COVARIANT NON-LINEAR NON-EQUILIBRIUM THERMODYNAMICS AND THE ERGODIC THEORY OF STOCHASTIC AND QUANTUM FLOWS

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Abstract - We give a covariant theory of non-linear non-equilibrium thermodynamics in terms of non-Riemannian geometries. We give a gauge potential characterization of irreversibility. We extend our theory to supersymmetric systems. We present the ergodic structures of the stochastic flows: the Koopman and Perron-Frobenius stochastic semigroups, and the Lyapunov stochastic exponents. We study the problem of instability of the stochastic flows. We indicate the extension of the formalism to quantum mechanics.

I. The Torsion Geometry of Fokker-Planck Diffusions

In this article M denotes a smooth compact orientable n-dimensional manifold, further provided with a linear connection ∇ which we assume to be compatible with a Riemannian metric g on M; i.e. $\nabla g = 0$. The Christoffel coefficients of ∇ can be decomposed as [1,2]

$$\Gamma^{\alpha}_{\beta\gamma} = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} + K^{\alpha}_{\beta\gamma}. \tag{1.1}$$

The first term in (1.1) are the Christoffel coefficients of the Levi-Civita connection ∇^g associated to g, and

$$K^{\alpha}_{\beta\gamma} = T^{\alpha}_{\beta\gamma} + S^{\alpha}_{\beta\gamma} + S^{\alpha}_{\gamma\beta}, \tag{1.2}$$

is the cotorsion tensor, with $S^{\alpha}_{\beta\gamma} = g^{\alpha\nu} g_{\beta\kappa} T^{\kappa}_{\nu\gamma}$, and $T^{\alpha}_{\beta\gamma} = 1/2(\Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta})$ the skew-symmetric torsion tensor. We are interested in (one-half) the Laplacian operator associated to ∇, i.e. the operator acting on smooth functions on M defined as

$$H(\nabla) := 1/2\nabla^2 = 1/2 g^{\alpha\beta}\nabla_\alpha\nabla_\beta. \tag{1.3}$$

A straightforward computation shows that that $H(\nabla)$ only depends in the trace of the torsion tensor and g:

$$H(\nabla) \equiv H(g,Q) = 1/2\triangle_g + g^{\alpha\beta}Q_\beta\partial_\alpha, \tag{1.4}$$

with $Q := Q_\beta dx^\beta = T^\nu_{\nu\beta}dx^\beta$, the trace-torsion one-form; $\triangle_g$ is the Laplace-Beltrami operator of g: $\triangle_g f = \mathrm{div_g}\ \mathrm{grad} f$, $f \in C^\infty(M)$, with $\mathrm{div_g}$ the Riemannian divergence. Note that the conjugate vector-field of the trace-torsion Q, i.e. the vector-field $\hat{Q} = g^{\alpha\beta}Q_\beta$, is the covariant drift of the diffusion. Note further that if we rescale ∇ by $\eta\nabla$ in (1.3), η a real parameter (say $\hbar$), then the diffusion term and the drift are *both* quadratic in η; from now, we set $\eta = 1$. We remark that from all the terms of the irreducible decomposition of the torsion tensor, only the trace-torsion component is manifested in $H(\nabla)$ Then, to obtain a one-to-one correspondance between Markovian diffusion semigroups $\{P_x, x \in M\}$ on M with generator $H(\nabla)$, and linear connections ∇, for general dimension other than 2 (in this case the torsion is of trace type), we restrict ourselves to the so-called Riemann-Cartan-Weyl connections with torsion given by the trace component exclusively. In this case, the Christoffel coefficients of the metric compatible connection ∇ are of the form

$$\Gamma^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{2}{(n-1)}\left\{\delta^\alpha_\beta\ Q_\gamma - g_{\beta\gamma}\ Q^\alpha\right\} \tag{1.5}$$

Therefore, for the RCW (Riemann-Cartan-Weyl) connection ∇ defined in (1.5), we have a Markovian diffusion process $\{P_x, x \in M\}$ with generator the generalized Laplacian $H(g,Q)$ associated to ∇ by (1.4). $H(g,Q)$ is the backward Fokker-Planck (FP) operator of our theory. We have called these diffusions as (spin 0) *RCW diffusions* [2,3].

It is essential to remark that in formulating a covariant theory of thermodynamics, so that M is the manifold of the macroscopic coordinates of a thermodynamical system, these coordinates are *not* of a phase-space description, rather a configuration space. The reason for this is group theoretical. Connections with torsion are introduced in terms of the Cartan soldering one-form, i.e. a linear equivariant identification between the tangent space T_xM of M at every $x \in M$ with R^n conceived as an homogeneous space given by the quotient $Affine(n)/O(n)$, where $O(n)$ and $Affine(n)$ are the groups of orthogonal and affine transformations on R^n respectively [4,5]; these groups are unrelated to the symplectic group (of canonical transformations), which is the symmetry group for a phase-space description. We must stress the fact that one can formulate classical mechanics by applying the soldering method without recourse of a phase-space description [5]. Remarkably, by extending these methods of classical mechanics to non-linear stochastic fluctuations, the RCW diffusions can be introduced as the parallel transport with a RCW connection (1.5) of the canonical realization of a Wiener process on the bundle of orthogonal frames, further projected on M [6]. In other words, we are formulating a gauge-theory of diffusions on a configuration manifold M, with gauge group given by the local orthogonal group augmented by the local symmetry group of the trace-torsion Q. Section II will elaborate on this.

