ON AMENABILITY AND GEOMETRY OF SPACES OF BOUNDED REPRESENTATIONS

Gustavo Corach and José E. Galé

Abstract. - Several differential geometric aspects of the space of representations of amenable Banach algebras are studied: differential structure, natural connection, geodesics, reductive structure, and so on. As an application we get a characterization of amenable groups in terms of the existence of a reductive structure in that space.

Introduction.

It is known that spaces of group representations can sometimes be regarded as finite or infinite-dimensional Banach manifolds, and that this is a useful viewpoint which concerns a wide range of mathematical subjects, also including the study of spaces of representations themselves ([**11**], [**18**]).

The analysis of the differential geometry of certain sets of operators, or, more generally, subsets of Banach and C^*-algebras, has provided a different approach to the understanding of several classical questions and is a new source of problems and results ([**9**], [**10**], [**22**], [**23**], [**2**], [**3**] and references therein). We wish to emphasize on reference [**2**] because the procedure carried out there comprises most of the cases considered in previous works on the subject (see [**3**]). In that paper a geometric characterization of injective von Neumann algebras is given in the following terms : a von Neumann algebra $\mathcal{M}$ is injective if and only if every normal $*$-representation π of $\mathcal{M}$ has a reductive structure (see definitions below). The proof of this theorem relies upon the fact that the existence of a reductive structure for π is equivalent to the existence of a $\mathcal{M}$-valued conditional expectation. In one direction such an expectation can be obtained by a standard method of averaging on a certain multiplicative subgroup of unitaries of $\mathcal{M}$. A similar characterization

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for a nuclear C^*-algebra $\mathcal{A}$, where the condition on the representation π to be normal is removed, is also given in [**2**]. This time the proof follows from the first result by passing to the bidual $\mathcal{A}^{**}$.

Thus, we see that the versatile concept of amenability (see [**24**]) also underlies the above kind of questions and therefore seems suitable for intrinsic geometric implications. The present paper is understood as a first attempt to make the relationships involving geometry of representations and amenability explicit.

Let $\mathcal{A}$, $\mathcal{B}$ be two Banach algebras and assume that $\mathcal{A}$ is amenable and $\mathcal{B}$ is unital with unit e. Denote by $\mathcal{G}$ the group of invertible elements of $\mathcal{B}$. If π is a bounded algebra homomorphism (i.e., a *representation*) from $\mathcal{A}$ into $\mathcal{B}$ we denote by $O(\pi)$ the set of representations from $\mathcal{A}$ into $\mathcal{B}$ which are similar to π by conjugation with elements of $\mathcal{G}$, and by τ_π the conjugation map $U \in \mathcal{G} \longrightarrow U\pi U^{-1} \in O(\pi)$. Under appropriate conditions on $\mathcal{B}$ (see property (DM) in Section 2), the map τ_π admits local cross sections (Proposition 4.1) and then it becomes a Banach principal bundle with structure group $\mathcal{G} \cap \pi(\mathcal{A})'$ where $\pi(\mathcal{A})'$ is the commutant of $\pi(\mathcal{A})$ in $\mathcal{B}$.

In order to get notions like those of parallel transport, curvature, torsion and so on, associated to the bundle $\tau_\pi : \mathcal{G} \longrightarrow O(\pi)$, we need to define a *connection* on this bundle, or equivalently a *connection form* (see [**19**, volume 1] for the definitions and properties; even though the book only deals with finite-dimensional manifolds, the basic facts also hold in general). Note that for defining a connection form it suffices to define a bounded projection $\mathcal{E}_\pi : \mathcal{B} \longrightarrow \pi(\mathcal{A})'$ satisfying $\mathcal{E}_\pi(\nu_1\mu\nu_2) = \nu_1\mathcal{E}_\pi(\mu)\nu_2$ for every $\mu \in \mathcal{B}$ and $\nu_1, \nu_2 \in \mathcal{G} \cap \pi(\mathcal{A})'$, for then the map $U \longrightarrow U\mathcal{E}_\pi(U^{-1}\cdot)$ gives us the wanted 1-form ([**19**, volume 1], [**9**, pp. 220,221]). An alternative method to produce a connection on

$\tau_\pi : \mathcal{G} \longrightarrow O(\pi)$ consists of "lifting" the tangent space at each σ of the base space $O(\pi)$, so that it may be considered as a subspace of *horizontal vectors* of $\mathcal{B}$ which moves in correspondence with the variations of σ along $O(\pi)$. Here $\mathcal{B}$ is taken as the tangent space at $U \in \mathcal{G}$, if $\tau_\pi(U) = \sigma$. This in tuitive idea is made precise whenever we have a 1-form $\pi \rightarrow K_\pi : Der_\pi(\mathcal{A}, \mathcal{B}) \longrightarrow \mathcal{B}$ such that K_π is a right inverse of $(d\tau_\pi)_e$, $UH^\pi U^{-1} = H^\pi$ for every $U \in \mathcal{G} \cap \pi(\mathcal{A})'$ where $H^\pi = K_\pi(Der_\pi(\mathcal{A}, \mathcal{B}))$, and $K_\sigma = Ad_U \circ K_\pi \circ Ad_{U^{-1}}$ for $\sigma = U\pi U^{-1}$, $U \in \mathcal{G}$ ([**22**, Lemma 3.4]). In the above, $Der_\pi(\mathcal{A}, \mathcal{B})$ is the space of derivations from $\mathcal{A}$ into $\mathcal{B}$ with respect to the $\mathcal{A}$- bimodule operations in $\mathcal{B}$ defined through π and is regarded as the space of tangent vectors of $O(\pi)$ at π.

Since we are concerned with infinite-dimensional manifolds we must be careful of checking special requirements needed in this setting, such as existence of direct complements of subspaces and others. For this reason, and also for simplicity in obtaining the elements of the geometry involved, we follow the program proposed in [**22**] and start with the introduction of the 1-form $\pi \to K_\pi$ instead of giving the definition of the connection form $\mathcal{E}_\pi$ directly. For the construction of K_π we use an abstract version of an averaging process (Theorem 2.1) which was suggested by [**12**, proof of Th. 1] (see also [**28**], where the interest of averaging in cohomology is carefully explained). Then the projection $\mathcal{E}_\pi$ is easily defined from K_π and it turns out to be a *quasiexpectation* (Proposition 2.2) so it induces a structure of reductive homogeneous space on $O(\pi)$ (see [**19**, volume 2]). We think that this interpretation of quasiexpectations as geometric objects is noteworthy; in this regard $ker(\mathcal{E}_\pi)$ and $\pi(\mathcal{A})'$ appear as spaces of horizontal and vertical vectors, respectively.

Thus we are able to prove that if $\mathcal{A}$ is amenable then $O(\pi)$ is a reductive homogeneous space for every representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}$. We do not know if the converse holds, although it is possible to show some characterizations of the amenability of a locally compact group G within the same order of ideas. Namely, when $\mathcal{A} = L^1(G)$ and G is amenable we can choose a two sided invariant mean on G to obtain a quasiexpectation $\mathcal{E}_\pi$ as before which is moreover π-*invariant*, that is, $\mathcal{E}_\pi(\pi(f)\mu) = \mathcal{E}_\pi(\mu\pi(f))$ for every $f \in L^1(G)$, $\mu \in \mathcal{B}$, or equivalently, if we extend π as to be a representation of G, $\mathcal{E}_\pi(\pi(t)\mu\pi(t^{-1})) = \mathcal{E}_\pi(\mu)$ for every $t \in G$, $\mu \in \mathcal{B}$. Such a "tracial" property of $\mathcal{E}_\pi$ gives rise to a vector-valued version of the important concept of *amenability* of π in the sense defined in [**5**] (see also [**1**], [**17**] for instance), and it seems to have a geometric significance : G is amenable if and only if the mapping

$\tau_\pi : \mathcal{G} \longrightarrow O(\pi)$ is a principal bundle which admits a π-invariant connection form $\mathcal{E}_\pi$, that is, $[\pi(G), ker(\mathcal{E}_\pi)] \subseteq ker(\mathcal{E}_\pi)$ where $ker(\mathcal{E}_\pi)$ can be seen as space of horizontal vectors of $\mathcal{B}$ (Theorem 4.5). Let us remark that the last result does not appeal to any reductive structure on $O(\pi)$. However, if G is assumed to be *inner amenable*, see §3, then it turns out that G is amenable if and only if $O(\pi)$ is a reductive homogeneous space for every representation π (Theorem 4.6). This is a geometric version of a characterization of the class of amenable groups within the bigger one formed by all the inner amenable groups, [**24**, p.85], in which we do not need to refer to the π-invariance of $\mathcal{E}_\pi$.

The organization of the paper is as follows. In §1 we write some definitions and properties of a preliminary character about representations, amenability, and differential geometry in infinite dimensions. In §2 we prove the main result, Theorem 2.1, on the existence of the 1-form K_π and deduce its first consequences in a homological language. Section 3 is devoted to analyzing the link between amenability and existence of quasiexpectations or π-invariant projections associated to representations. In §4 we undertake the study of the differential geometry of sets of representations using the information collected in sections 2 and 3. Finally, in §5 we apply the results of Section 4 to automorphisms of amenable algebras.

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§**1. Preliminaries**.

If X and Y are Banach spaces, denote by $L(X,Y)$ the space of bounded linear operators from X into Y endowed with the topology of the operator norm. Thus $L(X,Y)$ becomes a Banach space. Abbreviate $L(X) = L(X,X)$.

If E is a reflexive Banach space and $X = E\hat{\otimes}E^*$, the projective tensor product of E and its topological dual E^*, then the dual X^* of X is isometrically isomorphic to $L(E)$. If E has in addition the approximation property, then $E\hat{\otimes}E^*$ is exactly the space of nuclear operators on E ([**14**, p.92 and 96]).

