THE PERIODIC PARABOLIC EIGENVALUE PROBLEM WITH L^∞ WEIGHT

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ABSTRACT. In this paper we study existence, uniqueness and simplicity of the principal eigenvalue for the Neumann and the Dirichlet periodic parabolic eigenvalue problem with a bounded, possibly discontinuous, weight and suitable regularity conditions on the coefficients.

1. Introduction.

Let Ω be a bounded domain in $I\!R^n$ with $C^{2+\theta}$ boundary, $0 < \theta < 1$, let $\{a_{i,k}(x,t)\}_{1\leq i,k\leq n}$; $\{a_j(x,t)\}_{1\leq j\leq n}$ be two families of $(\theta, \theta/2)$ Hölder continuous functions on $\Omega \times I\!R$. Suppose $a_{i,k}(x,t)$, $a_j(x,t)$ are T-periodic functions in t, satisfying the symmetry condition $a_{i,k} = a_{k,i}$ and such that for some $c > 0$ and all $(x,t) \in \overline{\Omega} \times I\!R$, $(\xi_1, \xi_2, ...\xi_n) \in I\!R^n$

$$\sum_{i,k} a_{i,k}(x,t)\xi_i\xi_j \geq c\sum_i \xi_i^2.$$

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We consider the periodic parabolic boundary eigenvalue problem

$$\left.\begin{aligned} &\partial u/\partial t - \sum a_{i,k}(x,t)D_{i,k}u - \sum a_j(x,t)D_j u = \lambda m(x,t)u \\ &Bu = 0 \\ &u(x,t) = u(x,t+T) \text{ for } (x,t) \in \overline{\Omega} \times I\!R \end{aligned}\right\} \tag{1.1}$$

where $Bu = u_{|\partial\Omega\times I\!R}$ or $Bu = \partial u/\partial\nu$ along $\partial\Omega \times I\!R$. ($\nu$ the exterior normal to Ω). The case $m \in C^{\theta,\theta/2}(\Omega \times I\!R)$, $m(x,t)$ T-periodic in t, is solved, for $Bu = u_{|\partial\Omega\times I\!R}$ by Beltramo-Hess in **[4]** and for general boundary conditions (that includes the Neumann condition), in **[3]** by Beltramo. They find necessary and sufficient conditions for the existence, uniqueness and simplicity of the principal eigenvalue. In **[3]**, the key for existence result is that

$$\int_0^T \sup_{x\in\Omega} m(x,t)dt > 0 \tag{1.2}$$

implies the existence of a Hölder continuous function $c : [0,T] \to I\!R$ such that $\int_0^T c(t)dt > 0$ and such that in a suitable tubular subregion of $\Omega \times [0,T]$ $m(x,t) \geq c(t)$. In this paper, we show that, under the additional assumption $D_i a_{i,j} \in C(\overline{\Omega} \times I\!R)$ these results can be extended for an arbitrary T-periodic function $m \in L^\infty(\Omega \times I\!R)$. The main difficulty is that such a c may not exist. However we prove that (1.2) is equivalent to have m with positive integral in a suitable tubular subregion of $\Omega \times I\!R$. This is sufficient to obtain the desired results.

2. Notation and Preliminaries.

We set, for $u \in C^{2,1}(\overline{\Omega} \times I\!R)$, $Lu = \partial u/\partial t + A(x,t,D)u,$ where

$$A(x,t,D) = -\sum a_{i,k}(x,t)D_{i,k}u - \sum a_j(x,t)D_j u$$

Let $a(x,t)$, $f(x,t)$ be two T-periodic in t functions belonging to $C^{\theta,\theta/2}(\overline{\Omega}\times I\!R)$, $0 < \theta < 1$. We start recalling some well known facts concerning the existence of solutions for the parabolic boundary problem

$$(L + aI)u = f \text{ in } \Omega \times I\!R$$
$$Bu = 0$$
$$u(x,0) = u_0(x)$$

with $Bu = \partial u/\partial \nu$ or $Bu = u_{|\partial\Omega \times I\!R}$.

For $p > 1$, let $W_B^{2,p}(\Omega) = \{f \in W^{2,p}(\Omega) : Bf = 0\}$. Let E be a vector space of functions on $\Omega \times I\!R$, we set $E_T = \{f \in E : f(x,t) = f(x,t+T)$ a.e. $(x,t) \in \Omega \times I\!R\}$ and $E_B = \{f \in E \cap \mathrm{Dom}(B) : Bf = 0\}$. The norm on $L_T^p(\Omega \times I\!R)$ will be the norm $\|f\|_{L_T^p(\Omega\times I\!R)} = \left(\int_{\Omega\times(0,T)} |f|^p\right)^{1/p}$.

We fix, for the whole paper, $n + 2 < p < \infty$. Let $X = L^p(\Omega)$. We consider $A_a(t) : W_B^{2,p}(\Omega) \subseteq X \to X$, $t \in I\!R$, given by

$$A_a(t)u = -\sum a_{i,k}(\,\cdot\,,t)D_{i,k}u - \sum a_j(\,\cdot\,,t)D_j u + a(\,\cdot\,,t)u.$$

Each $A_a(t)$ is a closed, linear and densely defined operator, with domain independent of t. Moreover for k large enough, say $k \geq 1 + \|a\|_\infty$, we set $A = A_{a+k}(0)$. For $0 \leq \alpha \leq 1$ let A^α be defined as in **[7]**. Let X_α be the domain of A^α. For $x \in X_\alpha$ we set $\|x\|_\alpha = \|A^\alpha x\|_{L^p(\Omega)}$. Provided with this norm X_α is a Banach space. Let $\|\ \|_{\alpha\beta}$ denotes the norm in the space of the bounded linear operators from X_α into X_β, $0 \leq \alpha, \beta \leq 1$. Then we have

$$X_\alpha \subseteq X_\beta \text{ for } 0 \leq \beta \leq \alpha \leq 1,\ X_0 = L^p(\Omega),\ X_1 = W_B^{2,p}(\Omega)$$

and for $\beta < \alpha$ the inclusion $i_{\alpha,\beta} : X_\alpha \to X_\beta$ is a compact operator. Moreover for $1/2 + n/(2p) < \alpha \leq 1$ we have $X_\alpha \subseteq C_B^{1+\gamma}(\overline{\Omega})$ for some $0 < \gamma = \gamma(\alpha) < 1$ where $C_B^{1+\gamma}(\overline{\Omega})$ denotes the subspace of the elements in $C^{1+\gamma}(\overline{\Omega})$ satisfying the boundary condition and this inclusion is compact. [cf. **[2]**, p. 16; **[7]**, p. 33].

