The Integral Representation of the Elementary Solution of the n-Dimensional Ultrahyperbolic Klein-Gordon Operator Iterated k-Times.

Susana Elena Trione

SUMMARY. The kernel $G_\alpha(P\pm i0,m,n)$ (formula (I,1;3)) can be expressed by an integral representation by means of a simple (symbolic) Fourier integral (cf. Theorem 1, form. (II,1;7)). Theorem 2 expresses the useful particular case $\alpha=2k$ where $G_{2k}(P\pm i0,m,n)$ is the elementary (causal, anticausal) solution of the Klein-Gordon operator (cf. [1], p. 38, form. (II, 2;12)). The particular case of the Theorem 2, corresponding to $n=4$, $q=1$, $k=1$ (cf. Theorem 3) is very interesting because the formula (III,1;4), when we adopt the plus sign, has been established by Bogoliubov-Parasiuk (cf. [5], p. 233) and plays an essential role in the theory, due to these authors, of causal distributional products. Moreover, we know that $G_0(P\pm i0,m,n)=\delta$ (cf. [1], p. 37, form. (II,2;7)); therefore by taking limits in both members of (II,1;7) for $k\to 0$, we obtain an integral representation of the δ measure of Dirac. We know that $G_\alpha(P\pm i0,m=0,n)=H_\alpha(P\pm i0,n)$ (cf. [1], p. 49, Th.1, form. (II,7;1)). We give in V.1 an integral representation of $H_\alpha(P\pm i0,n)$ by means of an inverse Laplace transform (cf. Theorem 4, formula (V,1;9)). Finally, we study the particular cases $H_{2k}(P\pm i0,n)$ and $H_0(P\pm i0,n)$ (cf. (V,1;11) and Note).

I.1. Introduction.

Let $x=(x_1,x_2,...,x_n)$ be a point of the n-dimensional Euclidean space $I\!R^n$. Consider a non-degenerate quadratic form in n variables of the form

$$P=P(x)=x_1^2+...+x_p^2-x_{p+1}^2-...-x_{p+q}^2, \qquad (\mathrm{I},1;1)$$

where $n = p + q$. The distribution $(P \pm i0)^\lambda$ is defined by

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \left\{ P \pm i\varepsilon |x|^2 \right\}^\lambda , \qquad (\mathrm{I}, 1; 2)$$

where $\varepsilon = 0$, $|x|^2 = x_1^2 + ... + x_n^2$, $\lambda \in \mathbb{C}$.

The distributions $(P \pm i0)^\lambda$ are an important contribution of Gelfand (cf. **[2]**, p. 274).

The distributions $(P \pm i0)^\lambda$ are analytic in λ everywhere except at $\lambda = -\frac{n}{2} - k$, $k = 0, 1, ...,$ where they have simple poles (cf. **[2]**, p. 275).

Let $G_\alpha(P \pm i0, m, n)$ be the causal (anticausal) distribution defined by

$$G_\alpha(P \pm i0, m, n) = H_\alpha(m, n)(P \pm i0)^{\frac{1}{2}\left(\frac{\alpha - n}{2}\right)} K_{\frac{n-\alpha}{2}} \left(\sqrt{m^2(P \pm i0)} \right) , \qquad (\mathrm{I}, 1; 3)$$

where m es a positive real number, $\alpha \in \mathbb{C}$, K_μ designates the well-known modified Bessel function of the third kind (cf. **[3]**, p. 78, formulae (6) and (7)):

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu}, \qquad (\mathrm{I}, 1; 4)$$

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+\nu}}{m!\Gamma(m + \nu + 1)}, \qquad (\mathrm{I}, 1; 5)$$

$H_\alpha(m, n)$ is the constant defined by

$$H_\alpha(m, n) = \frac{e^{i\frac{\pi}{2}\alpha} 2^{1-\frac{\alpha}{2}} e^{\pm\frac{\pi}{2}qi} (m^2)^{\frac{1}{2}\left(\frac{n-\alpha}{2}\right)}}{(2\pi)^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)}, \qquad (\mathrm{I}, 1; 6)$$

$\nu \neq$ integer.

II.1. The integral representation of the causal (anticausal) distribution $G_\alpha(P \pm i0, m, n)$.