In this paper we are dealing with representations of Banach algebras and locally compact groups, so we need to recall several basic facts about these objects. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach space. A *representation* from $\mathcal{A}$ into E is a bounded linear map $\pi : \mathcal{A} \to L(E)$ such that $\pi(ab) = \pi(a)\pi(b)$ for every $a, b \in \mathcal{A}$. It is *non-degenerate* if $\pi(\mathcal{A})E$ is dense in E. Of course, a representation π corresponds to a structure of a *left* Banach $\mathcal{A}$-module on E and by duality gives rise to the *dual* (bounded) *anti-representation* $\pi^* : \mathcal{A} \to L(E^*)$ (or, equivalently, to the *dual* right Banach $\mathcal{A}$-module structure on E^*) defined as $\langle x, \pi(a)^* x^* \rangle = \langle \pi(a)x, x^* \rangle$ for all $x \in E$, $x^* \in E^*$, $a \in \mathcal{A}$. Note that π^* is an anti-representation in the sense that $\pi^*(ab) = \pi^*(b)\pi^*(a)$ for every $a, b \in \mathcal{A}$. There is a similar interplay between (non-degenerate) anti-representations and dual representations. The set of non-degenerate representations of $\mathcal{A}$ into E will be denoted by $Rep(\mathcal{A}, E)$. It is clearly a closed subset, and therefore a complete, metric subspace, of $L(\mathcal{A}, L(E))$. Similarly, let G be a locally compact group. Then a *bounded strongly continuous* representation φ from G into E is a map $\varphi : G \to L(E)$ such that $\sup_{s\in G} \|\varphi(s)\|_{L(E)} < \infty$, $\varphi(ts) = \varphi(t)\varphi(s)$, ($t, s \in G$) and $\varphi(t)x \to \varphi(s)x$, as $t \to s$, for every $x \in E$. The set of bounded strongly continuous representations of G into E will be denoted by $R(G, E)$, and throughout this paper it is endowed with the metric topology given by the distance $d(\varphi, \psi) := \|\varphi - \psi\| = \sup_{s\in G} \|\varphi(s) - \psi(s)\|_{L(E)}$. It is easy to prove that $R(G, E)$ becomes a complete metric space. The following result is well known ([**18**, p.144], [**24**, p.42]).

Proposition 1.1. *Let G be a locally compact group with a fixed left Haar measure ds. Then the spaces $Rep(L^1(G), E)$ and $R(G, E)$ are isometric under the isometries given as follows:*

(i) $\varphi \in R(G, E) \to \pi_\varphi \in Rep(L^1(G), E)$, *where*

$$\pi_\varphi(f)x = \int_G f(s)\varphi(s)x\, ds,$$

($x \in E$, $f \in L^1(G)$).

(ii) $\pi \in Rep(L^1(G), E) \to \varphi_\pi \in R(G, E)$, where

$$\varphi_\pi(s)x = \lim_j \pi(\delta_s * f_j)x,$$

for $s \in G$, $x \in E$, where δ_s is the Dirac mass at $s \in G$ and $(f_j)_j$ is a suitably chosen approximation of the identity in $L^1(G)$.

Let Z be a Banach $\mathcal{A}$-bimodule. We will say that a bounded linear map $D : \mathcal{A} \to Z$ is a *derivation* if $D(ab) = a \cdot Db + Da \cdot b$ ($a, b \in \mathcal{A}$). The set $Der(\mathcal{A}, Z)$ of derivations from $\mathcal{A}$ into Z is a closed subspace of $L(\mathcal{A}, Z)$. A derivation $D \in Der(\mathcal{A}, Z)$ is called *inner* if there exists $z \in Z$ such that $D(a) =$

$a \cdot z - z \cdot a$ for every $a \in \mathcal{A}$. Suppose that $\mathcal{B}$ is another Banach algebra and $\pi : \mathcal{A} \to \mathcal{B}$ is a Banach algebra homomorphism. Then $\mathcal{B}$ becomes a Banach $\mathcal{A}$-module via π in the obvious way, and the corresponding space of derivations will be denoted by $Der_\pi(\mathcal{A},\mathcal{B})$. This means that $Der_\pi(\mathcal{A},\mathcal{B}) = \{D \in L(\mathcal{A},\mathcal{B}) : D(ab) = \pi(a)\cdot D(b) + Da \cdot \pi(a), (a, b \in \mathcal{A})\}$. Abbreviate $Der_\pi(\mathcal{A}) = Der_\pi(\mathcal{A},\mathcal{A})$. Note that $Der(\mathcal{A},\mathcal{A}) = Der_\pi(\mathcal{A})$ when π is the identity on $\mathcal{A}$; in this case we shall simply write $Der(\mathcal{A})$.

Let us now recall some definitions and characterizations concerning amenability. Our references for this paragraph are [**6**], [**16**], [**24**]. A Banach algebra $\mathcal{A}$ is said to be *amenable* if every derivation of $\mathcal{A}$ into a dual Banach $\mathcal{A}$-bimodule is inner. Any amenable Banach algebra must have a bounded approximate identity. Given a Banach algebra $\mathcal{A}$, a Banach $\mathcal{A}$-bimodule which is important in order to study the possible amenability of $\mathcal{A}$ is $E = \mathcal{A}\hat{\otimes}\mathcal{A}$, with module operations defined by $a \cdot (b \otimes c) = (ab) \otimes c$, $(b \otimes c) \cdot a = b \otimes (ca)$ for every $a, b, c \in \mathcal{A}$. As usual, we get $(\mathcal{A}\hat{\otimes}\mathcal{A})^*$ and $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ as the corresponding dual and bidual modules on $\mathcal{A}$. Then a *virtual diagonal* for $\mathcal{A}$ is any element M of $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $M \cdot a = a \cdot M$ and $\gamma^{**}(M) \cdot a = a$ ($a \in \mathcal{A}$), where γ^{**} is the bi-transpose of the generic "multiplication" $\gamma : b \otimes c \in \mathcal{A}\hat{\otimes}\mathcal{A} \to bc \in \mathcal{A}$. On the other hand, an *approximate diagonal* for $\mathcal{A}$ is a bounded net $(m_j)_j$ in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $m_j \cdot a - a \cdot m_j \to_j 0$ and $(\gamma m_j) \cdot a \to_j a$ ($a \in \mathcal{A}$). We have the following characterization.

Proposition 1.2. *A Banach algebra $\mathcal{A}$ is amenable if and only if there is a virtual diagonal for $\mathcal{A}$, if and only if there is an approximate diagonal for $\mathcal{A}$.*

In the case above, virtual diagonals can be obtained as cluster points of approximate diagonals, in the weak* topology on $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ with respect to $(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Conversely, approximate diagonals are obtained as the bounded nets in $\mathcal{A}\hat{\otimes}\mathcal{A}$ which converge to virtual diagonals in the weak* topology.

When $\mathcal{A} = L^1(G)$ for a locally compact group G, then $L^1(G)$ is amenable in the sense just considered if and only if G is amenable *as a group*, that is, there exists a left invariant *mean* on G : a positive, linear functional $m : L^\infty(G) \to \mathbf{C}$ such that $m(f_s) = m(f)$ for every $s \in G$, $f \in L^\infty(G)$, where $f_s(t) = f(st)$ for all $t \in G$. The definition of *right* invariant mean is similar and also characterizes the amenability of G. It is remarkable that there always exist means on amenable

G which are left and right invariant simultaneously.

To end these preliminaries we refer to the theory of infinite-dimensional homogeneous spaces such as given in [**26**, pp.368-373], which will be used later to describe the geometric implications of amenability. Let $\mathcal{G}$ be a Banach Lie group with identity e and suppose that $\mathcal{G}$ acts holomorphically and transitively on a Banach manifold $\mathcal{W}$; denote by $\tau_p(t)$ the action of $t \in \mathcal{G}$ over $p \in \mathcal{W}$. For $p \in \mathcal{W}$, let $\mathcal{W}_p$ be the tangent space of $\mathcal{W}$ at p and let $(d\tau_p)_e$ be the tangent map of τ_p at e. Then $\mathcal{W}$ is said to be a *Banach homogeneous space* under the action of $\mathcal{G}$ if there is $p \in \mathcal{W}$ such that

(1) $ker(d\tau_p)_e$ is a complemented subspace of $\mathcal{G}_e$.

(2) $(d\tau_p)_e : \mathcal{G}_e \to \mathcal{W}_p$ is surjective.

This definition and the following result have been taken from [**26**].

Proposition 1.3. *Let $\mathcal{G}$ be a Banach-Lie group which acts holomorphically on a Banach space X. Let $\mathcal{W} = \tau_p(\mathcal{G})$ be an orbit of the action of $\mathcal{G}$ on X, $p \in X$. Suppose that there exists $q \in \tau_p(\mathcal{G})$ such that*

(a) $ker(d\tau_q)_e$ *is a complemented subspace of* $\mathcal{G}_e$.

(b) $im(d\tau_q)_e$ *is a complemented subspace of* X.

(c) $\tau_q : \mathcal{G} \to \mathcal{W}$ *is open.*

Then $\mathcal{W}$ is a Banach homogeneous space and a submanifold of X.

§2. A general principle for constructing 1-forms and certain exact sequences.

Let $\mathcal{A}$ be an amenable Banach algebra with an associated virtual diagonal M and approximate diagonal $(m_j)_j$, $M = \lim_j m_j$ in the weak* topology of $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$, as in §1, which we fix henceforth. Let $\mathcal{B}$ be a Banach algebra which is also the topological dual X^* of a Banach space X such that X is a Banach $\mathcal{B}$-bimodule for which

$$(DM) \qquad \langle \mu\nu, x\rangle = \langle \mu, \nu \cdot x\rangle = \langle \nu, x \cdot \mu\rangle \ (x \in X; \mu, \nu \in \mathcal{B}).$$

Before the end of this section, we shall describe two relevant examples of Banach algebras satisfying property (DM).

For $T, S \in L(\mathcal{A}, \mathcal{B})$ we define $T \otimes S \in L(\mathcal{A}\hat{\otimes}\mathcal{A}, \mathcal{B}\hat{\otimes}\mathcal{B})$ by $(T \otimes S)(a \otimes b) = (Ta) \otimes (Sb)$, if $a, b \in \mathcal{A}$. Put $T \otimes_\gamma S = \gamma \circ T \otimes S$ (γ is here the "multiplication" in $\mathcal{B}$). If $x \in X$ and $T, S \in L(\mathcal{A}, \mathcal{B})$ we denote by $\delta_x \circ T \otimes_\gamma S$ the element of $(\mathcal{A}\hat{\otimes}\mathcal{A})^*$ defined by

$$(\delta_x \circ T \otimes_\gamma S)(\alpha) = \langle (T \otimes_\gamma S)(\alpha), x \rangle$$

for every $\alpha \in \mathcal{A}\hat{\otimes}\mathcal{A}$. Note that $(\delta_x \circ T \otimes_\gamma S)(M) = \lim_j \langle (T \otimes_\gamma S)(m_j), x \rangle$.