The inhomogeneous linear evolution equation

$$\begin{cases} \frac{du}{dt} + A_{a+k}(t)u(t) = f(t) & f \in C^\theta([0,T+\omega],X), \quad 0 < \theta \leq 1 \\ u(0) = u_0 & u_0 \in X \end{cases}$$

has an unique solution u satisfying

$$\begin{cases} u \in C([0,T+\omega],X) \cap C^1((0,T+\omega],X & \text{for } u_0 \in X \\ u \in C^1([0,T+\omega],X) & \text{if } u_0 \in X_1. \end{cases}$$

Moreover, for $0 \leq t \leq T + \omega$, $u(t)$ is given by

$$u(t) = U_{a+k}(t,0)u_0 + \int_c^t U_{a+k}(t,\tau)f(\tau)d\tau \tag{2.1}$$

where $U_{a+k} \in B(X)$, $0 \leq \tau \leq t \leq T+\omega$, is the associated evolution operator.

We denote $\Delta = \{(t,\tau) \in [0,T+\omega] \times [0,T+\omega] : 0 \leq \tau \leq t \leq T+\omega\}$ and we consider $U_a(t,\tau) = e^{k(t-\tau)}U_{a+k}(t,\tau)$. Known properties of U_{a+k} (see **[2]**, lemma 2.1) imply, for $(t,\tau) \in \Delta' = \{(t,\tau) \in \Delta : \tau < t\}$, that

$$\|U_a(t,\tau)\|_{\alpha,\beta} \leq c'(\alpha,\beta,\gamma)(t-\tau)^{-\gamma} \text{ for } 0 \leq \alpha \leq \beta < 1, \beta - \alpha < \gamma < 1 \tag{2.2}$$

And for $0 \leq \alpha < \beta \leq 1$, $0 \leq \gamma < \beta - \alpha$, and $(t,\tau),(s,\tau) \in \Delta$

$$\|U_a(t,\tau) - U_a(s,\tau)\|_{\beta,\alpha} \leq c'(\alpha,\beta,\gamma)|t-s|^{\gamma} \tag{2.3}$$

We put, for $2^{-1} + (2p)^{-1} \ \ n < \alpha \leq 1$, $K_{a,\alpha} = U_a(T,0)_{|X_\alpha} : X_\alpha \to X_\alpha$.

Remark 2.1. We observe that $f \in L^p(\Omega)$, $f > 0$ and $(t,\tau) \in \Delta$ imply $U_a(t,\tau)f$ belongs to the interior of the positive cone in $C_B^{1+\gamma}(\overline{\Omega})$, (**[7]**, lemma 13.4).

3. Auxiliary Lemmas.

For $\lambda > 0$ in $I\!R$ it is natural to have a generalized solution operator

$$(I\!L + \lambda)^{-1} : L_T^p(\Omega \times I\!R) \to L_T^p(\Omega \times I\!R)$$

compact and positive. Moreover the restriction to $C_T^{\mu,\mu/2}(\Omega \times I\!R)$ coincides with the classical solution operator. Our aim is to prove that the same result holds for $(I\!L + a)^{-1}$ with $a(x,t) \in L_T^{\infty}(\Omega \times I\!R)$ such that $0 < \delta < a(x,t) < d < \infty$, for some δ, $d \in I\!R$.

Since $p > n+2$, we can fix, from now on, $0 < \alpha < 1$ such that $\frac{1}{2} + \frac{n}{2p} < \alpha < 1$ and $\frac{1}{1-\alpha} < p < \alpha$.

We will need the following

Lemma 3.1. *Suppose as above* $Bu = \partial u/\partial \nu$ *or* $Bu = u_{|\partial\Omega \times I\!R}$. *Let* $a \in C^{\theta,\theta/2}(\overline{\Omega} \times I\!R)$, $0 < \theta < 1$, $a(x,t)$ T*-periodic in* t *satisfying*

$$a \geq 0 \ \text{ and } \ a \not\equiv 0 \ \text{ if } \ Bu = \partial u/\partial \nu_{|\partial\Omega \times I\!R}$$
$$a \geq 0 \ \text{ if } \ Bu = u_{|\partial\Omega}.$$

Let $X_0 = L^p(\Omega)$ *and* $X_1 = W^{2,p}_B(\Omega)$ *in the preceding construction. Then there exists* $0 < \gamma < 1$ *such that the operator*

$$S_a : L^p_T(\Omega \times I\!R) \to C^\gamma([0, T+\omega], X_\alpha)$$

defined by

$$S_a(g)t = U_a(t,0)[I - K_a]^{-1}\left(\int_0^T U_a(T,\tau)g(\tau)d\tau\right) + \int_c^t U_a(t,\tau)g(\tau)d\tau$$

is an injective, positive, and bounded operator.

Proof. We fix β such that $1 > \beta > \alpha$ and $1/(1-\beta) < p$, also we fix δ such that $0 < \delta < \beta - \alpha$, and γ', $1 > \gamma' > \beta$, such that $p > 1/(1-\gamma')$. We set

$$S_{a,1}(g)(t) = \int_c^t U_a(t,\tau)g(\tau)d\tau$$
$$S_{a,2}(g)(t) = U_a(t,0)[I - K_{a,\alpha}]^{-1}\left(\int_0^T U_a(T,\tau)g(\tau)d\tau\right)$$

We note that the integrals exist in the Bochner sense. The strong continuity of the evolution operator implies the measurability of the application from $[0, T+\omega]$ into X_α given by $\tau \to U_a(t,\tau)g(\tau)$. (2.2) and Hölder inequality give us

$$\sup_{t\in[0,T+\omega]} \int_c^t \|U_a(t,\tau)g(\tau)\|_\alpha d\tau < c\|g\|_{L^p_T(\Omega\times I\!R)}.$$