II. de Rham-Kodaira-Hodge Decomposition of the Drift and Maxwell-de Rham Equations

Consider the Hilbert space of square summable ω of smooth differential forms of degree q (henceforth, q-forms) on M, with respect to the Riemannian density vol_g. We shall denote this space as $L^{2,q}$. The inner product is $< \omega, \phi >= \int_M < \omega(x), \phi(x) > \mathrm{vol}_g$, where the integrand is given by the natural pairing between the components of ω and the conjugate tensor: $g^{\alpha_1\beta_1} \ldots g^{\alpha_q\beta_q} \phi_{\beta_1\ldots\beta_q}$. The de Rham-Kodaira operator on $L^{2,q}$ is defined as

$$\triangle = -(d + \delta)^2 = -(d\delta + \delta d), \tag{2.1}$$

where δ is the formal adjoint defined on $L^{2,q+1}$ of the exterior differential operator d defined on $L^{2,q}$: $< \delta\phi, \omega >=< \phi, d\omega >$, for $\phi \in L^{2,q+1}$ and $\omega \in L^{2,q}$; vol_g is the Riemannian volume element. From the elementary fact that $d^2 = 0$, follows that $\delta^2 = 0$. In the case of $q = 0$, $\delta = -\mathrm{div}_g$, so that on smooth functions, the de Rham-Kodaira operator coincides with the Laplace-Beltrami operator. In the general case, there is a coupling of the curvature to the q-form, as we shall see below.

The de Rham-Kodaira-Hodge theorem [9] states that $L^{2,1}$ admits the following invariant decomposition. For any $\omega \in L^{2,1}$ we have the decomposition:

$$\omega = d\ f + A_{cocl} + A_{harm}, \tag{2.2}$$

where $f \in C^\infty(M, R)$, A_{cocl} is a co-closed smooth 1-form:

$$\delta A_{cocl} = -div_g \hat{A}_{cocl} = 0,$$

(from now onwards, given a 1-form A, $\hat{A}$ denotes its conjugate vector-field with components $\hat{A}^\alpha = g^{\alpha\beta}A_\beta$) and A_{harm} is a co-closed and closed smooth 1-form:

$$\delta A_{harm} = 0, dA_{harm} = 0. \tag{2.3}$$

Otherwise stated, A_{harm} is an harmonic one-form, i.e. $\triangle A_{harm} = 0$. Furthermore, this decomposition is orthogonal in $L^{2,1}$, i.e.:

$$< d\ f,\ A_{cocl} >=< d\ f,\ A_{harm} >=< A_{cocl}, A_{harm} >= 0. \tag{2.4}$$

We know apply the above theorem to Q, the trace-torsion of the connection ∇, so we can write as above the Hilbert-space orthogonal decomposition

$$Q = df + A_{cocl} + A_{harm}. \tag{2.5}$$

To actually determine the terms in the decomposition (2.5), we recall that $\mu(dx) = \rho \mathrm{vol_g}$ is an invariant measure for $\{P_x, x \in m\}$ if for any smooth function f on M we have

$$\int [H(g,Q)f(x)]\rho(x)\mathrm{vol}_g = 0. \tag{2.6}$$

Thus, ρ is a weak solution of the τ-independent Fokker-Planck-Kolmogorov forward equation:

$$H(g,Q)^\dagger(\rho) = 1/2\triangle_g \rho - \mathrm{div}_g(\rho\hat{Q}) = 0. \tag{2.7}$$

Here, $H(g,Q)^\dagger$ is the adjoint of $H(g,Q)$ with respect to the pairing introduced by vol_g. If we look for a positive smooth ρ, by the Weyl lemma, it results that it is unique [22]. It is easy to prove that $\rho = \psi^2$ if and only if the trace torsion is of the form

$$Q = \mathrm{d}\ \mathrm{ln}\psi + A_{cocl} + A_{harm}, \tag{2.8}$$

where $A_{cocl} = \delta\beta_2/\rho$, and $A_{harm} = \omega_{harm}/\rho$, where β_2 and ω_{harm} are smooth forms on M of degree 2 and 1, respectively, and ω_{harm} is harmonic. We naturally interpretate these 1-forms as "electromagnetic" potentials. We remark that the form of the exact component of Q can be obtained alternatively by Einstein's λ transformations which extend the Weyl conformal transformations of the metric g defined by multiplication by ψ^2 [2,3]; furthermore, the electromagnetic terms correspond to a $U(1)$ symmetry group. It is easy to check from formula (6.3) below, that this factorization of A_{cocl} and A_{harm} is equivalent to the Riemannian orthogonality of $d\,ln\,\psi$ with A_{cocl} and A_{harm} respectively, and consequently with

A; this last orthogonality condition was obtained in [8]. Note that still A_{cocl} and A_{harm} remain to be orthogonal in Hilbert space. (This decomposition of the non-potential drift is absent in previous treatments.)

Then, we have determined the orthogonal decomposition of the trace-torsion drift:

$$\hat{Q} = grad\ ln\psi + \hat{A}, \quad \text{with} \quad A := \hat{A}_{cocl} + \hat{A}_{harm}. \tag{2.9}$$

(2.9) is the constitutive equation of this theory; the scalar $-ln\ (\psi^2).\mathrm{det(g)}$ is the generalized thermodynamic potential of the non-equilibrium theory [8,26].

The transition density $p_\tau^\nabla(x,y)$ is determined as the fundamental solution of the "heat" equation on the first variable x:

$$\frac{\partial u}{\partial \tau} = H(g,Q)(x)u\ . \tag{2.10}$$

It is well known that one can solve for the transition density in terms of a covariant Onsager-Machlup lagrangian [8]; due to the lack of space, we shall elaborate on this elsewhere.