The following result is the key of the paper.

Theorem 2.1. *Let $\mathcal{A}$ be an amenable Banach algebra and let $\mathcal{B}$ be a unital Banach algebra, with unit e, which satisfies property (DM). Suppose that $\pi : \mathcal{A} \to \mathcal{B}$ is a bounded algebra homomorphism such that $(\pi\gamma)(m_j) \to_j e$ in the weak* topology of $\mathcal{B}$ with respect to X. Then the mapping $K_\pi : L(\mathcal{A}, \mathcal{B}) \to \mathcal{B}$ given by*

$$\langle K_\pi(T), x \rangle = (\delta_x \circ T \otimes_\gamma \pi)(M),$$

for $x \in X$, $T \in L(\mathcal{A}, \mathcal{B})$, satisfies the following properties.

(i) *K_π is linear and bounded with $\|K_\pi\| \leq \|M\|\|\pi\|$.*

(ii) *For every $\mu \in \mathcal{B}$ and $T \in L(\mathcal{A}, \mathcal{B})$, $K_\pi(\mu T(\cdot)) = \mu K_\pi(T)$.*

(iii) *For every $a \in \mathcal{A}$ and $T \in L(\mathcal{A}, \mathcal{B})$, $K_\pi(T_a) = K_\pi(T)\pi(a)$, where $T_a(b) = T(ab)$ for all $b \in \mathcal{A}$.*

(iv) *$K_\pi(\pi)e$.*

Proof: Part (i) is obvious.

(ii) We have

$$\langle K_\pi(\mu T(\cdot)), x \rangle - \langle \mu K_\pi(T), x \rangle$$

$$= (\delta_x \circ (\mu T(\cdot) \otimes_\gamma \pi))(M) - (\delta_{x \cdot \mu} \circ (T(\cdot) \otimes_\gamma \pi))(M)$$

$$= \lim_j (\langle (\mu T(\cdot) \otimes_\gamma \pi)(m_j), x \rangle - \langle (T \otimes_\gamma \pi)(m_j), x \cdot \mu \rangle)$$

$$= \lim_j (\langle \mu (T \otimes_\gamma \pi)(m_j), x \rangle - \langle \mu (T \otimes_\gamma \pi)(m_j), x \rangle)$$

$$= \lim_j 0 = 0.$$

(iii)
$$\begin{aligned}
&\langle K_\pi(T_a), x\rangle - \langle K_\pi(T)\pi(a), x\rangle\\
&= \lim_j(\langle (T_a \otimes_\gamma \pi)(m_j), x\rangle - \langle (T \otimes_\gamma \pi)(m_j), \pi(a)x\rangle)\\
&= \lim_j(\langle (T \otimes_\gamma \pi)(am_j), x\rangle - \langle (T \otimes_\gamma \pi)(m_j)\pi(a), x\rangle)\\
&= \lim_j(\langle (T \otimes_\gamma \pi)(am_j - m_j a), x\rangle) = 0.
\end{aligned}$$

(iv)
$$\begin{aligned}
\langle K_\pi(\pi), x\rangle &= \lim_j \langle (\pi \otimes_\gamma \pi)(e_j), x\rangle\\
&= \lim_j \langle (\pi\gamma)e_j, x\rangle = \langle e, x\rangle.
\end{aligned}$$

Let us observe that, from (ii) and (iv) above, it follows that $K_\pi(\mu\pi(\cdot)) = \mu$ for every $\mu \in \mathcal{B}$.

The introduction of K_π has been suggested by [**12**, proof of Th. 1]. The map K_π is in fact an averaging operator (see [**28**, p. 77, p.81 and p. 51]).

Now, for $\mathcal{A}$ and $\mathcal{B}$ as in the theorem, we denote by $Hom(\mathcal{A}, \mathcal{B})$ the space of bounded algebra homomorphisms $\pi : \mathcal{A} \to \mathcal{B}$ for which $\pi(\mathcal{A}) \cdot X$ is dense in X, or, which is the same because of Cohen's factorization theorem ([**6**, p.61]), $\pi(\mathcal{A}) \cdot X = X$. The space $Hom(\mathcal{A}, \mathcal{B})$ is metrizable and complete with respect to the topology defined by the operator norm. It is clear from the factorization and condition (DM) that $(\pi\gamma)(m_j) \to_j e$ in the weak* topology of $\mathcal{B}$.

If π is a fixed element of $Hom(\mathcal{A}, \mathcal{B})$ we define $\mathcal{E}_\pi : \mathcal{B} \to \mathcal{B}$ by $\mathcal{E}_\pi(\mu) = K_\pi(\pi(\cdot)\mu)$, for every $\mu \in \mathcal{B}$.

Proposition 2.2. *For $\mathcal{A}$, $\mathcal{B}$, π, $\mathcal{E}_\pi$ as before we have :*

(i) $\|\mathcal{E}_\pi(\mu)\| \le \|\pi\|^2\|\mu\|$ *for all* $\mu \in \mathcal{B}$.

(ii) $im(\mathcal{E}_\pi) = \pi(\mathcal{A})' := \{\nu \in \mathcal{B}; \nu\pi(a) = \pi(a)\nu, a \in \mathcal{A}\}$.

(iii) $\mathcal{E}_\pi^2 = \mathcal{E}_\pi$ *and* $\mathcal{E}_\pi(e) = e$

(iv) $\mathcal{E}_\pi(\nu_1\mu\nu_2) = \nu_1\mathcal{E}_\pi(\mu)\nu_2$ *for every* $\mu \in \mathcal{B}$ *and* $\nu_1, \nu_2 \in Im(\mathcal{E}_\pi)$.

Proof: (i) It is clear from the definition.

(ii) For $a \in \mathcal{A}$ and $\mu \in \mathcal{B}$ one gets $\pi(a)\mathcal{E}_\pi(\mu) = \pi(a)K_\pi(\pi(\cdot)\mu) = K_\pi(\pi(a\cdot)\mu) = K_\pi(T_a)$ by Theorem 2.1 (ii), where $T(\cdot) = \pi(\cdot)\mu$, and $K_\pi(T_a) = K_\pi(T)\pi(a) =$

$\mathcal{E}_\pi(\mu)\pi(a)$ by Theorem 2.1 (iii). Conversely, if $\nu \in \pi(\mathcal{A})'$ then $\mathcal{E}_\pi(\nu) = K_\pi(\pi(\cdot)\nu) = K_\pi(\nu\pi(\cdot)) = \nu$.

(iii) It is apparent from (ii) and the definition.

(iv) $\mathcal{E}_\pi(\nu_1\mu\nu_2) = K_\pi(\pi(\cdot)\nu_1\mu\nu_2) = K_\pi(\nu_1\pi(\cdot)\mu\nu_2) = \nu_1 K_\pi(\pi(\cdot)\mu\nu_2) = \nu_1 K_\pi(\pi(\cdot)\mu)\nu_2$.

Define $\Delta_\pi : \mathcal{B} \to L(\mathcal{A}, \mathcal{B})$ by $\Delta_\pi(\mu)(a) = \mu\pi(a) - \pi(a)\mu$, for every $a \in \mathcal{A}$, $\mu \in \mathcal{B}$. Clearly, the image of Δ_π is the set of inner derivations from $\mathcal{A}$ into $\mathcal{B}$, via π, but in this way $\mathcal{B}$ is in fact a *dual* Banach $\mathcal{A}$-bimodule (for X is a $\mathcal{A}$-bimodule, via π) and so, since $\mathcal{A}$ is amenable, $im(\Delta_\pi)$ equals $Der_\pi(\mathcal{A}, \mathcal{B})$.

From(ii) of Proposition 2.2, $im(\mathcal{E}_\pi) = ker(\Delta_\pi)$. Moreover, $K_\pi \circ \Delta_\pi = I_\mathcal{B} - \mathcal{E}_\pi$, where $I_\mathcal{B}$ is the identity in $L(\mathcal{B})$, because if $\mu \in \mathcal{B}$ then $(K_\pi \circ \Delta_\pi)(\mu) = K_\pi(\mu\pi(\cdot)) - K_\pi(\pi(\cdot)\mu) = \mu - \mathcal{E}_\pi(\mu)$. Then we have the following.

Proposition 2.3. *Let $\mathcal{P}_\pi : L(\mathcal{A}, \mathcal{B}) \to L(\mathcal{A}, \mathcal{B})$ be the map defined as $\mathcal{P}_\pi = \Delta_\pi \circ K_\pi$. Then $\mathcal{P}_\pi$ is a bounded projection such that $im(\mathcal{P}_\pi) = Der_\pi(\mathcal{A}, \mathcal{B})$.*

Proof: By composition, $\mathcal{P}_\pi$ is linear and bounded. If $T \in L(\mathcal{A}, \mathcal{B})$ then $(\Delta_\pi \circ K_\pi)[(\Delta_\pi \circ K_\pi)(T)] = \Delta_\pi[(I - \mathcal{E}_\pi)(K_\pi(T))] = (\Delta_\pi \circ K_\pi)(T) - (\Delta_\pi \circ \mathcal{E}_\pi(K_\pi(T)) = (\Delta_\pi \circ K_\pi)(T)$, whence $\mathcal{P}_\pi$ is a projection. Furthermore, $\Delta_\pi(\mu) = \Delta_\pi(\mu - \mathcal{E}_\pi(\mu) = \Delta_\pi(K_\pi(\Delta_\pi(\mu)))$ and so $im\mathcal{P}_\pi = im\Delta_\pi$.

Now, Proposition 2.3 together with Proposition 2.2 yields an exact sequence of Banach spaces, naturally associated to a fixed representation of an amenable algebra.

