between a commutative Banach algebra and its maximal ideal space, J. Funct. Anal., 25 (1977), 366-390.

[T] M. Takesaki, Conditional expectations in von Neumann algebras, J. Funct. Anal. 9 (1972), 306-321.

[V] A. Varela, Geometría de órbitas de estados en álgebras de operadores, tesis doctoral, Facultad de Ciencias Exactas, Universidad de Buenos Aires, noviembre de 1996.

[Vo] D.V. Voiculescu, A non commutative Weyl-von Neumann theorem, Rev. Roum. Math. Pures et Appl., 21 (1976), 97-113.

[W] Y. Watatani, Index for C^*-subalgebras, Memoirs AMS vol. 83, no. 424 (1990).

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