It is quite remarkable that the condition of existance of a stationary solution of the Fokker-Planck equation (2.10) leads to a decomposition of the trace-torsion in which there appears two potentials, one of which is harmonic, which are further normalized by $1/\psi^2$. The corresponding "electromagnetic" fields are

$$F_{cocl} = d(\frac{\delta\beta_2}{\psi^2}) = \frac{1}{\psi^2}d\delta\beta_2 + d(\frac{1}{\psi^2}) \wedge \delta\beta_2 = 1/\psi^2(d\delta\beta_2 - 2d\ ln\psi \wedge \delta\beta_2)). \tag{2.11}$$

The complete Maxwell equations for A_{cocl} are, in addition of (2.11), the equation

$$\delta F_{cocl} = j, \tag{2.12}$$

where j is the current one-form; after some algebra using (2.1) we can rewrite this equation as a non-homogeneous spin 1 wave equation, the so-called Maxwell-de Rham equation [10]

$$\triangle A_{cocl} = trace((\nabla^g)^2)A_{harm} - R^\beta_\alpha(g)\ A_{harm\ \beta}dx^\alpha = -j. \tag{2.13}$$

The first identity in (2.13) with $R^\beta_\alpha(g) = R_{\mu\alpha}{}^{\mu\beta}(g)$ the Ricci curvature tensor associated to g, is the Weitzenbock formula [9a], which is fundamental in topological quantum field theory and monopole theory [17,18]). For the harmonic one-form

$A_{harm} = \omega_{harm}/\psi^2$, it follows from its closedness that its "electromagnetic" field two-form vanishes:

$$F_{harm} = dA_{harm} = d(\frac{\omega_{harm}}{\psi^2}) \equiv 0. \tag{2.14}$$

Thus we call $A_{harm} = \omega_{harm}/\psi^2$ the *Aharonov* $-$ *Bohm* (AB) *potential*: it has a zero field. Finally, the fact that A_{harm} is co-closed, can be expressed in the form of the conservation equation (Lorentz gauge condition):

$$div_g \hat{A}_{harm} = -\delta A_{harm} = 0. \tag{2.15}$$

Equivalently, from the harmonicity of the AB potential, instead of $(2.14, 2.15)$ we can write the *homogeneous* spin 1 wave equation version of (2.13) for A_{harm} .

III. Electromagnetic Potentials and the Breaking of Detailed Balance

We shall introduce the probability vector associated to the RCW diffusion. Consider the vector field

$$J_\tau := p^\nabla_\tau \hat{Q} - \frac{1}{2} grad\ p^\nabla_\tau \qquad (3.1).$$

Then, the Fokker-Planck equation can be written as a seemingly Liouville equation

$$\frac{\partial p^\nabla_\tau}{\partial \tau} + div_g J = 0, \tag{3.2}$$

In the stationary state the probability vector-field is

$$J_{st} = \rho \hat{Q} - \frac{1}{2} grad\ \rho, \tag{3.3}$$

with $\hat{Q}$ given by (2.8). Therefore, J_{st} is

$$J_{st} = \hat{A}, \quad \text{with} \quad A = \delta\beta_2 + \omega_{harm} \tag{3.4}$$

Then, since $\delta\beta_2$ and ω_{harm} are divergenceless (they are co-closed), we have

$$div_g J_{st} = -\delta J_{st} = -\delta \hat{A} = 0. \tag{3.5}$$

This is the Liouville equation. Note that in the case in which we set $A \equiv 0$, or equivalently by orthogonality, when both $\delta\beta_2$ and ω_{harm} vanish, we have a null probability vector.

We can now characterize the irreversibility (i.e., breaking of detailed balance) for an RCW diffusion, without introducing the usual time-reversal operator [7,8]. For this we introduce the equivalent notion to reversibility known as symmetrizability of a diffusion process. In general, a diffusion with generator L and invariant measure μ is symmetrizable iff L is a symmetric operator in $L^2(\mu)$ [11]. Remarkably, this characterization goes back to Kolmogorov [12]. Therefore, for a RCW diffusion, detailed balance is equivalent to the conditions that A vanishes, or equivalently, that $\delta\beta_2$ and ω_{harm} vanish.

IV. The Perron-Frobenius Stochastic Semigroups

Consider the canonical Wiener space Ω of maps $\omega : R \to R^n, \omega(0) = 0$, with the canonical realization of the Wiener process $W_\tau(\omega) = \omega(\tau)$. The stochastic motions associated to a RCW diffusion are described by the Itô stochastic differential equation (s.d.e.),

$$dB_\tau = \hat{Q}(B_\tau)d\tau + Y(B_\tau)dW_\tau, \tag{4.1}$$

where $YY^\dagger = g$, i.e. Y is a square root of g. A flow of the s.d.e. (4.1) is a mapping $F_\tau : M \times \Omega \to M, \quad \tau \geq 0$, such that for each $\omega \in \Omega$, the mapping $F_.(.\ ,\omega) : [0,\infty) \times M \to M$ is continuous and such that $\{F_\tau(x) : \tau \geq 0\}$ is a solution of (4.1) with $F_0(x) = x$, for any $x \in M$. In other words, $F_\tau(x,-)$ is the solution of (4.1) starting at $x \in M$. Assume now that all components Y^α_β, $\hat{Q}^\alpha$, $\alpha = 1,\ldots,n$ of the vector fields Y and $\hat{Q}$ on M in (4.1), lie in the Sobolev spaces $H^{s+2}(M)$ and $H^{s+1}(M)$ respectively, where $H^s(M) = W^{2,s}(M)$, with $s > n/2 + k$. In this case, the flow of (4.1) induces a *diffeomorphism* in $H^s(M,M)$ and hence (by Sobolev's embedding theorem) a diffeomorphism in $C^k(M,M) = \{f : M \to M : f$ and f^{-1} are k times continously differentiable $\}$: i.e.