Proposition 2.4. *Let $\mathcal{A}$ be an amenable Banach algebra*

and let $\mathcal{B}$ be a unital Banach algebra satisfying (DM). For $\mathcal{E}_\pi$, Δ_π, $\mathcal{P}_\pi$ as above the sequence

$$0 \to ker\mathcal{E}_\pi \xrightarrow{i} \mathcal{B} \xrightarrow{\mathcal{E}_\pi} \mathcal{B} \xrightarrow{\Delta_\pi} L(\mathcal{A}, \mathcal{B}) \xrightarrow{I - \mathcal{P}_\pi} L(\mathcal{A}, \mathcal{B})/Der_\pi(\mathcal{A}, \mathcal{B}) \cong K_\pi^{-1}(\pi(\mathcal{A})') \to 0$$

is exact.

Proof: Straightforward.

Let us apply the results above to the following two main examples.

Examples

(1) For a reflexive Banach space E we take $X = E\hat{\otimes}E^*$ and $\mathcal{B} = L(E)$. As said before, $\mathcal{B} = X^*$ under the duality given by $\langle A, x\otimes x^*\rangle = \langle Ax, x^*\rangle$, for $x \in E$, $x^* \in E^*$ and $A \in L(E)$ ([**14**, p.92]). Moreover, X is a Banach $\mathcal{B}$-bimodule with respect to the module operations defined as

$$A \cdot (x \otimes x^*) = (Ax) \otimes x^*,$$

$$(x \otimes x^*) \cdot A = x \otimes (A^* x^*),$$

for $x \in E$, $x^* \in E^*$, $A \in L(E)$, where $A^* : E^* \to E^*$ is the transpose of A.

Now let $\mathcal{A}$ be an amenable Banach algebra and let $\pi : \mathcal{A} \to L(E)$ be a non-degenerate representation. It is apparent that the properties required as hypotheses in Theorem 2.1 are satisfied in this case. For instance, $x\otimes x^* = (\pi(a)y)\otimes x^* = \pi(a) \cdot (y \otimes x^*)$ for every $x, y \in E$, $x^* \in E^*$ and $a \in \mathcal{A}$ with $\pi(a)y = x$. Hence $\pi(\mathcal{A}) \cdot X$ is dense in X.

Let $Rep(\mathcal{A}, E)$ be the closed subset of $Hom(\mathcal{A}, L(E))$ formed by the non-degenerate representations $\pi : \mathcal{A} \to L(E)$. In the example under consideration, we will apply propositions 2.2 and 2.3 (as well as other consequences of Theorem 2.1 which we will see later) to the space $Rep(\mathcal{A}, E)$ rather than to $Hom(\mathcal{A}, L(E))$.

(2) Let G be a locally compact group, and then take $X = C_0(G)$, $\mathcal{B} = M(G)$, the convolution algebra of regular Borel measures on G. Recall that $M(G) = C_0(G)^*$ via the duality $\langle \nu, h\rangle = \int_G h(u)\, d\nu(u)$, and also that $C_0(G)$

is a Banach $M(G)$-bimodule for the module operations

$$(\nu \cdot h)(s) = \int_G h(su)\, d\nu(u),$$

$$(h \cdot \nu)(s) = \int_G h(us)\, d\nu(u),$$

for every $s \in G$, $h \in C_0(G)$, and $\nu \in M(G)$.

§3. Amenability, π-invariant projections, and quasi-expectations.

For a couple of Banach algebras $\mathcal{C}$ and $\mathcal{D}$ such that $\mathcal{D}$ is a Banach subalgebra of $\mathcal{C}$ we say that a mapping $Q : \mathcal{C} \to \mathcal{D}$ is a *quasi-expectation* if it is a bounded linear projection onto $\mathcal{D}$ such that $Q(d_1cd_2)=d_1Q(c)d_2$ for all $c\in\mathcal{C}$ and $d_1,d_2\in\mathcal{D}$. Under the assumptions of Theorem 2.1 we can obtain a large family of quasi-expectations with domain in $\mathcal{B}$. Namely, each $\mathcal{E}_\pi$ associated to $\pi\in Hom(\mathcal{A},\mathcal{B})$ provides such a quasi-expectation, as Proposition 2.2 shows.

In the theory of operator algebras the existence of a quasi-expectation from $L(H)$, where H is a Hilbert space, onto a von Neumann algebra $\mathcal{M}\subset L(H)$ is equivalent to the injectivity of $\mathcal{M}$ ([**8**]); on the other hand, injectivity is an appropriate notion of amenability for von Neumann algebras (see [**24**], [**14**] for a complete discussion). We wonder if a similar equivalence takes place in our setting, via representations. Although, for general Banach algebras $\mathcal{A}$, we do not know how to recover amenability from the existence of suitable quasi-expectations, we do have some consistent results about this question when $\mathcal{A}=L^1(G)$, for a locally compact group G.

In this case, if we choose a left invariant mean m on $L^\infty(G)$ then a virtual diagonal M in $(L^1(G\hat{\otimes}G))^{**}=L^1(G\times G)^{**}=L^\infty(G\times G)^*$ is given by

$$M:h\in L^\infty(G\times G)\to M(h)=\int_G h(t,t^{-1})dm(t),$$

where the formal integral represents the action of m on the function $h(t,t^{-1})$. Clearly, $\|M\|=1$. If $\pi\in Hom(L^1(G),\mathcal{B})$, then $(\delta_x\circ(\pi(\cdot)\mu\otimes\pi))(f\otimes g)=\langle\pi(f)\mu\pi(g),x\rangle$ for every $x\in X$, $f,g\in L^1(G)$, and then the mapping $(f,g)\to\langle\pi(f)\mu\pi(g),x\rangle$ identifies to an element h_π of $L^\infty(G\times G)$. For $\mathcal{B}=L(E)$ we get from Proposition 1.1 that $h_\pi(s,t)=\langle\pi(s)A\pi(t)x,x^*\rangle$ where $A\in L(E)$, $x\otimes x^*\in E\hat{\otimes}E^*$ (note in passing that the left-hand member of this equality is a bounded continuous function in both variables s,t). Then it follows that

$$\mathcal{E}_\pi(A)=K_\pi(\pi(\cdot)A)=(\delta_x\circ(\pi(\cdot)A\otimes\pi))(M)$$

$$=M(h_\pi)=\int_G\langle\pi(s)A\pi(s^{-1})x,x^*\rangle dm(s),$$

for every $A\in L(E)$.

It will be used below the fact that if we choose m to be simultaneously right and left invariant then $\mathcal{E}_\pi(\pi(t)A\pi(t^{-1}))=\mathcal{E}_\pi(A)$ or, equivalently, $\mathcal{E}_\pi(\pi(t)A)=\mathcal{E}_\pi(A\pi(t))$, ($t\in G$, $A\in L(E)$). Such a property will be called here π*-invariance.*

Observe that the π-invariance of $\mathcal{E}_\pi$ supplies a vector version for the interpretation of $\pi(f)$, $f \in L^1(G)$, as a *finite* operator on E ([**7**]), or of π itself as an *amenable* representation ([**5**]). For every given amenable Banach algebra $\mathcal{A}$ and not only for $L^1(G)$, we would like to find a quasiexpectation $\mathcal{E}_\pi$ which were also π-invariant. It would be possible if we had a virtual diagonal M for $\mathcal{A}$ such that $M \circ a = a \circ M$ for every $a \in \mathcal{A}$, where $(b \otimes c) \circ a = b \otimes ac$, $a \circ (b \otimes c) = ba \otimes c$ for every $a, b, c \in \mathcal{A}$. Unfortunately, the existence of such M is unclear for us.

There is of course another way to find a projection $P : L(E) \longrightarrow \pi(\mathcal{A})'$ making π amenable "with respect to P", i.e., such that $P(\pi(a)A) = P(A\pi(a))$ for every $a \in \mathcal{A}$, $A \in L(E)$, for amenable algebras $\mathcal{A}$ ([**14**, p.257]) but in this case the projection lacks, in general, the $\pi(\mathcal{A})'$-module homomorphism property which is characteristic of quasiexpectations.

The following result is a vector version of Theorem 2.2 of [**5**].

Theorem 3.1. *Let G be a locally compact group. Then the following assertions are equivalent.*

(i) G is amenable.

(ii) For every reflexive Banach space E and every $\pi \in Rep(L^1(G), E)$ there exists a π-invariant, bounded, linear projection $\mathcal{Q} : L(E) \to \pi(L^1(G))'$ onto $\pi(L^1(G))'$ of norm $\leq \|\pi\|^2$.

(iii) For every reflexive Banach space E and every $\pi \in R(G, E)$ there exists a π-invariant, bounded, linear projection $\mathcal{Q} : L(E) \to \pi(G)'$ onto $\pi(G)'$ of norm $\leq \|\pi\|^2$.

(iv) If λ is the left regular representation of G into $L^2(G)$ given by $\lambda(s)f = f(s^{-1}\cdot)$ for every $s \in G$, $f \in L^2(G)$, then there exists a λ-invariant norm-one linear projection $\mathcal{Q} : L(L^2(G)) \to \lambda(G)'$ onto $\lambda(G)'$.

Proof: (i) $\Rightarrow$ (ii) This is part of Proposition 2.2 applied to $\mathcal{A} = L^1(G)$ and $\mathcal{B} = L(E)$.

(ii) $\Rightarrow$ (iii) As it has been indicated in §1 there is an identification between $Rep(L^1(G), E)$ and $R(G, E)$ by means of explicit formulas which imply that $\pi(G)' = \pi(L^1(G))'$ in $L(E)$.

(iii) $\Rightarrow$ (iv) Obvious.