Also, for $0 \le s \le t \le T+\omega$

$$\|(S_{a,1}g)(t) - (S_{a,1}g)(s)\|_\alpha \le \int_0^s \|[U_a(t,s) - U_a(s,s)]\|_{\alpha,\beta}\ \|U_a(s,\tau)]\|_{0,\beta}\ \|g(\tau)\|_0 d\tau$$
$$+ \int_s^t \|U_a(s,\tau)\|_{0,\alpha}\ \|g(\tau)\|_0 d\tau.$$

A straightforward computation using (2.2) and (2.3) shows that, for some $c > 0$, $\varepsilon > 0$

$$\|(S_{a,1}g)(t) - (S_{a,1}g)(s)\|_\alpha \le c|t-s|^\varepsilon \|g\|_{L^p_T(\Omega\times I\!R)}$$

$K_a : K_\alpha \to X_\alpha$ is a compact, and strongly positive operator with spectral radius $0 < \rho(K_a) < 1$ (**[7]**, Remark 14.1 and Lemma 14.2), so $(I - K_a)^{-1} : X_\alpha \to X_\alpha$ is bounded, then $(S_{a,2}g)(t)$ is well defined. We have

$$\begin{aligned} &\|(S_{a,2}g)(t) - (S_{a,2}g)(s)\|_\alpha \\ \leq &\|U_a(t,0) - U_a(s,0)\|_{\beta,\alpha} \; \|(I - K_a)^{-1}\|_{\beta,\beta} \int_0^T \; \|U_a(T,\tau)\|_{\beta,0} \; \|g(\tau)\|_0 d\tau \\ \leq &c|t-s|^\delta \|g\|_{L^p_T(\Omega\times I\!R)}. \end{aligned}$$

Also

$$\|(S_{a,2}g)(t)\|_\alpha \leq \|(S_{a,2}g)(t) - (S_{a,2}g)(0)\|_\alpha + \|(S_{a,2}g)(0)\|_\alpha$$

then

$$\sup_{t\in[0,T+\omega]} \|(S_{a,2}g)(t)\|_\alpha \leq c\|g\|_{L^p_T(\Omega\times\, I\!R\,)}.$$

So, for some $\gamma \in (0,1)$, $S_a : L^p_T(\Omega \times\, I\!R\,) \to C^\gamma([0, T+\omega], X_\alpha)$ is bounded.

The positivity assertion follows from remark 2.1.

To prove the injectivity we note that for $g \in L^p_T(\Omega\times\, I\!R\,)$, $S_a(g) = 0$ implies $S_a(g)(t) = 0$ in $C(\overline{\Omega})$ for all t, $t = 0$ gives $(I - K_a)^{-1}(\int_0^T U_a(T,\tau)g(\tau)d\tau) = 0$ and so $\int_0^t U_a(t,\tau)g(\tau)d\tau = 0$ for $0 \leq t \leq T\omega$. Then for $s < t$

$$\begin{aligned} 0 &= \int_c^t U_a(t,\tau)g(\tau)d\tau \\ &= U_a(t,s)\int_0^s U_a(s,\tau)g(\tau)d\tau + \int_s^t U_a(t,\tau)g(\tau)d\tau \\ &= \int_s^t U_a(t,\tau)g(\tau)d\tau = 0. \end{aligned}$$

So $U_a(t,\tau)g(\tau) = 0$ a.e. $\tau \in [0,t]$, for all $0 < t < T+\omega$, then $g = 0$. ∎

We note that, for $g \in C_T^{\theta,\theta/2}(\overline{\Omega}\times\, I\!R\,)$ and $t \in (0,\omega)$, $S_a(g)(t+\omega) = S_a(g)(t)$ and so, by density, the same holds for $g \in L^p_T(\Omega\times\, I\!R\,)$. So $S_a(g)$ has an unique T-periodic extension to $\overline{\Omega}\times\, I\!R$, we will denote this extension also by $S_a(g)$.

Corollary 3.2. *Under the assumption of the Lemma 3.1* $S_a : L^p_T(\Omega \times\, I\!R\,) \to L^p_T(\Omega\times\, I\!R\,)$ *is a compact operator. Moreover, there exists* γ'', $0 < \gamma'' < 1$, *such that*

$$S_a : C^{1+\gamma'',\gamma''}_{T,B}(\overline{\Omega}\times\, I\!R\,) \to C^{1+\gamma'',\gamma''}_{T,B}(\overline{\Omega}\times\, I\!R\,)$$

is a compact operator.

Proof. $1/2 + n/(2p) < \alpha < 1$ implies that there exists $0 < \sigma < 1$ such that $X_\alpha \subseteq C^{1+\sigma}(\overline{\Omega})$. Moreover, for some $0 < \gamma'' < 1$ we have

$$C_T^\gamma(\mathbb{R}, X_\alpha) \subseteq C_{T,B}^{1+\gamma'',\gamma''}(\overline{\Omega} \times \mathbb{R}) \subseteq L_T^p(\overline{\Omega} \times \mathbb{R})$$

with continuous inclusions and the last inclusion is a compact operator by Ascoli Arzela theorem. ∎

Remark 3.3. We set $Y = C_{T,B}^{1+\gamma'',\gamma''}(\overline{\Omega} \times \mathbb{R})$. Then $S_a : L_T^p(\Omega \times \mathbb{R}) \to Y$ is a strongly positive operator. Indeed, for a positive g in $L_T^p(\Omega \times \mathbb{R})$, $S_a g$ belongs to Y, moreover for $t \in \mathbb{R}$ remark 2.1 and the definition of S_a imply that, for the Neumann boundary condition, $S_a(g)(t)$ is a never zero function in $C(\overline{\Omega})$, so $S_a(g)$ belongs to the interior of the positive cone in $C(\overline{\Omega} \times \mathbb{R})$. For the Dirichlet boundary condition, we note that $S_a(g)(t)$ belongs to the interior of the positive cone in $C_B^{1+\gamma}(\overline{\Omega})$ and $\partial(S_a(g)/\partial\nu$ is a continuos and never zero function on $\partial\Omega \times \mathbb{R}$, so $S_a(g)$ belongs to the interior of the positive cone in $C_{T,B}^{1+\gamma'',\gamma''}(\overline{\Omega} \times \mathbb{R})$.