$$F_\tau(\omega) : M \to M, \quad F_\tau(\omega)(x) = F_\tau(x,\omega)$$

is a diffeomorphism of M almost surely for $\tau \geq 0$ and $\omega \in \Omega$ [13]. Then, its derivative mapping (also called the tangent extension) [19] $TF_\tau(.,\omega) : T_.M \to T_{F_\tau(.,\omega)}M$ is a diffeomorphism of TM of class C^{r-1}. We shall assume in the following, that these analytical conditions are satisfied.

It is most remarkable that the infinite-dimensional group of diffeomorphisms of M, which is the imprint of any covariant theory, say relativity, is the symmetry group of the stochastic flows. This sets the basis for the ergodic theory of the RCW diffusions we shall elaborate in the last sections of this article.

For stochastic flows the usual composition rules are unvalid. We have instead the co-cycle rule:

$$F_{\tau+\tau'}(\omega) \;=\; F_\tau(\theta_{\tau'}(\omega))\circ F_{\tau'}(\omega), \tau, \tau' \geq 0, \tag{4.2}$$

almost surely for $\omega \in \Omega$, where $\theta_\tau : \Omega \to \Omega$ is the canonical shift on Wiener space: For any $s \mapsto \omega(s)$, $\theta_\tau(\omega)(s) = \omega(\tau + s) - \omega(\tau)$. Consider the enlarged space $S = M \times \Omega$, and the mapping

$$\Theta_\tau : S \to S,\ \Theta_\tau(x,\omega) := (F_\tau(\omega)(x), \theta_\tau(\omega)),\ (x,\omega) \in M \times \Omega. \tag{4.3}$$

Then, since $F_\tau(F_{\tau'}(x,\omega), \theta_{\tau'}(\omega)) = F_{\tau+\tau'}(x,\omega), a.s.$ we have, $\Theta_{\tau+\tau'} = \Theta_\tau \circ \Theta_{\tau'}, \tau, \tau' \geq 0$, a.s.. Furthermore, if $\mu = \rho \mathrm{vol}_g$ is an invariant measure for (4.1), and P^W denotes the Wiener measure on Ω, then the measure $\mu^S = \mu \otimes P^W$ is **invariant** by the flow: $\mu^S(\Theta_\tau^{-1}(B \times \Lambda)) = \mu^S(B \times \Lambda)$. for any Borel measurable sets $B \in \mathcal{B}(M), \Lambda \in \mathcal{B}(\Omega)$; indeed, for any such B and Λ we have:

$$\begin{aligned} &\mu \otimes P^W(\Theta_\tau^{-1}(B \times \Lambda)) = \mu \otimes P^W\{(x,\omega) : F_\tau(\omega)x \in B,\ \theta_\tau(\omega) \in B\} \\ &= \mu \otimes P^W\{(x,\omega) : F_\tau(\omega)x \in B\}.\mu \otimes P^W\{(x,\omega) : \theta_\tau(\omega) \in \Lambda | F_\tau(\omega)x \in B\} \\ &= (\int_{x \in M} p_\tau(x,B)d\rho(x)).\rho \otimes P^W\{(x,\omega) : \theta_\tau(\omega) \in \Lambda\} = \mu(B).P(\Lambda) \end{aligned}, \tag{4.4}$$

since μ is invariant and the events of the conditional probability are independent. Consider the triple $(S, \mathcal{F}, \mu^S)$, where $\mathcal{F}$ is the σ-algebra $\mathcal{B}(M) \times \mathcal{B}(\Omega)$, the product of σ algebras of measurable sets on M and Ω respectively, $S = M \times \Omega$ and $\mu^S = \mu \otimes P^W$. Then, $(S, \mathcal{F}, \mu^S)$ is a *stochastic phase space* with μ^S a $\{\Theta_\tau : \tau \geq 0\}$-invariant measure representing an equilibrium measure. We introduce the *stochastic Koopman semigroup* of operators: $(V_\tau f)(y) = f(\Theta_\tau(y)), y \in S, \tau \geq 0$ and $f \in L^\infty(S)$. Now, for any density r on S and $F \in \mathcal{F}$, define $\int_F (U_\tau r)(y) d\mu(y) := \int_{\Theta_\tau^{-1}(F)} r(y) d\mu(y)$; this is the *stochastic Perron-Frobenius (PF)* semigroup of operators. These semigroups are adjoint with respect to the pairing defined by the measure μ.

V. The Stochastic Lyapunov Spectra

For any $h \in L^2(\mu)$ and for $n \in N$, we put

$$P^n f(x) = \int p_n^\nabla(x,dy) f(y)$$

where $p^{\nabla}_{\tau}(x,y), \tau > 0$, is the transition density of a RCW diffusion on M generated by $H(g,Q)$, with Q given by (2.8) and unique invariant density given by $\mu = \rho \mathrm{vol}_g$. We introduce further the Cesaro sums

$$f^n(x) = \frac{1}{N} \sum_{i=0}^{N-1} P^n f(x).$$

By von Neumann's Ergodic Theorem, $f_n \to f^*, n \to \infty$, where the limit is taken in the $L^2(\mu)$ sense. For $f \in L^1(\mu)$, define $P^n f$ and f^n as above. Then, by Birkhoff's Ergodic Theorem we have: $f^n(x) \to f^*(x), n \to \infty$, for ρ almost everywhere $x \in M$. As usual, $f^*(x)$ is the (internal) time average of f.