We shall give a representation of the kernel $G_\alpha(P \pm i0, m, n)$ (cf. (I,1;3)) by means of a simple (symbolic) Fourier integral.

We start from the formula (40), p. 283, **[4]**, vol I,

$$\begin{aligned}\int_0^\infty e^{-pt}\tfrac{1}{2}t^{-\nu-1}e^{-\frac{\alpha}{t}}\,dt &\overset{\text{def}}{=} f(p,\alpha,\nu)\\ &= \alpha^{-\frac{\nu}{2}}p^{\frac{\nu}{2}}K_\nu\left[2\alpha^{\frac{1}{2}}p^{\frac{1}{2}}\right],\end{aligned}\qquad (\mathrm{II},1;1)$$

valid for Re p > 0 and Re $\alpha > 0$

By making the change of parameters:

$p = \pm i\left(m^2 \mp i\varepsilon\right),$

$\alpha = \mp\frac{i}{4}\left(P \pm i\varepsilon|x|^2\right),$

where $\varepsilon > 0$, $m > 0$ and $P = P(x)$ defined by (I,1;1); formula (II,1;1) can be written

$$\begin{aligned}&f\left(\pm i\left(m^2 \mp i\varepsilon\right), \mp\frac{i}{4}\left(P \pm i\varepsilon|x|^2\right), \nu\right)\\ &= 2^\nu e^{\pm i\frac{\pi}{2}\nu}\left(m^2 \mp i\varepsilon\right)^{\frac{\nu}{2}}\left(P \pm i\varepsilon|x|^2\right)^{-\frac{\nu}{2}}\cdot K_\nu\left[\sqrt{\left(m^2 \pm i\varepsilon\right)\left(P \pm i\varepsilon|x|^2\right)}\right].\end{aligned}\qquad (\mathrm{II},1;2)$$

We obtain, taking limits in both members of (II,1;2) for $\varepsilon \to 0$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^\infty e^{\pm i(m^2\mp i\varepsilon)t}t^{-\nu-1}e^{\mp\frac{i}{4t}}(P \pm i\varepsilon|x|^2)dt\\ &= 2^{\nu+1}e^{\pm i\frac{\pi}{2}\nu}m^\nu(P \pm i0)^{-\frac{\nu}{2}}K_\nu\left[\sqrt{m^2(P \pm i0)}\right].\end{aligned}\qquad (\mathrm{II},1;3)$$

By putting $\nu = \frac{n-\alpha}{2}$, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^\infty e^{\pm i(m^2\mp i\varepsilon)t}t^{-\left(\frac{n-\alpha}{2}\right)-1}e^{\mp\frac{i}{4t}(P\pm i\varepsilon|x|^2)}dt\\ &= 2^{\frac{n-\alpha}{2}+1}e^{\pm i\frac{\pi}{2}\left(\frac{n-\alpha}{2}\right)}m^{\frac{n-\alpha}{2}}(P \pm i0)^{-\frac{\left(\frac{n-\alpha}{2}\right)}{2}}\cdot K_{\frac{n-\alpha}{2}}\left(\sqrt{m^2(P \pm i0)}\right).\end{aligned}\qquad (\mathrm{II},1;4)$$

By multiplying both members of (II,1;4) by adequate constants we can write, equivalently,

$$\begin{aligned}&C_\pm(n,\alpha,q)\lim_{\varepsilon\to 0}\int_0^\infty e^{\pm i(m^2\mp i\varepsilon)t}t^{-\left(\frac{n-\alpha}{2}\right)-1}e^{\mp\frac{i}{4t}(P\pm i\varepsilon|x|^2)}dt\\ &= H_\alpha(m,n)P(\pm i0)^{\frac{1}{2}\left(\frac{\alpha-n}{2}\right)}K_{\frac{n-\alpha}{2}}\left(\sqrt{m^2(P \pm i0)}\right),\end{aligned}\qquad (\mathrm{II},1;5)$$

where the constants $C_{\pm}(n,\alpha,q)$ are

$$C_{\pm}(n,\alpha,q)=\frac{2^{1-\frac{n}{2}}e^{i\frac{\pi}{2}\alpha}e^{\mp i\frac{\pi}{2}\left(\frac{n-\alpha}{2}\right)}e^{\pm i\frac{\pi}{2}q}}{(2\pi)^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}. \qquad (\mathrm{II},1;6)$$

From the definitory formula (I,1;3) the right-hand member of (II,1;5) is, precisely, $G_\alpha(P\pm i0,m,n)$. Therefore, we have prove the following theorem.