In the sequel Krein Rutman Theorem refers to the version stated in [**1**], Th. 3.2.

Remark 3.4. Under the hypothesis of the Lemma 3.1 the spectral radius of the operator $S_a : L_T^p(\Omega \times \mathbb{R}) \to L_T^p(\Omega \times \mathbb{R})$ agrees with the spectral radius of its restriction $S_a : Y \to Y$.

Indeed, the spectrum of $S_a : L_T^p(\Omega \times \mathbb{R}) \to L_T^p(\Omega \times \mathbb{R})$ is the point spectrum (except by the zero element), and every eigenfunction belongs to Y, so both spectra agree (except perhaps by the zero element).

Krein Rutman theorem, corollary 3.2 and remark 3.3 imply that these spectral radius agree with a positive eigenvalue and no other eigenvalue has positive eigenfunction.

Remark 3.5. Let λ be a positive real number; for $a = \lambda$ we consider the bounded operator $S_\lambda : L_T^p(\Omega \times \mathbb{R}) \to Y$. We observe that $W = S_\lambda(L_T^p(\Omega \times \mathbb{R}))$

is independent of λ. Moreover, for $\lambda, \mu \in I\!R^{>0}$ we have $S_\lambda^{-1} - \lambda I = S_\mu^{-1} - \mu I$ on W.

Definition 3.6. *We define* $I\!L : W \to L_T^p(\Omega \times I\!R)$ *by*

$$I\!L = S_\lambda^{-1} - \lambda I, \qquad \lambda > 0.$$

$I\!L$ is an extension of the differential operator L, such that $I\!L + \lambda : W \to L_T^p(\Omega \times I\!R)$ is a bijective operator with positive inverse. We consider W endowed with the Y-topology. It follows that $I\!L : W \to L_T^p(\Omega \times I\!R)$ is a closed operator.

Let P be the positive cone in Y and let T_1, T_2 be operators on Y, we say $T_1 \ll T_2$ if $(T_2 - T_1)(P) \subseteq (P)^\circ$.

Lemma 3.7. *Suppose $a \in L_T^\infty(\Omega \times I\!R)$ satisfies $\delta < a(x,t) < d$ for some positive constants $0 < \delta < d$ and W, Y as in remarks 3.6 and 3.3 respectively. Then*

(1) $I\!L + a : W \to L_T^p(\Omega \times I\!R)$ is a bijection with continuous inverse.

(2) $(I\!L + a)^{-1} : Y \to Y$ is a strongly positive and compact operator with positive spectral radius r.

(3) $(I\!L + a)^{-1} : L_T^p(\Omega \times I\!R) \to L_T^p(\Omega \times I\!R)$ is a compact operator and its spectral radius agrees with r.

(4) This spectral radius is an eigenvalue with positive eigenfunction and no other eigenvalue has positive eigenfunction.

Proof. We choose $\eta \in I\!R$, $\eta > d$ and we set

$$T_i : L_T^p(\Omega \times I\!R) \to L_T^p(\Omega \times I\!R), \qquad i = 1, 2, 3$$

given by

$$T_1 = (\eta - d)S$$
$$T_2 = S_\eta \circ (\eta - a)$$
$$T_3 = (\eta - \delta)S_\eta$$

where $\eta - a$ denotes the operator multiplication by $\eta - a$. Each T_i is a positive and compact operator, then the spectrum $\sigma(T_i)$ is the point spectrum (except perhaps by the zero element). For $i = 1, 2, 3$ $T_i(L^p_T(\Omega \times I\!R))$ is contained in Y, then the spectrum $\sigma(T_i)$ agrees with the spectrum of the restriction $T_{i|Y} : Y \to Y$ (except perhaps by the zero element). Also, we note that these restrictions are strongly positive operators. Let r_i denotes the spectral radius of T_i. Now $0 < \eta - d < \eta - a < \eta - \delta$ and then, as operators on Y, $T_1 \ll T_2 \ll T_3$. Suppose the Neumann condition, since $(I\!L + \eta)(1) = \eta 1$, the Krein Rutman thorem says that $1/\eta$ is the spectral radius of S_η. The same theorem gives us $r_1 < r_2 < r_3$, and so $0 < r_2 < 1$. For the Dirichlet condition, let λ_0, u_0 be the principal eigenvalue and the positive eigenfunction associated, respectively for L, i.e. $(I\!L + \eta)u_0 = (\eta + \lambda_0)u_0$. So $1/(\eta + \lambda_0)$ is the spectral radius of S_η, then $0 < r_2 < 1$. From this we obtain, in both cases

$$(I\!L + a)^{-1} = [I - (I\!L + \eta)^{-1}(\eta - a)]^{-1}(I\!L + \eta)^{-1}$$

which implies (1)-(4). ■

Suppose a as in Lemma 3.7. We set

$$S_a = (I\!L + a)^{-1} : L^p_T(\Omega \times I\!R) \to L^p_T(\Omega \times I\!R)$$

Note that, for $a \in C^{\theta,\theta/2}(\Omega \times I\!R)$, S_a agrees with the operator defined in the statement of the Lemma 3.1.

Remark 3.8. Suppose the Neumann boundary condition. We consider

$$(I\!L + 1)^{-1} : L^p_T(\Omega \times I\!R) \to L^p_T(\Omega \times I\!R)$$

The Krein Rutman theorem implies that

$$(I\!L + 1)^{-1*} : L^{p'}_T(\Omega \times I\!R) \to L^{p'}_T(\Omega \times I\!R)$$

has a positive eigenvector Ψ with eigenvalue 1. We normalize Ψ such that $\langle \Psi, 1 \rangle = 1$.

Remark 3.9. Let $m(x,t)$ be a T-periodic in t function in $L^\infty_T(\Omega \times I\!R)$ satisfying $\|m\|_\infty \leq 1/2$. Suppose $\lambda \in I\!R^{>0}$, then (by Lemma 3.7)

$$S_{\lambda(1-m)} : L^p_T(\Omega \times I\!R) \to L^p_T(\Omega \times I\!R)$$

is a compact and positive operator with positive spectral radius $\rho_m(\lambda)$. We define $\mu : I\!R^{\geq 0} \to I\!R$ by $\mu_n(\lambda) = \rho_m(\lambda)^{-1} - \lambda$ for $\lambda > 0$ and $\mu_m(0) = 0$. It is known that, for a Holder continuous m, μ_m is a concave function. Now we extend this result to a bounded m.