(iv) $\Rightarrow$ (i) We include the following adaptation of a standard argument ([**7**, p.147], [**5**, p.387], [**4**, pp.605]) for the sake of completeness. Take $E = L^p(G)$, $1 < p < \infty$ and choose $x \in E$, $x^* \in E^*$ with $\langle x, x^* \rangle = 1$. For $h \in L^\infty(G)$, let denote by M_h the multiplication operator in $L(E)$ given as $M_h(f) = fh$, ($f \in E$). Now

define $m : L^\infty(G) \to \mathbf{C}$ by $m(h) = \langle \mathcal{Q}(M_h)x, x^* \rangle$ for every $h \in L^\infty(G)$. Our claim is that m is a left invariant mean on $L^\infty(G)$. To see this, we start by noting that $\mathcal{Q}(I) = I$, where I is the identity operator on E: since $I \in \pi(G)'$ there exists $A \in L(E)$ with $\mathcal{Q}(A) = I$ and therefore $I = \mathcal{Q}(A) = \mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(I)$. Then, besides the clear fact that m is a bounded functional on $L^\infty(G)$, we get $m(1) = \langle \mathcal{Q}(M_1)x, x^* \rangle = \langle \mathcal{Q}(I)x, x^* \rangle = \langle x, x^* \rangle = 1$ and $\|m(h)\| \leq \|\mathcal{Q}(M_h)\|\|x\|\|x^*\| \leq \|\pi\|^2\|h\| = \|h\|$. So, it only remains to show that m is left invariant, because the positiveness of m follows automatically (see [**24**]). If $t \in G$ and $h \in L^\infty(G)$ the (left) t-translated function of h is $\pi(t^{-1})h$ and $M_{\pi(t^{-1})h} = \pi(t^{-1}) \circ M_h \circ \pi(t)$ so that $\mathcal{Q}(M_{\pi(t^{-1})h}) = \mathcal{Q}(\pi(t^{-1}) \circ M_h \circ \pi(t)) = \mathcal{Q}(M_h)$ by hypothesis. Thus $m(\pi(t^{-1})h) = m(h)$ as required. The final conclusion is that G is amenable.

Corollary 3.2. *Let G be a locally compact group. Then G is amenable if and only if for every reflexive Banach space and every $\pi \in R(G, E) \cong Rep(L^1(G), E)$ there exists a π-invariant quasi-expectation $\mathcal{E} : L(E) \to \pi(G)'$ of norm $\leq \|\pi\|^2$.*

Note that the module property which is asked for in the concept of a quasi-expectation is not strictly necessary in order to prove the amenability of the group G in Corollary 3.2, as Proposition 3.1 shows. On the contrary, we cannot remove the "π-invariance" condition on $\mathcal{E}$ in this corollary (or on $\mathcal{Q}$ in the quoted proposition), at least concerning the single left regular representation λ of G on $L^2(G)$, since the existence of a (non necessarily λ-invariant) quasi-expectation onto $\lambda(G)'$ only implies that $\lambda(G)'$ is injective, and this is not enough to establish the amenability of G, in general ([**4**, p.604], [**24**, 1.31 and 2.35]). However, the equivalence between amenable groups, and *inner amenable* groups G for which $\lambda(G)''$ is injective, ([**24**, p.85]) implies that the amenability of an inner amenable locally compact group G can be characterized by the existence, for a given representation π, of the corresponding quasi-expectation, non necessarily π-invariant. Recall that a locally compact group G is said to be inner amenable if there exists a mean m on G such that $m({}_{s^{-1}}f_s) = m(f)$, for every $s \in G$ and $f \in L^\infty(G)$, where ${}_{s^{-1}}f_s = f(sts^{-1})$ for all $t \in G$ ([**24**, p.84]). We thank Professor A.T.M. Lau for having pointed out to us a mistake in the original statement of the following Theorem 3.3, and also for having given us references [**20**], [**21**].

Theorem 3.3. *Let G be an inner amenable, locally compact group. The following are equivalent.*

(i) *G is amenable.*

(ii) *For every unital, Banach algebra $\mathcal{B}$ satisfying (DM) and every $\pi \in Hom(L^1(G), \mathcal{B})$ there exists a quasi-expectation $\mathcal{E} : \mathcal{B} \to \pi(L^1(G))'$.*

(iii) *For every unital, Banach algebra $\mathcal{B}$ satisfying (DM) and every $\pi \in Hom(L^1(G), \mathcal{B})$ there exists a bounded, linear projection $\mathcal{Q} : \mathcal{B} \to \pi(L^1(G))'$ such that $U\mathcal{Q}(\mu)U^{-1} = \mathcal{Q}(U\mu U^{-1})$ for every $\mu \in \mathcal{B}$ and every invertible element U of $\pi(L^1(G))'$.*

(iv) *For every reflexive Banach space E and every $\pi \in Rep(L^1(G), E) \cong R(G, E)$ there exists a quasi-expectation $\mathcal{E} : L(E) \to \pi(L^1(G))'$.*

(v) *For every reflexive Banach space E and every $\pi \in R(G, E)$ there exists a bounded, linear projection $\mathcal{Q} : L(E) \to \pi(G)'$ onto $\pi(G)'$ such that $U\mathcal{Q}(A)U^{-1} = \mathcal{Q}(UAU^{-1})$ for every $A \in L(E)$ and every invertible element U of $\pi(G)'$.*

Proof: (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (v),(iv) $\Rightarrow$ (v) are evident.

(v) $\Rightarrow$ (i) Take $\pi = \lambda$, the left regular representation of G on $L^2(G)$, and use the fact that $\lambda(G)'$ is a von Neumann algebra : in [**2**, Theorem 5.3] it is observed that, under the hypothesis assumed in (v), there is a quasi-expectation from $L(L^2(G))$ onto $\lambda(G)'$ indeed, and so $\lambda(G)'$ is injective. Therefore $\lambda(G)''$ is injective too ([**24**, p.79]), and then G must be amenable ([**21**, Corollary 3.2]).

A locally compact group is said to be a [IN]-*group* if it contains a compact neighbourhood of the identity which is invariant under conjugation. The class of the [IN]- groups G, which of course includes that of the discrete ones, is identified by the property that $L^p(G)$ has nonzero central elements for every $1 \leq p < \infty$ (see [**20**, Theorem 1]; we are grateful to A.T.M. Lau for this observation). Moreover, all [IN]- groups are inner amenable ([**21**]), so Theorem 3.3 can be rightly applied to them. We are going to show some improvement of this last result. Denote by λ_p the left regular representation of G on $L^p(G)$ for $1 \leq p < \infty$.

Proposition 3.4. *Let G be an [IN]- group. Then G is amenable if and only if, for some $1 \le p < \infty$, there exists a norm-one linear projection $\mathcal{Q}_p : L(L^p(G)) \to \lambda_p(G)'$ such that $U\mathcal{Q}_p(A)U^{-1} = \mathcal{Q}_p(UAU^{-1})$ for every $A \in L(L^p(G))$ and every invertible element U of $\lambda_p(G)'$.*

Proof: Take a nonzero central element $x \in L^p(G)$. It means that $\lambda(t)x = \rho(t^{-1})x$ where $[\rho(t^{-1})x] = x(\cdot\, t^{-1})$, for every $t \in G$. Note that λ and ρ commute when they act on $L^p(G)$. As in Proposition 3.1 we define $m(h) = \langle \mathcal{Q}_p(M_h)x, x\rangle$, ($h \in L^\infty(G)$). Then, if $t \in G$ and $h \in L^\infty(G)$, we have that $M_{\rho(t)h} = \rho(t)\circ M_h \circ \rho(t^{-1})$ and therefore if ω is the modular function on G,

$$m[\rho(t)h] = \langle \mathcal{Q}_p(\rho(t)\circ M_h \circ \rho(t^{-1}))x, x\rangle$$

$$\omega(t^{-1})\langle \mathcal{Q}_p(M_h)\rho(t^{-1})x, \rho(t^{-1})x\rangle = \omega(t^{-1})\langle \mathcal{Q}_p(M_h)\lambda(t)x, \lambda(t)x\rangle$$

$$\omega(t^{-1})\langle \lambda(t^{-1})\mathcal{Q}_p(M_h)\lambda(t)x, x\rangle = \omega(t^{-1})\langle \mathcal{Q}_p(M_h)x, x\rangle = \omega(t^{-1})m(h),$$

(see [**15**, Th. 20.10]). Since every [IN]-group is unimodular, we have that $\omega(t) \equiv 1$ indeed, and then m is right invariant. The remainder is clear.

Note that $p = 2$ is the case (v) $\Rightarrow$ (i) of Theorem 3.3, where G does not need to be [IN] but only an inner amenable group

§4. Reductive homogeneous spaces.

Let $\mathcal{A}$ and $\mathcal{B}$ be as in Theorem 2.1. For every U in the set $\mathcal{G}$ of invertible elements of $\mathcal{B}$ and every $\pi \in Hom(\mathcal{A}, \mathcal{B})$ we have $U\pi U^{-1} \in Hom(\mathcal{A}, \mathcal{B})$, where $(U\pi U^{-1})(a) = U\pi(a)U^{-1}$ for all $a \in \mathcal{A}$. This provides a natural action of $\mathcal{G}$ over $Hom(\mathcal{A}, \mathcal{B})$ and corresponding map $\tau_\pi : U \in \mathcal{G} \to U\pi U^{-1} \in Hom(\mathcal{A}, \mathcal{B})$ so that each orbit $O(\pi)$ of the action is the image $\tau_\pi(\mathcal{G})$. In this section we study the underlying differentiable structure of the space $Hom(\mathcal{A}, \mathcal{B})$ and the geometric features of the action $\mathcal{G} \times Hom(\mathcal{A}, \mathcal{B}) \to Hom(\mathcal{A}, \mathcal{B})$. The next result shows how the 1-form K_π constructed in Theorem 2.1 allows us to regard the orbits $O(\pi)$ as homogeneous spaces.

Proposition 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $\mathcal{A}$ amenable, and $\mathcal{B}$ unital, satisfying (DM). Then, if $\pi \in Hom(\mathcal{A}, \mathcal{B})$, there exists an open neighborhood $\mathcal{W}$*

of π in $Hom(\mathcal{A},\mathcal{B})$, such that $K_\pi(\sigma) \in \mathcal{G}$ and $K_\pi(\sigma)\pi K_\pi(\sigma)^{-1} = \sigma$ for all $\sigma \in \mathcal{W}$.

Proof: It is enough to take $\mathcal{W} = \{\sigma : \|\sigma - \pi\| < (\|M\|\|\pi\|)^{-1}\}$ for then $\|e - K_\pi(\sigma)\| = \|K_\pi(\pi - \sigma)\| \leq \|M\|\|\pi\|\|\pi - \sigma\| < 1$, so $K_\pi(\sigma) \in \mathcal{G}$. Moreover, by Theorem 2.1, we have $\sigma(a)K_\pi(\sigma) = K_\pi(\sigma(a)\sigma) = K_\pi(\sigma_a) = K_\pi(\sigma)\pi(a)$, for every $a \in \mathcal{A}$.