Theorem 1. *The causal (anticausal) distribution kernel* $G_\alpha(P\pm i0,n)$ *admits the integral representation*

$$G_\alpha(P\pm i0,m,n)=C_{\pm}(n,\alpha,q)\lim_{\varepsilon\to 0}\int_0^\infty e^{\pm i(m^2\mp i\varepsilon)t}t^{-\left(\frac{n-\alpha}{2}\right)-1}e^{\mp\frac{i}{4t}(P\pm i\varepsilon|x|^2)}dt, \qquad (\mathrm{II},1;7)$$

where the constants $C_{\pm}(n,\alpha,q)$ *are*

$$C_{\pm}(n,\alpha,q)=\frac{2^{1-\frac{n}{2}}e^{i\frac{\pi}{2}\alpha}e^{\mp i\frac{\pi}{2}\left(\frac{n-\alpha}{2}\right)}e^{\pm i\frac{\pi}{2}q}}{(2\pi)^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}. \qquad (\mathrm{II},1;8)$$

III.1. Particular cases.

The distributional functions $G_{2k}(P\pm i0,m,n)$ where $n=$ integer ≥ 2 and $k=1,2,...$ (called "Feynman distributions") are elementary causal (anticausal) solutions of the ultrahyperbolic Klein-Gordon operator, iterated k-times (cf. Theorem 3, p. 38, formula (II,2;12) of [**1**]):

$$K^k\left\{G_{2k}\left(P\pm i0,m,n\right)\right\}=\delta,$$

where K^k is the n-dimensional ultrahyperbolic Klein-Gordon operator iterated k-times:

$$K^k=\left\{\frac{\partial^2}{\partial x_1^2}+...+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-...-\frac{\partial^2}{\partial x_{p+q}^2}-m^2\right\}^k, \qquad (\mathrm{III},1;1)$$

$m>0$ and $p+q=n$.

It is useful to register the particular case of the Theorem 1, corresponding to $\alpha = 2k$.

Theorem 2. *The elementary solutions* $G_{2k}(P\pm i0,m,n)$ *admit the integral representation*

$$G_{2k}(P \pm i0, m, n) = C_{\pm}(n,k,q) \cdot \lim_{\varepsilon \to 0} \int_0^{\infty} e^{\pm i(m^2 \mp i\varepsilon)t} t^{k-\frac{n}{2}-1} \cdot e^{\mp \frac{i}{4t}(P \pm i\varepsilon|x|^2)} dt, \tag{III, 1; 2}$$

where the constants $C_{\pm}(n,k,q)$ *are*

$$C_{\pm}(n,k,q) = \frac{2^{1-\frac{n}{2}} e^{\pm i\frac{\pi}{2}q} e^{\mp i\frac{\pi}{2}\left(\frac{n}{2}-k\right)}}{(2\pi)^{\frac{n}{2}}\Gamma(k)}. \tag{III, 1; 3}$$

Moreover, it is useful to register the particular case of this theorem, corresponding to $n = 4$, $q = 1$, $k = 1$.

Theorem 3. *The solutions* $G_2(P \pm i0, m, 4)$ *of the Klein-Gordon operator admit the representation*

$$G_2(P \pm i0, m, 4) = \lim_{\varepsilon \to 0} \frac{1}{16\pi^2} \int_0^{\infty} e^{\pm i(m^2 \mp i\varepsilon)t} \frac{1}{t^2} e^{\mp \frac{1}{4t}(x_1^2+x_2^2+x_3^2-x_4^2 \pm i\varepsilon|x|^2)} dt. \tag{III, 1; 4}$$

This last formula, when we adopt the plus sign, has been stablished by Bogoliubov-Parasiuk (cf. [**5**], p. 233) and plays an essential role in the theory, due to these authors, of causal distributional products.

IV.1. Note.

We know (cf. [**1**], p. 37, formula (II,2;7)), that

$$G_0(P \pm i0, m, n) = \delta.$$

Therefore, the thesis of Theorem 2, permit us, by taking limits in both members of (II,1;6) for $k \to 0$, obtain an integral representation of the δ-measure of Dirac, by means of a simple (symbolic) Fourier integral.