Lemma 3.10. *Let m be a function in $L^\infty_T(\Omega \times I\!R)$. Then μ_m is a concave function on $[0, \infty)$ and μ_m is analytic on $(0, \infty)$.*

Proof. Without lost of generality we can suppose $\|m\|_\infty \leq 1/2$. We consider the following norm on W.

$$\|f\|_G = \|f\|_{C^{1+\gamma'',\gamma''}_{T,B}(\overline{\Omega}\times[0,T])} + \|(I\!L + 1)f\|_{L^p_T(\Omega\times I\!R)}$$

$W_{\|\cdot\|_G}$ is a Banach space. We consider $T_0 : W_{\|\cdot\|_G} \to L^p_T(\Omega \times I\!R)$ given by

$$T_0 = I\!L + \lambda(1 - m),$$

T_0 is bijective and bicontinuous. Let K be the inclusion $K : W_{\|\cdot\|_G} \to L^p_T(\Omega \times I\!R)$. K is compact, so $T_0 - (\mu_m(\lambda) + \lambda)K$ is a compact perturbation of an isomorphism and then it is a Fredholm operator with zero index. Lemma 3.7 and the Krein Rutman theorem imply that dim Ker $(T_0 - (\mu_m(\lambda) + \lambda)K) = 1$ and if u_0 is a generator of Ker $(T_0 - (\mu_m(\lambda) + \lambda)K)$ then $u_0 \notin R(T_0 - (1\mu_m(\lambda) + \lambda)K)$. The Crandall Rabinowitz lemma (see **[5]**, Lemma 1.3, p. 163) implies that $\mu_m(\lambda)$ is a real analytic function of λ for $\lambda > 0$.

We choose $\{m_j\}_{j\in N}$ a sequence in $C^\infty(\Omega \times I\!R)$, with supp $(m_j) \subseteq K_j \times I\!R$, for some compact subset K_j of Ω, and satisfying $\|m_j\|_\infty \leq 1/2$ and such that m_j converges to m in the L^p sense. Each μ_{m_j} is a concave function on $[0, \infty)$, (**[7]**, Lemma 15.2). We set $T_j : W_{\|\cdot\|_G} \to L^p_T(\Omega \times I\!R)$ given by

$$L_j = I\!L + \lambda(1 - m_j)$$

so $T_j - T_0$ tends to zero in $B(W_{\|\cdot\|_G} \to L^p_T(\Omega \times I\!R)$. Now $T_0 u_0 = (\mu_m(\lambda) + \lambda)u_0$. The Crandall Rabinowitz lemma implies that there exists $\alpha_j(\lambda)$ and u_j satisfying $T_j u_j = \alpha_j(\lambda)u_j$ and such that $u_j \to u_0$ in $W_{\|\cdot\|_G}$ and $\alpha_j(\lambda) \to \mu_m(\lambda) + \lambda$ as j tends to ∞, so $u_j \gg 0$ for a large enough j. By the Krein Rutman theorem $\alpha_j(\lambda) = \mu_{m_j}(\lambda) \to +\lambda$. So $\lim_{j\to\infty} \mu_{m_j}(\lambda) = \mu_m(\lambda)$, for $\lambda > 0$. Also $\mu_{m_j}(0) = \mu_m(0)$. Then $\mu_m(\lambda)$ is a concave function on $[0, \infty)$. ∎

Remark 3.11. The Crandall Rabinowitz lemma implies that for $\lambda > 0$ $\mu_m(\lambda) + \lambda$ is a K-simple eigenvalue of the operator $I\!\!L + \lambda(1-m)$. Now, for $u \in W$ $I\!\!L\, u - \lambda m u - \mu_m(\lambda) u = T_0 u - (\mu_m(\lambda) + \lambda) K u$. Suppose $\mu_m(\lambda) = 0$, let M be the operator $M : W \to L^p_T(\Omega \times I\!\!R)$ given by $Mu = mu$. Then, as in **[4]**, lemma 3.7, λ is an M-simple eigenvalue of $I\!\!L$.

4. Main results.

In this section we will assume that the coefficients $a_{i,j}$, $1 \le i$, $j \le n$ belongs to $C^1(\overline{\Omega} \times I\!\!R)$. Let m be a function in $L^\infty(\Omega \times [a,b])$ such that $\|m\|_\infty \le 1$. We set $m^\sim : [a,b] \to I\!\!R$ defined by $m^\sim(t) = \operatorname*{ess\,sup}_{x\in\Omega} m(x,t)$.

Let π denote the projection $\pi : I\!\!R^n \times I\!\!R \to I\!\!R$ given by $\pi(x,t) = t$. For $B \subseteq I\!\!R^{n+1}$ and $t \in I\!\!R$ we put $B_t = \{x \in I\!\!R^n : (x,t) \in B\}$. Also we set, for a domain Ω and for $\delta > 0$ $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$.

Lemma 4.1. *Let m be a function in $L^\infty(\Omega \times (a,b))$. Suppose $c \in I\!\!R$ such that*

$$\int_a^b m^\sim(t)dt > c.$$

Given $\delta > 0$ such that $\Omega_\delta \ne \emptyset$, there exists a finite disjoint set $\{Q_r\}_{1\le r\le N}$ of congruent open cubes in $I\!\!R^{n+1}$ with edges of length ℓ and parallel to the coordinates axis satisfying

(1) $\ell \le \delta/(2(n+1))$, $Q_r \subseteq \Omega_{\delta/2} \times [a,b]$, $1 \le r \le N$.

(2) $\{\pi(Q_r)\}_{1\le r\le N}$ *is disjoint.*

(3) $\sum_{1\le r\le N} |\pi(Q_r)| = b - a$.

(4) $\int_{\bigcup_{r=1}^N Q_r} m(x,t)dxdt > c\,\ell^n$.