The Markov system with transition density $p^{\nabla}_{\tau}(x,y), x, y \in M$ is called ergodic with respect to the invariant measure μ, if for any $f \in L^1(\mu)$, $f^*(x)$ is μ-constant a.e.. Consequently, $f^*(x) = \hat{f}(x)$, with $\hat{f}(x) = \int f(x) d\mu(x) = \int f(x) \psi^2 vol_g$, a.e..

Let us assume as above, that we have a Markov system written in the form of (4.1). We already know that with appropiate regularity conditions on Y and Q, eqt.(17) has an integral flow $F_{\tau}(\omega) : M \to M$, for a.e. ω, and any $\tau > 0$, which is a diffeomorphism of M, with unique invariant measure μ, and on the augmented space $M \times \Omega$ we have the unique invariant measure $\mu \otimes P$, $\mu(B) = \int_B \psi^2(x) vol_g(x)$, $B \in \mathcal{B}(M)$. It was proved by Carverhill ([16]) that there exists a sub-set $\Gamma \subset M \times \Omega$ of full measure (i.e.: $\mu \otimes P(\Gamma) = 1$), such that for $(x,\omega) \in \Gamma$, there exists a filtration of $T_x M$ by linear sub-spaces:

$$0 = V^{(r)}_{(x,\omega)} \subset V^{(r-1)}_{(x,\omega)} \subset \ldots \subset V^{(1)}_{(x,\omega)} = T_x M,$$

together with real numbers $\lambda^{(r)}_{(x,\omega)} < \lambda^{(r-1)}_{(x,\omega)} < \ldots < \lambda^{(1)}_{(x,\omega)}$, such that, for each $j \in \{1, \ldots, r\}$, we have:

$$v \in V^{(j)}_{(x,\omega)} - V^{(j+1)}_{(x,\omega)} \text{ iff } lim_{t \to \infty} \frac{1}{t} ln \| T_x F_{\tau}(\omega) v \| = \lambda^{(j)}_{(x,\omega)}.$$

Note that since M is compact, this definition is independant of the metric in TM, which can then be taken to be the original g. The numbers $\{\lambda^j_{(x,\omega)}, j = 1, \ldots, r\}$ are the *characteristic* or *Lyapunov exponents* of the system (4.1) with respect to ρ. When the system is ergodic with respect to μ, the exponents are $\mu \otimes P$ almost surely independent of (x,ω); in this case we shall simply denote them as $\{\lambda^j, j = 1, \ldots, r\}$. We shall say that the RCW diffusion is **stable** if its biggest Lyapunov exponent λ^1 is negative.

VI. The Witten ψ-Deformed Laplacian and the RCW Diffusions

Assume given a everywhere positive $\psi \in C^\infty(M)$. We then have an induced smoooth density $\rho = \psi^2 \mathrm{vol}_g$ on M. Consider the Hilbert space $L^{2,q,\rho} = L^2\Omega^q(M,\rho)$, of differential forms on M of degree q, square integrable with respect to ρ, with inner product:

$$< \phi_1, \phi_2 >^\rho = \int_M < \phi_1(x), \phi_2(x) > \rho, \tag{6.1}$$

for $\phi_1, \phi_2 \in L^{2,q,\rho}$. Note that in the case of exact 1-forms, the quadratic form $q(df) = 1/2 < df, df >^\rho$, f a smooth function on M, corresponds to the RCW diffusion with generator $H(g,Q)$, with $Q = d\ ln\ \psi$ [3]. Consider now the formal adjoint δ^ψ of d defined on $L^{2,q+1,\rho}$ by:

$$< \delta^\psi \omega, \phi >^\rho = < \omega, d\phi >^\rho, \tag{6.2}$$

for any $\omega \in L^{2,q+1,\rho}$ and $\phi \in L^{2,q,\rho}$. Since $d^2 = 0$, then $(\delta^\psi)^2 = 0$. Note that $\delta^\psi = \psi^{-2}\delta\psi^2$. For any $f \in C^\infty(M)$, and ω a smooth q-form, by integration by parts on $L^{2,q,0}$, we have:

$$\delta(f\omega) = f\delta\omega - i_{\mathrm{grad\ f}}\omega, \tag{6.3}$$

where i_X is the interior product derivation on q-forms: $i_X\omega(X_1, \ldots, X_{q-1}) = \omega(X, X_1, \ldots, X_{q-1})$, ω a q-form, $X, X_1, \ldots, X_{q-1}$ smooth vector-fields on M [19]. Then, using the fact that $f i_X = i_{fX}$, f and X a smooth function and vector-field respectively, we get

$$\delta^\psi \omega = \delta\omega - 2 i_{\mathrm{grad\ ln}\psi}\omega. \tag{6.4}$$

Define on $L^{2,q,\rho}$ the operator: $\triangle^{\psi,q} = -(d+\delta^\psi)^2$ which is equal to $-(d\delta^\psi + \delta^\psi d)$. Recalling that the Lie-derivative operator is $L_X = di_X + i_X d$, X a smooth vector field on M, we finally have

$$\triangle^{\psi,q} = \triangle^q + 2L_{\mathrm{grad\ ln}\psi}, \tag{6.5}$$

Define now the deformed exterior differential operator d^ψ mapping q-forms into $q+1$-forms, $d^\psi := \psi d \psi^{-1}$ so that $d^\psi\omega = d\omega - d\ ln\psi \wedge \omega$. Then, $(d^\psi)^2 = 0$. Its formal adjoint $(d^\psi)^*$ is $(d^\psi)^* := \psi^{-1}\delta\psi$; then, $(d^\psi)^{*2} = 0$. We introduce Witten's deformed Laplacian operator defined as [9,17a] : $L^{\psi,q} := -(d^\psi + d^{\psi *})^2 =$