V.1. The integral representation of the causal (anticausal) distribution $H_\alpha(P \pm i0, n)$.

We define the causal (anticausal) distribution $H_\alpha(P \pm i0, n)$ as follows:

$$H_\alpha(P \pm i0, n) = \frac{e^{\frac{i\pi\alpha}{2}} e^{\pm i\frac{\pi}{2}q} \Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} (P \pm i0)^{\frac{\alpha-n}{2}}, \qquad \text{(V,1;1)}$$

where $\alpha \in \mathbb{C}$, P is defined by (I,1;1) and q is the number of negative terms of the quadratic form P.

We know that $G_\alpha(P \pm i0, m = 0, n) = H_\alpha(P \pm i0, n)$. (cf. [**1**], page 49, Theorem 1, formula (II,7;1)).

The distributional functions H_α are the causal (anticausal) analogues of the elliptic kernel of M. Riesz ([**6**], pp. 16-21) and have analogue properties, that we use to obtain causal (anticausal) solutions of the n-dimensional ultrahyperbolic operator, iterated k-times (k integer ≥ 1),

$$L^k = \left\{ \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - ... - \frac{\partial^2}{\partial x_{p+q}^2} \right\}^k . \qquad \text{(V,1;2)}$$

We shall give a representation of the kernel $H_\alpha(P \pm i0, n)$ (cf. (V,1;1)) by means of an inverse Laplace transform.

We start from the formula (31), p. 264, [**4**], vol.1,

$$\begin{aligned} \int_0^\infty e^{-pt} \exp\left(-\alpha e^{-t}\right) dt &\overset{\text{def}}{=} f(\alpha, p) \\ &= \alpha^{-p} \gamma(p, \alpha), \end{aligned} \qquad \text{(V,1;3)}$$

here (cf. [**4**], p. 387)

$$\begin{aligned} \gamma(\alpha, x) &= \int_0^x e^{-t} t^{\alpha-1} dt \\ &= \alpha^{-1} x^\alpha \; {}_1F_1(\alpha; \alpha+1; -x), \end{aligned} \qquad \text{(V,1;4)}$$

is the incomplete gamma function and ${}_1F_1(a;c;z)$ is the Kummer's confluent hypergeometric series (cf. [**4**], p. 373).

By making the change of parameters

$$\begin{aligned} \alpha &= \left(P \pm i\varepsilon|x|^2\right), \\ p &= -\left(\frac{\alpha - n}{2}\right), \end{aligned} \tag{V, 1; 5}$$

where $\varepsilon > 0$ and $P = P(x) = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2$, $p + q = n$; formula (V,1;3) can be written

$$\begin{aligned} f\left[(P \pm i\varepsilon|x|^2), -\left(\frac{\alpha - n}{2}\right)\right] &= \int_0^\infty e^{-\left(\frac{\alpha-n}{2}\right)t} \exp\left[-(\mathrm{P} \pm \mathrm{i}\varepsilon|\mathrm{x}|^2)\mathrm{e}^{-\mathrm{t}}\right] \mathrm{dt} \\ &= (P \pm i\varepsilon|x|^2)^{\frac{\alpha-n}{2}} \gamma\left(-\frac{\alpha - n}{2}, P \pm i\varepsilon|x|^2\right). \end{aligned} \tag{V, 1; 6}$$

We obtain, taking limits in both members of (V,1;6), for $\varepsilon \to 0$,

$$\lim_{\varepsilon \to 0} \int_0^\infty e^{-\left(\frac{\alpha-n}{2}\right)t} \exp\left[-(\mathrm{P} \pm \mathrm{i}\varepsilon|\mathrm{x}|^2)\mathrm{e}^{-\mathrm{t}}\right] \mathrm{dt} = (\mathrm{P} \pm \mathrm{i}0)^{\frac{\alpha-\mathrm{n}}{2}} \gamma\left(\frac{\mathrm{n} - \alpha}{2}, \mathrm{P} \pm \mathrm{i}0\right). \tag{V, 1; 7}$$