<u>*Proof.*</u> Without lost of generality we assume that $\|m\|_\infty \le 1$. For $k \in N$ we define $m^\sim_k(t) = \operatorname*{ess\,sup}_{x\in\Omega_{1/k}} m(x,t)$. Each $m^\sim_j$ is a measurable function on $[a,b]$. We

have $m_j^{\sim}(t) \leq m_{j+1}^{\sim}(t)$ and $\lim_{j \to \infty} m_j^{\sim}(t) = m^{\sim}(t)$. So

$$\lim_{j \to \infty} \int_a^b m_j^{\sim}(t) dt > c$$

We fix $k \in N$ large enough such that $\int_a^b m_k^{\sim}(t)dt > c$ and $k > 1/\delta$. Let $E(\eta) = \{(x,t) \in \Omega_{1/k} x[a,b] : m(x,t) \geq m_k^{\sim}(t) - \eta\}$. Also we set $(E(\eta))^d = \{(x,t) \in E : (x,t)$ is a density point of $E_\eta\}$.

We fix $\alpha \in (0, 1/2)$. Then we consider for $r \in I\!N$ the set $E(\eta)^{(r)}$ of the points in $(E(\eta))^d$ such that $|Q \cap E(\eta)|/|Q| \geq 1 - \alpha$ for each open cube Q containing (x,t) with diameter less than $1/r$ and edges parallel to the coordinate axis. It is easy to see that $E(\eta)^{(r)}$ is a measurable set. Also $E(\eta)^{(r)} \subseteq E(\eta)^{(s)}$ for $r < s$ and $(E(\eta))^d \subseteq \bigcup_{r \in I\!N} E(\eta)^{(r)}$. Moreover, we have $|(E(\eta))_t| \neq 0$ a.e. $t \in [a,b]$, so $|(E(\eta)^d)_t| \neq 0$ a.e. $t \in [a,b]$ and then $|\pi(E(\eta)^d)| = b - a$. So $\lim_{r \to \infty} |\pi(E(\eta)^{(r)})| \geq |\pi(E(\eta)^d)| = b - a$. Then $\lim_{r \to \infty} |\pi(E(\eta)^{(r)})| = b - a$.

Given $\varepsilon > 0$, we fix $r > 2k$ such that $|\pi(E(\eta)^{(r)})| \geq b - a - \varepsilon$, then we choose ℓ, $0 < \ell < 1/(r(n+1))$ such that $N\ell = b - a$ for some $N \in I\!N$. Let $\{t_i\}_{0 \leq i \leq N}$ be the partition of $[a,b]$ given by $t_i = a + i\ell$, $1 \leq i \leq N$. For $1 \leq i \leq N$, we take a cube Q_i with edges parallel to the coordinate axis and of length ℓ, chosen as follows: If the strip $I\!R^n \times (t_{i-1}, t_i)$ meets $E(\eta)^{(r)}$ we take Q_i such that $Q_i \cap E(\eta)^{(r)} \neq \emptyset$ and $\pi(Q_i) = (t_{i-1}, t_i)$. In the other cases, we choose Q_i such that $Q_i \cap \Omega_{1/k} \neq \emptyset$. Since $E(\eta)^{(r)} \subseteq \Omega_{1/k}$ and $\operatorname{diam}(Q_i) < 1/(2k\sqrt{n+1})$ we have $Q_i \subseteq \Omega_{1/(2k)} \times (t_{i-1}, t_i)$, $1 \leq i \leq N$. Let $I = \{i : 1 \leq i \leq N$ and $(I\!R^n \times (t_{i-1}, t_i)) \cap E(\eta)^{(r)} \neq \emptyset\}$ and let I^c be its complement. Since $|\pi(E(\eta)^{(r)})| \geq b - a - \varepsilon$, I^c satisfies $\sum_{i \in I^c} (t_i - t_{i-1}) < \varepsilon$.

We have, for $i \in I$

$$\int_{Q_i} m(x,t) dx dt = \int_{Q_i \cap E(\eta)} m(x,t) dx dt + \int_{Q_i \cap E(\eta)^c} m(x,t) dx dt.$$

Now

$$
\begin{aligned}
&\int_{Q_i\cap E(\eta)} m(x,t)dxdt \\
&\geq \int_{Q_i\cap E(\eta)} m_k^{\sim}(t)dxdt - \eta|Q_i \cap E_(\eta)| \\
&= \int_{t_{i-1}}^{t_i} m_k^{\sim}(t)(|(Q_i\cap E(\eta))_t| - |(Q_i)_t|)dt + \int_{t_{i-1}}^{t_i} m_k^{\sim}(t)|(Q_i)_t|dt \\
&\quad - \eta|Q_i \cap E_(\eta)| \\
&\geq \int_{t_{i-1}}^{t_i} (|(Q_i\cap E(\eta))_t| - |(Q_i)_t|)dt + \ell^n \int_{t_{i-1}}^{t_i} m_k^{\sim}(t)dt - \eta\ell^{n+1} \\
&= |Q_i \cap E(\eta)| - |Q_i| + \ell^n \int_{t_{i-1}}^{t_i} m_k^{\sim}(t)dt - \eta\ell^{n+1} \\
&\geq -\alpha\ell^{n+1} - \eta\ell^{n+1} + \ell^n \int_{t_{i-1}}^{t_i} m_k^{\sim}(t)dt
\end{aligned}
$$

on the other hand

$$
\begin{aligned}
\left|\int_{Q_i\cap E(\eta)^c} m(x,t)dxdt\right| &\leq |Q_i \cap E_\eta^c| = |Q_i| - |Q_i \cap E(\eta)| \\
&\leq |Q_i|(1-(1-\alpha)) = \alpha\ell^{n+1}
\end{aligned}
$$

So

$$
\sum_{i\in I} \int_{Q_i} m(x,t)dxdt \geq \ell^n \sum_{i\in I} \int_{t_{i-1}}^{t_i} m_k^{\sim}(t)dt - \#(I^c)\alpha\ell^{n+1} - \#(I)\eta\ell^{n+1}
$$

where $\#(I)$ means cardinal of I and, since $\ell\#(I^c) \leq \varepsilon$

$$
\sum_{i\in I^c} \left|\int_{Q_i} m(x,t)dxdt\right| \leq \#(I^c)\ell^{n+1} \leq \varepsilon\ell^n
$$