$-(d^{\psi}d^{\psi *}+d^{\psi *}d)$. The two Laplacian operators families defined above are conformally conjugate:

$$\triangle^{\psi,q}=\psi^{-1}L^{\psi,q}\psi,\quad q=0,\ldots,n. \tag{6.6}$$

Consider the semigroups in $L^{2,q,\rho}$, $q=0,\ldots,n$, with generators given by $\frac{1}{2}\triangle^{\psi q}=\frac{1}{2}\triangle^{q}+L_{\text{grad }\ln\psi}$; we shall denote these semigroups as $P^{q}_{\tau}, q=0,\ldots,n.$ Clearly, P^{0}_{τ} is the stochastic process with infinitesimal generator given by $H(g,Q)$, with $Q=d\ ln\ \psi$.

Since d commutes with the Lie derivative and $\triangle^{q}$, then, d commutes with $\triangle^{\psi,q}$, $q=0,\ldots,n$. Consequently, if ω is a smooth closed 1-form, i.e. $d\omega=0$, then $\triangle^{\psi,q}d\omega=0$. Consequently, we obtain that $dP^{q-1}_{\tau}=P^{q}_{\tau}d$, $q=1,\ldots,n$.

From the identity $(d^{\psi})^{2}=0$, we can introduce a deformed de Rham complex: $H^{q}_{\psi}(M,R):=Ker(d^{\psi}:\Lambda^{q}\rightarrow\Lambda^{q+1})/Ran(d^{\psi}:\Lambda^{q-1}\rightarrow\Lambda^{q})$. Here Λ^{q}, denotes the space of smooth q-differential forms on M, $q=0,\ldots,n$. It is easy to prove that $H^{q}_{\psi}(M,R)\cong H^{q}(M,R)$, where $H^{q}(M,R)$ is de Rham's q-cohomology group constructed from d. The Hodge theorem [9] states that $dimH^{q}(M,R)=dim(Ker(\triangle^{q}))$, which is further equal to $dim(Ker(\triangle^{\psi,q}))$.

VII. RCW Supersymmetric Diffusions

Consider a Hamiltonean operator H on a Hilbert space $\mathbf{H}$, together with a self-adjoint operator Q and a bounded self-adjoint operator P both defined on $\mathbf{H}$, such that

$$H=Q^{2}\geq 0, P^{2}=1, \text{ and } \{Q,P\}=QP+PQ=0. \tag{7.1}$$

Then, the triple $\{H,P,Q\}$ is said to be a supersymmetric system [17a], or still, to have supersymmetry. Since P is self-adjoint and $P^{2}=1$, then P has for eigenvalues 1 and -1. Define $\mathbf{H}_{\text{ferm}}=\{\phi\in\mathbf{H},P\phi=-\phi\}$ and $\mathbf{H}_{\text{bos}}=\{\phi\in\mathbf{H},P\phi=\phi\}$, which are called the fermionic and bosonic states, respectively. Then, $Q:\mathbf{H}_{\text{ferm}}\rightarrow\mathbf{H}_{\text{bos}}$ and $Q:\mathbf{H}_{\text{bos}}\rightarrow\mathbf{H}_{\text{ferm}}$, or in other words, Q maps fermionic states into bosonic states and viceversa.

In our theory we take as in [18]: $\mathbf{H}=\oplus^{n}_{q=0}L^{2,q,\mu}$, and H is $1/2\triangle^{\psi}=\frac{1}{2}\triangle+L_{\text{grad }\ln\psi}$ as an operator on forms of arbitrary degree, where $\triangle_{g}=-(d\delta+\delta d)^{2}$. Put further $Q=i(d+\delta^{\psi})$ and P defined on $\mathbf{H}$ by its restriction to q-forms as the operator of multiplication by $(-1)^{q}$. Then, $\{H,P,Q\}$ is a supersymmetric system. Fermionic (bosonic) states are given by odd (even) forms. Note that the knowledge of g,ψ, determines *both* the symmetrizable diffusions of bosons and fermions.

We would like to note as a closing observation, that the RCW geometry is crucial to the definition of the pre-symplectic structure of the loop space on M [2], from which the Atiyah-Singer theorem [24] follows quite straightforwardly in the Riemannian case [25].

VIII. RCW Diffusions and Instability

This Section will be dedicated to show that the present geometrical formalism is tailored to study the instability of the RCW diffusions as a topological problem. We shall start by defining mean exponents of the flow F_τ of (4.1) generated by $H(g,Q)$.

For $v \in S_xM := \{v \in T_xM :< v, v >= 1\}$, and for $p \in R$ define [20,21,14]

$$\mu(v,p) = lim_{t\to\infty}\frac{1}{t}ln\, I\!E\,|TF_\tau(v)|^p,$$

where $|.|$ denotes the operator norm of the linear transformation defined by the derivative mapping $T_xF_\tau : T_xM \to T_{F_\tau(x)}M$. Define further the **moment exponents**:

$$\mu_x(p) = lim_{t\to\infty}\frac{1}{t}ln\, I\!E\,\|T_xF_\tau\|^p.$$

where $I\!E\,(z) = \int_\Omega z(\omega)P^W(d\omega)$ and $z : R^n \to R$ any random function on $R^n \cong T_xM$.