Taking into account the definitory formula of $H_\alpha(P \pm i0, n)$ (cf. (V,1;1)), by multiplying both members of (V,1;7) by adequate constants, we can write, equivalently,

$$\begin{aligned} &C_\pm(n, q, \alpha)\gamma^{-1}\left(\frac{n - \alpha}{2}, P \pm i0\right) \cdot \lim_{\varepsilon \to 0} \int_0^\infty e^{-\frac{\alpha-n}{2}t} \exp\left[-(\mathrm{P} \pm \mathrm{i}\varepsilon|\mathrm{x}|^2)\mathrm{e}^{-\mathrm{t}}\right] \mathrm{dt} \\ &= \frac{e^{i\frac{\pi}{2}\alpha} e^{\pm i\frac{\pi}{2}q} \Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} (P \pm i0)^{\frac{\alpha-n}{2}}. \end{aligned} \tag{V, 1; 8}$$

From the definitory formula (V,1;1) the right-hand member of (V,1;8) is, precisely, $H_\alpha(P \pm i0, n)$. Therefore, we have proved the following theorem

Theorem 4. *The kernels $H_\alpha(P \pm i0)$ admit the integral representation*

$$H_\alpha(P \pm i0, n) = C_\pm(n, q, \alpha)\gamma^{-1}\left(\tfrac{n-\alpha}{2}, P \pm i0, n\right) \lim_{\varepsilon \to 0} \int_0^\infty e^{-(P \pm i\varepsilon|x|^2)t} e^{\left(\frac{\alpha-n}{2}\right)e^{-t}} dt, \tag{V, 1; 9}$$

where the constants $C_\pm(n,q,\alpha)$ *are*

$$C_\pm(n,q,\alpha)=\frac{e^{i\frac{\pi}{2}\alpha}e^{\pm i\frac{\pi}{2}q}\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}. \qquad (\mathrm{V},1;10)$$

It is useful to register the particular case of the Theorem 4 corresponding to $\alpha=2k$. We observe that the distributions $H_{2k}(P\pm i0,n)$ are (causal, anticausal) elementary solutions of the homogeneous ultrahyperbolic operator iterated k-times (cf. **[1]**, p. 41, formulae (II,3;13) and (II,4;4)).

Theorem 5. *The elementary solutions* $H_{2k}(P\pm i0,n)$ *admit the integral representation*

$$H_{2k}(P\pm i0,n)=C_\pm(n,k,q)\gamma^{-1}\left(\tfrac{n-2k}{2},P\pm i0,n\right)\lim_{\varepsilon\to0}\int_0^\infty e^{-(P\pm i\varepsilon|x|^2)t}e^{\left(\frac{2k-n}{2}\right)e^{-t}}dt, \qquad (\mathrm{V},1;11)$$

where the constants $C_\pm(n,k,q)$ *are*

$$C_\pm(n,k,q)=\frac{e^{i\pi k}e^{\pm i\frac{\pi}{2}q}\Gamma\left(\frac{n-2k}{2}\right)}{2^\alpha\pi^{\frac{n}{2}}\Gamma(k)}. \qquad (\mathrm{V},1;12)$$

Note. We know that $H_0(P\pm i0,n)=\delta$ (cf. form. (II,1;8), p. 323, **[7]**).

Therefore, the thesis of Theorem 5, permit us, by taking limits in both members of (V,1;9) for $k\to0$, obtain an integral representation of the δ- measure of Dirac by means of an inverse Laplace transform.

References.

[1] Trione, S.E., *Distributional Products,* Cursos de Matemática, **3**, IAM - CONICET, Buenos Aires, Argentina, 1980.

[2] Gelfand, I.M. and G.E. Shilov, *Generalized Functions, Vol.I,* Academic Press, New York, 1964.

[3] Watson, G.N., *A treatise on the theory of Bessel functions,* Second Edition, Cambridge, University Press, 1947.

[**4**] Bateman, *Manuscript proyect, Tables of Integral Transforms,* Vol.I, McGraw-Hill, New York, 1954.

[**5**] Bogoliubov, N.N. and O.S. Parasiuk, *Uber die multiplication der Kausal - functionen in der Quanten-theorie der Felder,* Acta Mathematica, **97**, 227-266, 1957.

[**6**] Riesz, M., *L'integrale de Riemann-Liouville et le probléme de Cauchy*, Acta Mathematica, **81**, pp. 1-223, 1953.

[**7**] Trione, S.E., *On the Fourier Transform of Causal Distributions,* Studies in Applied Mathematics, **55**, 315-326, 1976.