The proposition proves that the map K_π restricted to $\mathcal{W}$ defines a *continuous local cross section* at π of the map τ_π. By a simple translation we get, for every $\sigma = U\pi U^{-1} \in O(\pi)$, a local cross section of τ_π at σ: in fact, it suffices to take the map $K_\pi(\cdot U)$ defined over the neighborhood $U\mathcal{W}U^{-1}$ of σ.

Remarks. (i) Proposition 4.1 applies to Example (1) of §2 such as it stands. But, moreover, the result is also true for *dual* representations π, σ, of *non necessarily* reflexive Banach spaces F, provided that we consider the space of all such dual representations from $\mathcal{A}$ into F^* (so $\mathcal{B} = L(F^*)$) instead of $Hom(\mathcal{A},\mathcal{B})$. The reason for this is that part (ii) of Theorem 2.1 holds for $\mu = \sigma(a)$ even though F^* is not reflexive.

(ii) Let G be a *connected*, locally compact group, and suppose that each $\pi \in Hom(\mathcal{A},\mathcal{B})$ has a neighborhood $\mathcal{W}$ of π such that for every $\sigma \in \mathcal{W}$ there exists U invertible in $\mathcal{B}$ with $\sigma = U\pi U^{-1}$. Then G must be amenable ([**27**, Theorem 2]). This fact together with results of [**12**] suggest that maybe the existence of continuous local cross sections in $Hom(\mathcal{A},\mathcal{B})$ characterizes the amenability of $\mathcal{A}$.

(iii) It is clear from Proposition 4.1 that each orbit $O(\pi)$ is a clopen subset of $Hom(\mathcal{A},\mathcal{B})$ and a homogeneous space of $\mathcal{G}$, $O(\pi) \cong \mathcal{G}/\mathcal{G}_\pi$, where $\mathcal{G}_\pi := \{U \in \mathcal{G} : U\pi U^{-1} = \pi\}$ (see [**29**]). Moreover, each element of the connected component of $\pi \in O(\pi) \subset Hom(\mathcal{A},\mathcal{B})$ has the form $exp(\mu_1)...exp(\mu_n)\pi exp(-\mu_n)...exp(-\mu_1)$ for $\mu_1,...,\mu_n \in \mathcal{B}$: the principal component of $\mathcal{G}$, denoted by $\mathcal{G}_0$, consists of a ll finite products of exponentials of elements of $\mathcal{B}$ [**28**]; also, the invertible element $K_\pi(\sigma)$ of Proposition 4.1 lies in $\mathcal{G}_0$ indeed. Hence, by considering the action of $\mathcal{G}_0$ over $Hom(\mathcal{A},\mathcal{B})$ instead of the action of the whole group $\mathcal{G}$ we obtain this time that the connected component of $\pi \in Hom(\mathcal{A},\mathcal{B})$ is $\tau_\pi(\mathcal{G}_0)$.

A straightforward application of the last remark is that if $\Phi : \mathcal{B} \to \mathcal{C}$ is a bounded epimorphism, where $\mathcal{B},\mathcal{C}$ are unital Banach algebras which satisfy condition (DM) then for every amenable Banach algebra $\mathcal{A}$ the map $\Phi^\# :$

$Hom(\mathcal{A},\mathcal{B}) \to Hom(\mathcal{A},\mathcal{C})$ defined by $\Phi^{\#}(\pi) = \Phi \circ \pi$ induces a continuous surjection from the connected component of π in $Hom(\mathcal{A},\mathcal{B})$ onto the connected component of

$\Phi \circ \pi$ in $Hom(\mathcal{A},\mathcal{C})$. It can be proven that $\Phi^{\#}$ is a Serre fibration, so that it has very useful homotopy lifting properties, but we shall not need this fact.

Next, we analyse the differentiable structure of $Hom(\mathcal{A},\mathcal{B})$ using the information given in propositions 2.4 and 4.1.

Theorem 4.2. *Let $\mathcal{A}$ be an amenable Banach algebra and let $\mathcal{B}$ be a unital Banach algebra satisfying (DM). Then, each orbit $O(\pi)$ of the action $\mathcal{G} \times Hom(\mathcal{A},\mathcal{B}) \to Hom(\mathcal{A},\mathcal{B})$ considered above is a Banach homogeneous space, in particular a Banach submanifold of $L(\mathcal{A},\mathcal{B})$, and the tangent space of $O(\pi)$ at $\sigma \in O(\pi)$ is isomorphic to $Der_\sigma(\mathcal{A},\mathcal{B})$. Moreover, the map*

$\tau_\pi : \mathcal{G} \to O(\pi)$ is a Banach principal bundle with structure group $\mathcal{G}_\pi$.

Proof: For the first part it suffices to observe that the mapping $(U,T) \in \mathcal{G} \times L(\mathcal{A},\mathcal{B}) \to UTU^{-1} \in L(\mathcal{A},\mathcal{B})$ is holomorphic, the map $\tau_\pi : \mathcal{G} \to O(\pi)$ has Δ_π as its tangent map at e, the map τ_π is open and the subspaces $ker(\Delta_\pi)$ of $\mathcal{B}$, and $im(\Delta_\pi)$ of $L(\mathcal{A},\mathcal{B})$ are respectively complemented.

The result follows from Proposition 1.3.

Now, that τ_π defines a (Banach) principal bundle follows as usual from the existence of local cross sections for τ_π. In more detail, take $\sigma \in O(\pi)$ and choose a continuous local cross section S_π of τ_π, at σ, on a neighbourhood $\mathcal{W}$ of σ, as in the remark to Proposition 4.1. It is easy to check that $\tau_\pi^{-1}(\mathcal{W}) = S_\pi(\mathcal{W})\mathcal{G}_\pi$, that this decomposition of the elements of $\tau_\pi^{-1}(\mathcal{W})$ is unique, and that it is given by

$U = S_\pi(\tau_\pi(U))\varphi(U)$ for every $U \in \tau_\pi^{-1}(\mathcal{W})$, where $\varphi : \tau_\pi^{-1}(\mathcal{W}) \to \mathcal{G}_\pi$ can be made explicit as $\varphi(U) = (S_\pi(\tau_\pi(U)))^{-1}U$, if $U \in \tau_\pi^{-1}(\mathcal{W})$. Thus we may identify $\tau_\pi^{-1}(\mathcal{W})$ to $\mathcal{W} \times \mathcal{G}_\pi$ by means of the mapping $\Phi : U \in \tau_\pi^{-1}(\mathcal{W}) \to (\tau_\pi(U), \varphi(U)) \in \mathcal{W} \times \mathcal{G}_\pi$, where $\varphi(UV) = \varphi(U)V$ for every $U \in \tau_\pi^{-1}(\mathcal{W})$, $V \in \mathcal{G}_\pi$, and whose inverse Φ^{-1} is given as $\Phi^{-1} : (\rho, V) \in \mathcal{W} \times \mathcal{G}_\pi \to S_\pi(\rho)V \in \tau_\pi^{-1}(\mathcal{W})$. Moreover S_π is differentiable, for it is the restriction of a linear mapping between Banach spaces ($dK_\pi(\rho) \cong K_\pi$, if $\rho \in O(\pi)$, as it should be !). Hence $\tau_\pi : \mathcal{G} \to O(\pi)$ is a Banach principal bundle with structure group $\mathcal{G}_\pi$ ([**19**; volume 1, p.50]).

Let now $\mathcal{A}$ and $\mathcal{B}$ be *any* two Banach algebras, where

$\mathcal{B}$ is unital. We keep the same notations as above; thus $\mathcal{G}$

is the group of invertible elements of $\mathcal{B}$ and so on. Since we have in mind to apply the theory of reductive homogeneous spaces of classical differential geometry to $O(\pi)$ (see [**19**], [**2**]) we introduce the following definition for $\pi \in Hom(\mathcal{A},\mathcal{B})$.

Definition 4.3. *Suppose that $O(\pi)$ is a Banach homogeneous space. Then a reductive structure for π is a Banach subspace H^π of $\mathcal{B}$ such that*

(i) $\mathcal{B} = H^\pi \oplus \pi(\mathcal{A})'$

(ii) $UH^\pi U^{-1} = H^\pi$ *if U is invertible in $\pi(\mathcal{A})'$.*

Let $\mathcal{E} : \mathcal{B} \to \pi(\mathcal{A})'$ be the projection associated to the decomposition $\mathcal{B} = H^\pi \oplus \pi(\mathcal{A})'$ in the above definition. We will say that the reductive structure for π is *metric* if $\|\mathcal{E}\| \leq \|\pi\|^2$.

The spaces H^π will be called *horizontal spaces* and give rise to a connection on the homogeneous space $O(\pi)$ (see [**22**, section 3 and p.16] and [**9**, p.220]).

Theorem 4.4. *If $\mathcal{A}$ is amenable and $\mathcal{B}$ satisfies property (DM) then every $\pi \in Hom(\mathcal{A},\mathcal{B})$ has a metric reductive structure.*

Proof: It is enough to take $H^\pi = ker(\mathcal{E}_\pi)$, where $\mathcal{E}_\pi$ is as in Proposition 2.2.

The projection $\mathcal{E}_\pi$ in the preceding theorem is obtained from the mapping K_π of Theorem 2.1 as in §2. In fact, H^π can alternatively be presented under the form of $K_\pi(Der_\pi(\mathcal{A},\mathcal{B}))$ such as we indicate in the Introduction. For, it is shown in §2 that $K_\pi \circ \Delta_\pi = I = -\mathcal{E}_\pi$ and then K_π identifies to the isomorphism $Der_\pi(\mathcal{A},\mathcal{B}) \to ker(\mathcal{E}_\pi)$ defined by $\Delta_\pi(\mu) \to \mu - \mathcal{E}_\pi(\mu)$. Moreover, if $\sigma = Ad_U(\pi)$, that is $\sigma = U\pi U^{-1}$, with $U \in \mathcal{G}$, we have

$$\langle K_\sigma(T), x\rangle = \lim_j \langle (T \otimes_\gamma \sigma)(m_j), x\rangle$$

$$= \lim_j \langle (T \otimes_\gamma U\pi(\cdot)U^{-1})(m_j), x\rangle = \lim_j \langle (U^{-1}T(\cdot)U \otimes_\gamma \pi)(m_j), U^{-1}\cdot x \cdot U\rangle$$

$$= \langle K_\pi(U^{-1}T(\cdot)U), U^{-1}\cdot x\cdot U\rangle = \langle U[K_\pi(U^{-1}T(\cdot)U)]U^{-1}, x\rangle$$

$$= \langle (Ad_U \circ K_\pi \circ Ad_{U^{-1}})(T), x\rangle$$

for every $x \in X$, $T \in L(\mathcal{A},\mathcal{B})$.