We have the following properties (Prop. 5, [20]): $i)p \to \mu_x(p)$ is a convex function; $ii)p \to \frac{1}{p}\mu_x(p)$ is increasing, and finally $iii) - \frac{1}{p}\mu_x(-p) \leq \lambda^1 \leq \frac{1}{p}\mu_x(p)$ if $p > 0$. We recall that λ^1 is the biggest Lyapunov exponent (see Section V).

We shall say that the flow (or, the system) arising from (4.1) generated by $H(g,\psi)$ is **moment stable** if $\mu_x(1) < 0$, for ρ a.e. $x \in M$. From (iii) we see that if the flow is moment stable, then it is stable.

For our study of stability, we are interested in a semigroup induced by the lift of the diffeomorphisms of M given by the flows of (4.1) to their tangent mappings.

Let $\{F_\tau, \tau \geq 0\}$ be a flow of (4.1) such that $F_\tau(.,\omega) : M \to M$ is a diffeomorphism of class C^r, for almost all $\omega \in \Omega$. Let ϕ be a 1-form with coefficients in C^{r-1}; set

$$\tilde{P}^0_\tau(\phi) = I\!E\,(\phi \circ TF_\tau). \tag{8.2}$$

With the above assumptions, we have:

$$\tilde{P}^0_\tau(\phi) = P^1_\tau(\phi), \tag{8.3}$$

for ϕ a **closed** 1-form of class C^2. This formula follows from the application of the Ito formula to the closed (see Section IV) time-dependant 1-forms $P_{\tau-\tau'}\phi, 0 \leq \tau' \leq \tau$:

The Ito formula [20]: Suppose $\{F_\tau : \tau \geq 0\}$ is a smooth flow of (4.1). Then, if $x_\tau = F_\tau(x_0)$, $v_\tau = T_{x_0}F_\tau(v_0)$, for any 1-form ϕ of class C^2, we have

$$\begin{aligned} \phi(v_\tau) = \phi(v_0) + \int_0^\tau \nabla^g \phi(B(x_s)dW_s)v_s + \int_0^\tau \phi(\nabla^g B(v_s)dW_s) \\ + \frac{1}{2}\int_0^\tau tr\ d\phi(B(x_s)-, \nabla^g B(v_\tau)-)ds + \frac{1}{2}\int_0^\tau \triangle^{\psi,1}\phi(v_s)ds, \end{aligned} \tag{8.4}$$

The problem of moment stability of flows has for long been known to be related to the existance of harmonic forms on smooth compact manifolds [20]. Let us give topological obstructions on M for moment stability:

Theorem Let M be such that $H^1(M,R) \neq 0$. Given a RCW geometry determined by a smooth metric g and a C^2 wave function $\psi \geq 0$ with associated RCW Laplacian $H(g,Q)$, with $Q = d\ ln\ \psi$, then, the flow generated by $H(g,\psi,0)$ with unique invariant density ρ, is not moment stable.

Proof: By hypothesis and Section IV, we have

$$\dim(H^1(M,R)) = \dim(Ker\triangle^{\psi,1}) \neq 0$$

. Then, there exists a smooth 1-form ϕ such that $\triangle^{\psi,1}\phi = 0$ (or equivalently, since $< \triangle^{\psi,1}\phi, \phi >_\rho = < d\phi, d\phi >_\rho + < \delta^\psi\phi, \delta^\psi\phi >_\rho$, ϕ is closed and $\delta^\psi\phi = 0$), and $x \in M$ and $v \in S_xM$, with $\phi(v) > 0$. Since $\phi \in Ker\triangle^{\psi,1}$ iff $P^1_\tau\phi = \phi$, we have

$$0 = lim_{\tau\to\infty}\frac{1}{\tau}ln\phi(v) = lim_{\tau\to\infty}\frac{1}{\tau}ln\ P^1_\tau\phi(v),$$

which by (8.3) equals to $lim_{\tau\to\infty}\frac{1}{\tau}ln\,\mathbb{E}\,\phi(TF_\tau(v))$ which is smaller or equal than $\mu(v,1) \leq \mu_x(1)$, by property iii above (c.q.d.).

Recall that the existance of a non-null harmonic Aharonov-Bohm potential breaks detailed balance; this condition is an obstruction for the moment stability of the flow generated by $H(g,Q)$ with $Q = d\ ln\ \psi$, as if the Bohm-Aharonov would not be present at all! In the general case of the diffusions (4.1) generated by $H(g,Q)$ with Q given by (2.8), by essentially the same argument, we see that they are not moment stable provided there exists an harmonic one-form ϕ such that it preserves the full drift vector field given in (2.8), i.e.: $L_{\hat{A}}\phi = 0$, or, equivalently, $d(i_{\hat{Q}}\phi) = d(g(\hat{Q},\hat{\phi})) = 0$, since ϕ is closed.

IX. Conclusions

We have given a geometrical theory of non-linear non-equilibrium thermodynamics, in terms of the Laplacian operators associated to the RCW geometries. That Riemannian geometry underdescribes diffusion processes was first noted in [7], and the trace-torsion was identified for the description of the covariant drift, albeit with no identification of the RCW Laplacian as the Fokker-Planck operator which incorporates both the drift and the diffusion tensor. Our theory has further allowed to construct the diffusion processes associated to supersymmetric systems, starting from the spin 0 reversible diffusion. Yet, we remark that this supersymmetric property is not extensible to the inclusion of the electromagnetic potentials A in the full drift; i.e. by adding $L_{\hat{A}}$ to H in Section VI one cannot obtain a supersymmetric system, precisely because of the breaking of detailed balance due to A. We have also seen that the existance of the non-zero harmonic term in A is a topological obstruction for the moment stability of the RCW diffusions, even when one does not consider its active role in the drift (Theorem of Section VIII).