Hence

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{Q_i} m(x,t)dxdt \\
&\geq \ell^n \int_a^b m_k^{\sim}(t)dt - 2\alpha\ell^{n+1}N - \varepsilon\ell^n - \sum_{i\in I^c} \ell^n \left|\int_{t_{i-1}}^{t_i} m_k^{\sim}(t)dt\right| - \#(I)\eta\ell^{n+1} \\
&\geq \ell^n \int_a^b m_k^{\sim}(t)dt - 2\alpha\ell^{n+1}N - 2\varepsilon\ell^n - \eta(b-a)\ell^n.
\end{aligned}
$$

Finally $\sum_{i=1}^{N} \int_{Q_i} m(x,t)dxdt \geq c\ell^n$ for α, η and ε small enough. ∎

Remark 4.2. By the absolute continuity of the indefinite integral, in Lemma 4.1, Q_1 and Q_N can be chosen with the same projection on $I\!R^n$. Also for δ small enough, we can replace each Q_i by $Q_i^{\sim}$ where $Q_i^{\sim}$ is the parallelepiped with the same basis as Q_i and such that $\pi(Q_i^{\sim}) = (t_{i-1} + \delta, t_i - \delta)$.

Let A, B two sets, we will denote with $A \Delta B$ their symmetric difference $(A - B) \cup (B - A)$.

Remark 4.3 Suppose Ω_ε connected, let $\{Q_i\}_{i=1}^{N}$ be a family of congruent open cubes in $I\!R^{n+1}$ with edges of length $\ell < \varepsilon/2n$ and parallel to the coordinates axis satisfying $\bigcup_{1 \leq i \leq N} Q_i \subseteq \Omega_\varepsilon \times [a,b]$ and $\bigcup_{1 \leq i \leq N} \Pi(Q_i) = [a,b]$, then there exists a tube $B = \{(\gamma(t) + \Omega_0, t), 0 \leq t \leq T\} \subseteq \Omega \times [a,b]$ with $\gamma \in C^\infty([0,T])$, $\gamma^{(j)}(0) = \gamma^{(j)}(T)$ for all j, and Ω_0 a domain with C^∞ boundary such that $\left| \left(\bigcup_{1 \leq i \leq N} Q_i \right) \Delta B \right| \leq \delta$.

Lemma 4.4 *Let m be a function in $L^\infty(\Omega \times I\!R)$, $m(x,t)$ T-periodic in t, suppose*

$$P(m) = \int_0^T \operatorname*{ess\,sup}_{x \in \Omega} m(x,t)dt > 0.$$

Then there exist $\gamma \in C^2(I\!R, \Omega)$ a periodic curve in Ω and a domain Ω_0 in $I\!R^n$ with C^∞ boundary such that the tube $B = \{(\gamma(t) + z, t) : z \in \Omega_0,\ 0 \leq t \leq T\}$ satisfies: $B \subseteq \Omega \times [0,T]$ and $\int_B m(x,t)dxdt > 0$.

Proof. We can assume $\|m\|_\infty \leq 1$. Since Ω has regular boundary, there exists $\varepsilon > 0$ such that Ω_ε is a non empty and connected set. Let $\{Q_i\}_{i=1}^{N}$ be the family of cubes with edges of length ℓ, provided by lemma 4.1 such that $\sum_{i=1}^{N} \int_{Q_i} m(x,t)dxdt > \ell^n P(m)/2$, for this family and $\delta = 4^{-1}\ell^n P(m)$ we consider the tube B, provided by remark 4.3. Then

$$\left| \int_B m(x,t)dxdt - \sum_{i=1}^{N} \int_{Q_i} m(x,t)dxdt \right| \leq 2 \left| B \Delta \left(\bigcup_{1 \leq i \leq N} Q_i \right) \right| < 4^{-1}\ell^n P(m).$$

So $\int_B m(x,t)dxdt \geq \sum_{i=1}^{N} \int_{Q_i} m(x,t)dxdt - 4^{-1}\ell^n P(m) \geq 4^{-1}\ell^n P(m) > 0$ ∎

Theorem 4.5. *Let m be a T-periodic function in $L^\infty(\Omega\times \mathbb{R})$.*

(a) *Suppose $P(m)>0$ and $\langle \Psi, m\rangle < 0$. Then there exist $\lambda>0$, and $w>0$, $w\in C^{1+\gamma'',\gamma''}_{B,T}(\overline{\Omega}\times \mathbb{R})$ solution of the periodic Neumann eigenvalue problem*

$$\mathbb{L}\, w = \lambda m w$$

$$\partial w/\partial \nu_{|\partial\Omega\times \mathbb{R}} = 0.$$

(b) *Suppose $P(m)>0$. Then there exist $\lambda>0$, and $w>0$, $w\in C^{1+\gamma'',\gamma''}_{B,T}(\overline{\Omega}\times \mathbb{R})$ solution of the periodic Dirichlet eigenvalue problem*

$$\mathbb{L}\, w = \lambda m w$$

$$w_{|\partial\Omega\times \mathbb{R}} = 0$$

Proof. First, we treat the case Dirichlet boundary condition. We take $m_j \in C^\infty(\Omega\times \mathbb{R})$, T-periodic with $\mathrm{supp}(m_j)\subseteq K_j\times \mathbb{R}$ for some compact $K_j\subseteq\Omega$, and such that $\lim_{j\to\infty} m_j = m$ in $L^p_T(\Omega\times \mathbb{R})$. We may suppose $\|m\|_\infty \le 1/2$. If the tube B provided by lemma 4.4 is a cylinder $C=\Omega_0\times[0,T]$ the function $\mu^c_{m_j}(\lambda)$ defined by

$$Lu^c_j - \lambda m_j u^c_j = \mu^c_{m_j}(\lambda) u^c_j \quad \text{on} \quad \Omega_0\times \mathbb{R} \tag{4.1}$$

$$u^c_j \in C^{2,1}\left(\overline{\Omega}_0\times \mathbb{R}\right), \quad u^c_{j|\partial\Omega_0\times \mathbb{R}} = 0$$

$$u^c_j > 0 \text{ in } \Omega_0\times \mathbb{R} \text{ and } T-\text{periodic}$$