It follows that $K_\sigma = Ad_U \circ K_\pi \circ Ad_{U^{-1}}$, so that it defines a 1-form in the sense of [**22**] (see also [**2**, 5.7] and [**9**]). As usual, all invariants of the canonical connection on the principal bundle $\tau_\pi : \mathcal{G} \to O(\pi)$ can be computed in terms of K. This can be done in a rather general setting (see [**22**] for details) but we present a concise resumé of the results in the present example. For instance, the *curvature form* of the connection assigns to every $U \in \mathcal{G}$ and $X, Y \in (T\mathcal{G})_U = \mathcal{B}$ the element $\Omega_U(X,Y) = (1/2)h_1([U^{-1}Y, U^{-1}X]) + (1/2)[h_1(U^{-1}X), h_1(U^{-1}Y)]$, where h_1 is the projection onto the horizontal vectors at $e \in \mathcal{G}$, i.e., $h_1 = I - \mathcal{E}_\pi$. Thus,

$$\Omega_U(X,Y) = (-1/2)(I-\mathcal{E}_\pi)[U^{-1}X, U^{-1}Y] + (1/2)[(I-\mathcal{E}_\pi(U^{-1}X), (I-\mathcal{E}_\pi(U^{-1}Y)].$$

Given a smooth curve $\gamma : [0,1] \to O(\pi)$ with $\gamma(0) = \pi$, the *transport equation* for γ is the differential equation

$$\dot{\Gamma}(t) = K_{\gamma(t)}(\dot{\gamma}(t))\Gamma(t)$$

and the solution Γ with the initial condition $\Gamma(0) = 1$ is called the *horizontal lift* of γ. The name is not arbitrary: in fact, Γ is a lifting of γ with respect to the map $\tau_\pi : \mathcal{G} \to Hom(\mathcal{A}, \mathcal{B})$, in the sense that $\tau_\pi(\Gamma(t)) = \gamma(t), t \in [0,1]$ and the tangent vectors of the smooth map Γ belong to the corresponding horizontal spaces:

for all $t \in [0,1], \dot{\Gamma}(t) \in H^{\gamma(t)}$.

The invariants of the linear connection induced in the tangent bundle can also be easily computed. Thus, if $Y(t)$ is a tangent field along $\gamma(t)$ then the *covariant derivative* of Y is the vector field $\frac{DY}{dt}$ defined by the formula

$$K_\gamma(\frac{DY}{dt}) = K_\gamma(\dot{Y}) + [K_\gamma(Y), K_\gamma(\dot{\gamma})]$$

(recall that, when restricted to $ker(\mathcal{E}_\gamma)$, K_γ is an isomorphism). A curve γ is called a *geodesic* if $\frac{D\dot{\gamma}}{dt} = 0$. It can be shown that the unique geodesic ψ in $O(\pi)$ with initial value π and velocity vector $\dot{\psi}(0) = X$ is given by

$$\psi(t) = e^{tK_\pi(X)}\pi e^{-tK_\pi(X)}, t \in (-\infty, +\infty).$$

The *torsion tensor* T is defined, using the same notation as before, by the formula

$$K_\pi(T(X,Y)) = (I - \mathcal{E}_\pi)([K_\pi(X), K_\pi(Y)]),$$

for all $X, Y \in Der_\pi(\mathcal{A}, \mathcal{B})$, and, analogously, the *curvature tensor* is defined by

$$K_\pi(R(X,Y)Z) = (I - \mathcal{E}_\pi)([K_\pi(Z), (I - \mathcal{E}_\pi)[K_\pi(X), K_\pi(Y)])$$

for all $X, Y, Z \in Der_\pi(\mathcal{A}, \mathcal{B})$.

We do not know if the existence of a natural geometry in $Hom(\mathcal{A}, \mathcal{B})$ as the one considered here implies the amenability of the algebra $\mathcal{A}$, in general. However, this is the case when G is a locally compact group and $\mathcal{A}$ is assumed to be the group algebra $L^1(G)$, as we are going to see in several cases. For example, and according to Theorem 4.2 and [**27**] (see Remark (ii) of Proposition 4.1), we already know that, for G connected, G is amenable if and only if the mapping $\tau_\pi : \mathcal{G} \to O(\pi)$ is a Banach principal bundle. Other geometric characterizations follow from statements of Section 3.

Let us give the following definition. For every couple of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ a reductive structure for $\pi \in Hom(\mathcal{A}, \mathcal{B})$ will be called π-*invariant* if $[\pi(\mathcal{A}), H^\pi] \subseteq H^\pi$.

We denote by $GL(E)$ the group of invertible elements of $L(E)$.

Theorem 4.5. *Let G be a locally compact group. Then the following are equivalent.*

(i) *G is amenable.*

(ii) *For every reflexive Banach space E and every $\pi \in R(G, E)$*

the mapping $\tau_\pi : GL(E) \longrightarrow O(\pi)$ is a principal bundle with a π-invariant connection form $\mathcal{E} : L(E) \to \pi(G)'$ such that $\|\mathcal{E}\| \leq \|\pi\|^2$.

(iii) For every reflexive Banach space E each $\pi \in R(G, E)$ has a metric, π-invariant reductive structure.

Proof: It is a consequence of former considerations and Theorem 3.1.

Theorem 4.6. *Let G be an inner amenable, locally compact group. Then G is amenable if and only if for every reflexive Banach space E each $\pi \in R(G, E)$ has a reductive structure.*

Proof: If $\mathcal{E} : L(E) \to \pi(G)'$ is the projection associated to the π-invariant structure then, for $A \in L(E)$ and U invertible in $L(E)$, we have that $\mathcal{E}(U(A - \mathcal{E}(A))U^{-1}) = 0$, that is, $UAU^{-1} - U\mathcal{E}(A)U^{-1} \in ker(\mathcal{E})$. Hence $U\mathcal{E}(A)U^{-1} \in im(\mathcal{E})$ and $\mathcal{E}(UAU^{-1}) = U\mathcal{E}(A)U^{-1}$. Now the theorem is a simple re-statement of Theorem 3.3.

Proposition 4.7. *Let G be an [IN]- group. Then G is amenable if and only if there exists p, $1 \le p < \infty$, for which the representation $\lambda_p \in R(G, L^p(G))$ has a metric reductive structure.*

Proof: This is the geometric analogue to Proposition 3.4.

§5. Automorphism groups of amenable algebras.

Let $\mathcal{B}$ be a unital Banach algebra satisfying (DM) and let X be a "predual" $\mathcal{B}$-bimodule as in the definition of the property (DM). Throughout this section $\mathcal{A}$ is a closed two-sided ideal of $\mathcal{B}$ which is amenable as a Banach algebra and such that $\mathcal{A} \cdot X = X$. In this case $\mathcal{B}$ is isomorphic to the Banach algebra $M_r(\mathcal{A}) := \{T \in L(\mathcal{A}) : T(ab) = T(a)b$ for every $a, b \in \mathcal{A}\}$ of *right multipliers* of $\mathcal{A}$: given a bounded approximate unit (e_j) in $\mathcal{A}$ and T in $M_r(\mathcal{A})$ a standard argument shows that for any cluster point μ of $T(e_j)$ in the weak topology of

$\mathcal{B}$ we get $T(a) = \mu a$ for each $a \in \mathcal{A}$.

By $End(\mathcal{A})$ we denote the space of bounded non-degenerate endomorphisms $\pi : \mathcal{A} \to \mathcal{A}$ endowed with the topology defined by the operator norm. Since $\mathcal{A} \cdot X = X$ we have $\pi(\mathcal{A}) \cdot X = \pi(\mathcal{A})\mathcal{A} \cdot X = \mathcal{A} \cdot X = X$ and therefore $End(\mathcal{A}) \subseteq Hom(\mathcal{A}, \mathcal{B})$ as closed subspace. Moreover, the action $(U, \pi) \in \mathcal{G} \times End(\mathcal{A}) \to U\pi U^{-1} \in End(\mathcal{A})$ is well defined and $End(\mathcal{A})$ is formed by part of the orbits of $Hom(\mathcal{A}, \mathcal{B})$. More precisely, we have the following.

Proposition 5.1. *For $\mathcal{A}$, $\mathcal{B}$ as just before every orbit $O(\pi)$, $\pi \in End(\mathcal{A})$, of the action $(U, \pi) \in \mathcal{G} \times End(\mathcal{A}) \to U\pi U^{-1} \in End(\mathcal{A})$*

is a Banach homogeneous space and a submanifold of $L(\mathcal{A})$.

Proof: By Cohen's theorem ([**6**]), $\mathcal{A} = \mathcal{A}\mathcal{A}$ and so

$Der_\pi(\mathcal{A}, \mathcal{B}) = Der_\pi(\mathcal{A})$ whenever $\pi \in End(\mathcal{A})$. Then the restriction of the mapping $\mathcal{P}_\pi$ of Proposition 2.3 to

$L(\mathcal{A})$ is a bounded projection from $L(\mathcal{A})$ onto $Der_\pi(\mathcal{A}) \subset L(\mathcal{A})$, i.e.,

$im(\Delta_\pi) = Der_\pi(\mathcal{A})$ is complemented in $L(\mathcal{A})$. Thus, the mapping $(U,T) \in \mathcal{G} \times L(\mathcal{A}) \to U\pi U^{-1} \in L(\mathcal{A})$ being holomorphic, we can follow the same argument as in Theorem 4.2 to conclude the proof.

Clearly, if $\pi \in End(\mathcal{A})$ then π has a reductive structure with respect to $\mathcal{B}$, that is, there is a Banach subspace H^π of $\mathcal{B}$ such that $\mathcal{B}$ is decomposable as $\mathcal{B} = H^\pi \oplus \pi(\mathcal{A})'$.