We further constructed the "phase space" of the RCW diffusions by identifying the invariant measure $\mu \otimes P^W$. Remarkably, would not be because of the stationary measure of the process on M, the stochastic flows would appear as corresponding to a quantum free field theory described by the Wiener measure. It is important to remark that the stationary density may be associated to a non-linear functional. Indeed, in Witten's theory [17] M is an infinite-dimensional manifold, the *loop space* on M i.e. the space of mappings from the circle to a a finite-dimensional manifold (for instance, R^n or C^n). Then, ψ is a (Morse) function defined on loop space which can be chosen to be a polynomial on the elements ϕ of loop space, for instance, the ϕ^4 theory; this is of interest for studying phase transitions as done in detail by Graham [23].

The construction of the ergodic theory of stochastic flows presented in this article can be thought as an extension to stochastic systems of the probabilistic theory of classical dynamical systems [14]. Furthermore, if M is three or four dimensional space-time (in the latter case, τ is *not* to be confused with the relativistic time variable, while in the non-relativistic case it can be identified with the absolute time) , then ψ is Schroedinger field, and consequently we can define the Lyapunov exponents of quantum diffusions generated by a RCW geometry [2,14,18].

REFERENCES.

1. Hehl,F. et al, Review in Modern Physics, **48** (1976), 3; ibidem, Physics Reports vol. **258** (1995) 1-157,
2. D.Rapoport, Intern. J. Theor. Physics **35**, 2, (1996), 597.
3. D. Rapoport, Int. J.Theor. Physics **30**, 11 (1991) 1497.
4. S. Kobayashi & K.Nomizu, *Foundations of Differentiable Geometry I*, Interscience, New York, (1963).
5. D. Rapoport & S. Sternberg, Annals of Phys. **158** (1984) 447 (1984), and, Nuovo Cimento **80 A**, (1984), 371-383.
6. D.Rapoport, submitted, Procs. XXI Intern. Colloq. in Group Theor. Methods in Physics, Clausthal, Germany, 20-25 July 1996, H. Doebner et al (Eds.).
7. L.Garrido, Physica **100A**, (1980) 140; L.Garrido & J.Llosa, in *Field Theory, Quantization and Statistical Physics*, E.Tirapegui (ed.), Reidel Publs., Dordrecht,(1981).
8. R. Graham, Z.Physik B, **26**, 281, (1977); ibidem as in [7.b], and references therein.
9. a. H.Cycon, H.Froese, W.Kirsch & B.Simon, *Schroedinger Operators with Applications to Quantum Mechanics and Global Geometry*, Springer V., Berlin (1987). b. G. de Rham, *Variétés Différentiables*, Hermann, Paris, (1960).
10. C. Misner, K. Thorpe & J.A. Wheeler, *Gravitation*, Freeman Publs., San Francisco (1973).
11. V. Bogachev and M. Rockner, J. Funct. Analysis **133**, (1995), 168.
12. A.N.Kolmogorov, Math. Annalen **113** (1937), 776-772.
13. A.P.Carverhill & K.D. Elworthy, Zeitschrift für Wahrscheinlickeitstheorie **65**, 245 (1983); P.Baxendale, Compos.Math. **53**, 19-50, (1984). The choice of the Sobolev spaces has a technical status, we profit of its embedding theorem.
14. D. Rapoport, in *Dynamical Systems and Chaos, II, Proceedings*, Tokyo, May 1994, Y.Aizawa et al (edts.),vol. 2, World Scientific, Singapore, (1995).
15. A. Lasota & M. Mackey, *Probabilistic properties of dynamical systems*, Cambridge Univ. Press, Cambridge, 1985. I.Prigogine, as [14].
16. A. Carverhill, Stochastics **14**, 273 (1985)

17. a. E.Witten, J. Diff. Geometry **17**, (1982), 661-692; b. E. Witten, Mathematical Research Letters **1** (1994), 769.

18. D. Rapoport, Lecture Notes SMR 847/13-1995 (ICTP), Conf. on Topological and Geometrical Problems Related to Quantum Field Theory, March, 1995, ICTP, Trieste. D.Rapoport, *Quantum Geometry and Topological Quantum Field Theory*, submitted to Annals of Physics.

19. R. Abraham & J. Marsden, *Foundations of Mechanics*, Benjamin Publ., 1979.

20. K.D.Elworthy, *Geometric Properties of diffusions on manifolds*, in Lecture Notes in Mathematics **1362**, P.L Henequin (ed.), Springer Verlag, (1989).

21. P. Collet, in *Instabilities and Nonequilibrium structures III, Proceedings*, E.Tirapegui and W.Zeller (eds.), Kluwer, Dordrecht, 1991.

22. N. Ikeda & S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North Holland/Kodansha, Amsterdam and Tokyo, (1981). K.D. Elworthy, *Stochastic Differential Equations on Manifolds*, Cambridge Univ. Press (1982).

23. R.Graham & T.Tel, in *Instabilities and Nonequilibrium structures III*, E.Tirapegui and W.Zeller (eds.), Kluwer, Dordrecht, 1991.

24. N.Verline, E. Getzler & M. Vergne, *Heat kernels and Dirac operators*, Springer V., Berlin, 1990.

25. M. Atiyah, Asterisque **2**,43-60, 1985.