is such that $\mu^c_{m_j}(\eta) < 0$ for some $\eta>0$ independent of j. This holds because from $\int_C m(x,t)dxdt > 0$ (lemma 4.4), there exists $\varphi\in C^\infty_c(\Omega_0)$, $\varphi>0$, $\int_C \varphi^2(x)dx = 1$ and $c>0$ such that $\int_C m_j(x,t)\varphi^2(x)dxdt > c > 0$ for all j. Also $D_i a_{i,j}\in C_T(\overline{\Omega}\times \mathbb{R})$, so we can apply Prop. 3.1 in **[6]**, p. 110, to obtain that the principal eigenvalues $\lambda^c_{m_j}$ given by

$$Lv^c_j = \lambda^c_j(m_j) m_j v^c_j \quad \text{in} \quad \Omega_0\times \mathbb{R} \tag{4.2}$$

$$v^c_j \in C^{2,1}\left(\overline{\Omega}_0\times \mathbb{R}\right), \quad v^c_{j|\partial\Omega_0\times \mathbb{R}} = 0$$

$$v^c_J > 0 \text{ in } \Omega_0\times \mathbb{R} \text{ and } T-\text{periodic}$$

are uniformly bounded above by η, and from the concavity of $\mu^c_{m_j}(\lambda)$ we obtain $\mu^c_{m_j}(\eta) < 0$ for all j. We normalize v^c_j by $\|v^c_j\|_{L^\infty(C)} = 1$. From (4.2) and the compactness of $(\mathbb{L} + 1)^{-1}$ it follows that there exist (modulo a subsequence) $v^c = \lim_{j\to\infty} v^c_j$ in $L^p(C)$ and $\mu^c_m(\eta) = \lim_{j\to\infty} \mu^c_{m_j}(\eta)$. v^c is a solution of a Dirichlet problem in C of the type (4.1) with weigth $m^c = m_{|C}$ and eigenvalue $\mu^c_m(\eta)$. We denote v_j and v the extensions of v^c_j and v^c respectively, by zero to $\Omega \times \mathbb{R}$. From the maximun principle applied to $w_j = \eta(L+\eta)^{-1}(1+m_j)v_j$ ([**7**], p. 43) we obtain

$$\eta\,(\mathbb{L} + \eta)^{-1}\,((1+m)v) \geq v \tag{4.3}$$

Let $S_\eta : Y \to Y$ be the operator defined by $S_\eta u = \eta(\mathbb{L} + \eta)^{-1}((1+m)u)$, and let ρ, u_η be its spectral radius and a positive eigenfunction associated respectively. So, by (4.3) and the Krein rutman theorem, $\rho \geq 1$. Since $S_\eta u_\eta = \rho u_\eta$ we have $(\mathbb{L} + \lambda^\sim(1-m))^{-1} u_\eta = \left(2\rho^{-1}\eta - \eta\right)^{-1} u_\eta$, where $\lambda^\sim = \rho^{-1}\eta$ and so $\mu_m(\lambda^\sim) = \eta\rho^{-1} - \eta < 0$. Also $\mu_m(0) > 0$. Then we have a solution $u^D \in W$, $\lambda^D > 0$ of the Dirichlet problem

$$\mathbb{L}\,(u^D) = \lambda m u^D \text{ in } \Omega \times \mathbb{R}$$

$$u^D > 0 \text{ in } \Omega \times \mathbb{R} \text{ and } T-\text{periodic}, \quad u^D_{|\partial\Omega\times\mathbb{R}} = 0.$$

If the tube B is not a cylinder, by the change of coordinates $(y,t) = \Phi(x,t) = (x-\gamma(t),t)$ we have a similar problem to (4.1) in a cylinder C with a new operator L^Φ and a new weigth m^Φ with $\int_C m^\Phi(x,t)dxdt > 0$. We denote v_j and v, defined on B, extended by 0 to $\Omega \times \mathbb{R}$ corresponding to the functions v^c_j and v^c defined in the cylinder $C = \Phi(B) = \Omega_0 \times \mathbb{R}$. So we obtain (4.3) on $\Omega \times \mathbb{R}$ and we get the solution u^D in $\Omega \times \mathbb{R}$. We may remark that $\mu^N_{m_j}(\lambda) \leq \mu^D_{m_j}(\eta)$ (the supra index N, D refers to the Neumann or Dirichlet condition). So we have $\mu^N_{m_j}(\eta) < 0$ for all j. This gives that $\mu^N_m(\eta) \leq 0$, but $\mu^N_m(0) = 0$. Now the condition $\langle\Psi, m\rangle < 0$ gives $d\mu^N_m/d\lambda_{|\lambda=0} > 0$ which gives $\mu^N_m(\varepsilon) > 0$ for small enough $\varepsilon > 0$. Existence and uniqueness of the principal eigenvalues $\lambda^D > \lambda^N > 0$ follows from the concavity of $\mu^N_m(\lambda)$ and $\mu^N_m(\lambda)$.

Theorem 4.6. *Under the hypothesis of the theorem 4.1 the principal eigenvalue is an M-simple eigenvalue.*

Proof. Follows from remark 3.11.

Remark 4.7. Since for a T-periodic function $m \in L^\infty(\Omega \times I\!R), \mu_m$ is real analytic and concave, with the same proof give for the case $m \in C^{\theta,\theta/2}(\overline{\Omega} \times I\!R)$, $\theta > 0$ (see [**7**], Theorems 16.1 and 16.3) the following results holds.

Let m be a T-periodic function, $m \in L^\infty(\Omega \times I\!R)$ and let $\underline{m}(t) = \operatorname{ess\,inf} m(x,t)$, $\overline{m}(t) = \operatorname{ess\,sup} m(x,t)$. Suppose that there exists a positive eigenvalue λ with a positive eigenfunction $u_\lambda \in \mathrm{Dom}(I\!L)$ associated, solution of the periodic parabolic boundary eigenvalue problem $I\!L u = \lambda m u$, $Bu = 0$. Then if the boundary condition is the Dirichlet condition we have $P(m) > 0$, and for the Neumann condition we have

(1) $\underline{m} \neq \overline{m}$ in $L^\infty(I\!R)$ implies $P(m) > 0$ and $\langle \Psi, m \rangle < 0$.

(2) $\underline{m} = \overline{m}$ in $L^\infty(I\!R)$ (i.e. m is function of t alone) implies

$$\int_0^T m(t)dt = 0.$$

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