We can proceed with the space of bounded automorphisms of $\mathcal{A}$, $Aut(\mathcal{A})$, along the same lines as for $End(\mathcal{A})$. To begin with, let us recall several general facts concerning automorphism groups of Banach algebras.

First, *for every Banach algebra $\mathcal{C}$, $Aut(\mathcal{C})$ is a Banach-Lie group with Lie algebra $Der(\mathcal{C})$*: the exponential map $exp : Der(\mathcal{C}) \to Aut(\mathcal{C})$ is a local homeomorphism with local inverse $log : \mathcal{V} \to Der(\mathcal{C})$ given by $log(\sigma) = \sum_{n=1}^{\infty} \frac{(e-\sigma)^n}{n}$ for $\sigma \in \mathcal{V} = \{\sigma \in Aut(\mathcal{C}) : \|e-\sigma\| < 1\}$ ([**6**, p.87,91,92]). Therefore, the collection of pairs $\{(\pi\mathcal{V}, \varphi_\pi) : \pi \in Aut(\mathcal{C})\}$, where $\varphi_\pi : \sigma \in \pi\mathcal{V} \to log((\pi)^{-1}\sigma) \in Der(\mathcal{C})$, turns out to be a holomorphic atlas on $Aut(\mathcal{C})$ ([**26**, p.368]), so that $Aut(\mathcal{C})$ becomes a (complex) Banach manifold. Moreover, the multiplication $(\pi,\sigma) \in Aut(\mathcal{C}) \times Aut(\mathcal{C}) \to \pi\sigma^{-1} \in Aut(\mathcal{C})$ is holomorphic with respect to the atlas $(\pi\mathcal{V}, \varphi_\pi)_{\pi \in Aut(\mathcal{C})}$ and therefore $Aut(\mathcal{C})$ is a Banach-Lie group ([**26**, p.370]). The last statement is obvious.

Note that the result above does not ensure that $Aut(\mathcal{C})$ is a submanifold of $L(\mathcal{C})$. It is an interesting problem to characterize, or at least to have a number of examples of, those Banach algebras with this property. In this respect, see Theorem 5.2 below.

On the other hand, if $\mathcal{C}$ is also unital, we can consider the homomorphism of Banach-Lie groups $Ad : \mathcal{G}_\mathcal{C} \to Aut(\mathcal{C})$ where $\mathcal{G}_\mathcal{C}$ is the group of invertible elements of C and $Ad_U(c) = UcU^{-1}$ for $c \in C$, $U \in \mathcal{G}_\mathcal{C}$. Its kernel is the group of invertible elements of the center $\mathcal{Z}_\mathcal{C}$ of $\mathcal{C}$ and its image is the subgroup $Inn(\mathcal{C})$ of inner automorphisms of $\mathcal{C}$. The tangent map of Ad at the unit $1 \in \mathcal{G}_\mathcal{C}$ is the map $ad = (d\ Ad)_1 : \mathcal{C} \to Der(\mathcal{C})$ given by $c \to \Delta_I(c)$, thus the kernel of ad is $\mathcal{Z}_\mathcal{C}$ and the image of ad is the space $Der_{inn}(\mathcal{C})$ of inner derivations of $\mathcal{C}$. If the first Hochschild cohomology group $H^1(\mathcal{C},\mathcal{C})$ is trivial, i.e., $Der_{inn}(\mathcal{C}) = Der(\mathcal{C})$, then ad is onto. As a consequence, Ad is a submersion if and only if $\mathcal{Z}_\mathcal{C}$ is a complemented subspace in C; in this case $Aut(\mathcal{C})$ is a Banach homogeneous space of $\mathcal{G}_\mathcal{C}$

(this happens when C is a von Neumann algebra, for instance. See also Theorem 5.2). Under the same hypothesis, $H^1(\mathcal{C},\mathcal{C}) = 0$, the connected component of the identity in $Aut(\mathcal{C})$ is the image by Ad of the principal component of $\mathcal{G}_{\mathcal{C}}$ (see [**28**, Chapter 7]).

For unital C^*-algebras, it is worth mentioning the following important result, which is in the same spirit as this paper: if $\mathcal{C}$ is a unital, C^*-algebra then the map $Ad : \mathcal{U} \to Inn(\mathcal{C})$ is a fibre bundle if and only if $Der_{inn}(\mathcal{C})$ is a closed subspace of $L(\mathcal{C})$, where $\mathcal{U}$ denotes the group of unitary elements of $\mathcal{C}$ ([**25**, Theorem 2.1]).

All the above considerations fit well in our setting, i.e., for $\mathcal{A}$ amenable and $\mathcal{B}$ having property (DM) and being isomorphic to $M_r(\mathcal{A})$. Let us start by noting that if D is a bounded derivation from $\mathcal{B}$ into itself then the restriction of D to $\mathcal{A}$ gives us an inner derivation, implemented by some $\mu \in \mathcal{B}$, since $\mathcal{A}$ is amenable. Also, for $\nu \in \mathcal{B}$ and $a \in \mathcal{A}$, $D(\nu)a = D(\nu a) - \nu D(a) = \mu\nu a - \nu a\mu - \nu(\mu a - a\mu) = (\mu\nu - \nu\mu)a$ whence, looking at $\mathcal{B}$ as the multiplier algebra $M_r(\mathcal{A})$, we obtain that $D(\nu) = \mu\nu - \nu\mu$ and then $H^1(\mathcal{B},\mathcal{B}) = 0$. Hence the above remark on connected components ([**28**, Chapter 7]) applies to $\mathcal{B}$. As on the other hand every automorphism φ of $\mathcal{A}$ can be extended to an automorphism ϕ of $\mathcal{B} \cong M_r(\mathcal{A})$ by putting $\phi(\mu)a = \varphi(\mu\varphi^{-1}(a))$, for every $\mu \in \mathcal{B}$ and $a \in \mathcal{A}$, we have that each automorphism of the principal component of $\mathcal{A}$ is in $Ad(\mathcal{G}_0)$. Even more, we have the following theorem concerning $Aut(\mathcal{A})$.

Denote by $Aut(\mathcal{A})_\pi$ the connected component of an automorphism π in $Aut(\mathcal{A})$.

Theorem 5.2. *For $\mathcal{A}$, $\mathcal{B}$ as in 5.1 the Banach- Lie group $Aut(\mathcal{A})$ becomes a discrete union of Banach homogeneous spaces with respect to $\mathcal{G}$, and a closed submanifold of $L(\mathcal{A})$. Furthermore, the map*

$U \in \mathcal{G}_0 \to Ad_U \circ \pi \in Aut(\mathcal{A})_\pi$ is a fibre bundle for every $\pi \in Aut(\mathcal{A})$; in particular we have that an automorphism φ is in the component of π if and only if there exist $\mu_1, \cdots, \mu_n \in \mathcal{B}$ such that $\varphi(a) = e^{\mu_1}...e^{\mu_n}\pi(a)e^{\mu_n}...e^{\mu_1}$ for every $a \in \mathcal{A}$.

Proof: In view of the remark (iii) to Proposition 4.1 and Proposition 5.1 all we need to show is the following. If $\sigma = U\pi U^{-1}$ for some $\pi \in Aut(\mathcal{A})$ and $U \in \mathcal{G}$, then $\sigma \in Aut(\mathcal{A})$ with inverse σ^{-1} given by $\sigma^{-1}(a) = \pi^{-1}(U^{-1}aU)$ for every $a \in \mathcal{A}$. This implies that the orbit $O(\pi)$ of every automorphism π of $\mathcal{A}$ in

$End(\mathcal{A}) \subset Hom(\mathcal{A},\mathcal{B})$ consists entirely of automorphisms of $\mathcal{A}$, the tangent space $Der_{\pi}(\mathcal{A})$ at each point of $O(\pi)$ being complemented in $L(\mathcal{A})$. Also, $Aut(\mathcal{A})$ is closed as a subspace of $End(\mathcal{A})$ for, if $(\pi_n)_{n=1}^{\infty}$ is a sequence in $Aut(\mathcal{A})$ such that $\pi_n \to \sigma$ in the norm topology then $\|\pi_n - \sigma\| < 2^{-1}\|\sigma\|^{-1}\|M\|^{-1}$, $\|\pi_n\|^{-1} \geq 2^{-1}\|\sigma\|^{-1}$ for every $n \geq n_0$ and some $n_0 \in \mathbf{N}$. Hence $\|\pi_n - \sigma\| < \|M\|^{-1}\|\pi_n\|^{-1}$, ($n \geq n_0$), and then, by Proposition 4.1, we obtain that $K_{\pi_n}(\sigma) \in \mathcal{G}$ and $\sigma = K_{\pi_n}(\sigma)\pi_n K_{\pi_n}(\sigma)^{-1}$, ($n \geq n_0$). Thus $\sigma \in Aut(\mathcal{A})$. The remainder of the proof follows now readily.

Theorem 5.2 applies to the algebra $\mathcal{A} = L^1(G)$, for G amenable since it is a two-sided ideal of $\mathcal{B} = M(G)$ and the space $X = C_0(G)$ is a bimodule on $L^1(G)$ for the convolution, such that $L^1(G) * C_0(G) = C_0(G)$.

Other example which falls under the scope of the theorem is that of $Aut(\mathcal{K}(E))$, where E is a reflexive Banach space and $\mathcal{K}(E)$ is the space of compact operators on E, provided that $\mathcal{K}(E)$ is amenable (it happens, for instance, if E has a shrinking basis; see [**13**]). It is straightforward to verify that each of the technicalities demanded on $\mathcal{A}$, $\mathcal{B}$, X in this section is accomplished by $\mathcal{A} = \mathcal{K}(E)$, $\mathcal{B} = L(E)$, $X = E \otimes E^*$ (for instance, $\mathcal{K}(E)\cdot(E\otimes E^*) \subseteq E\otimes E^*$, $(E\otimes E^*)\cdot\mathcal{K}(E) \subseteq E\otimes E^*$).

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Gustavo Corach and José E. Galé
Instituto Argentino de Matemática
Viamonte 1636 1er piso
1055 Buenos Aires Argentina
and
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Zaragoza
50009 Zaragoza España

Instituto Argentino de Matemática
Saavedra 15 - 3er. Piso

1083 - Buenos Aires
Argentina.